CHAPTER 12

The laws of probability have applications in the science of genetics. The larger and more diverse the population, the greater is the probability that the species will have the characteristics necessary to adapt to changes in the environment. The cheetah, the world's fastest-running mammal, faces extinction because it lacks the genetic diversity necessary to survive disease. Once found worldwide, the species now lives wild in only a few areas of Africa.

PROBABILITY

Millions of Americans each year play state lotteries or play bingo. If you play the lottery, your hope is to beat the odds and be the person with the winning numbers. Mathematicians of the sixteenth, seventeenth, and eighteenth centuries, not satisfied with leaving things to chance, invented the study of probability to use mathematics to determine the likelihood of an event (choosing the winning lottery numbers, for example) occurring.

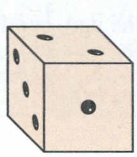
Although the rules of probability were first applied to gaming, they have many other applications. The quality of the food you eat, the pedigree of your cat or dog, and the cost of your car insurance involve probability. In the business of insurance underwriting, when determining the likelihood of an event such as an automobile accident, the age, gender, and location of the driver are facts used in setting the cost of the insurance.

12.1 THE NATURE OF PROBABILITY

PROFILE IN MATHEMATICS

JACOB BERNOULLI (1654–1705)

Swiss mathematician Jacob Bernoulli was considered a pioneer of probability theory. He was part of a famous family of mathematicians and scientists. In *Ars Conjectand,* published posthumously in 1713, he proposed that an increased degree of accuracy can be obtained by increasing the number of trials of an experiment. This theorem, called Bernoulli's theorem (of probability), is also known as the law of large numbers. Bernoulli's theorem of fluid dynamics, used in aircraft wing design, was developed by Daniel Bernoulli (1700–1782), Jacob's second son.



A die (one of a pair of dice) contains six surfaces, called faces. Each face contains a unique number of dots, from 1 to 6. The sum of the dots on opposite surfaces is 7.

History

Probability is used in many areas, including public finance, medicine, insurance, elections, manufacturing, educational tests and measurements, genetics, weather forecasting, investments, opinion polls, the natural sciences, and games of chance. The study of probability originated from the study of games of chance. Archaeologists have found artifacts used in games of chance in Egypt dating from about 3000 B.C.

Mathematical problems relating to games of chance were studied by a number of mathematicians of the Renaissance. Italy's Girolamo Cardano (1501–1576) in his *Liber de Ludo Aleae* (book on the games of chance) presents one of the first systematic computations of probabilities. Although it is basically a gambler's manual, many consider it the first book ever written on probability. A short time later, two French mathematicians, Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665), worked together studying "the geometry of the die." In 1657, Dutch mathematician Christian Huygens (1629–1695) published *De Ratiociniis in Luno Aleae* (on ratiocination in dice games), which contained the first documented reference to the concept of mathematical expectation (see Section 12.4). Swiss mathematician Jacob Bernoulli (1654–1705), whom many consider the founder of probability theory, is said to have fused pure mathematics with the empirical methods used in statistical experiments. The works of Pierre-Simon de Laplace (1749–1827) dominated probability throughout the nineteenth century.

The Nature of Probability

Before we discuss the meaning of the word *probability* and learn how to calculate probabilities, we must introduce a few definitions.

An **experiment** is a controlled operation that yields a set of results.

The process by which medical researchers administer experimental drugs to patients to determine their reaction is one type of experiment.

The possible results of an experiment are called its **outcomes.**

For example, the possible outcomes from administering an experimental drug may be a favorable reaction, no reaction, or an adverse reaction.

An **event** is a subcollection of the outcomes of an experiment.

For example, when a die is rolled, the event of rolling a number greater than 2 can be satisfied by any one of four outcomes: 3, 4, 5, or 6. The event of rolling a 5 can be satisfied by only one outcome, the 5 itself. The event of rolling an even number can be satisfied by any of three outcomes: 2, 4, or 6.

Probability is classified as either *empirical* (experimental) or *theoretical* (mathematical). ***Empirical probability*** is the relative frequency of occurrence of an event and is determined by actual observations of an experiment. ***Theoretical probability*** is determined through a study of the possible *outcomes* that can occur for the given experiment. We will indicate the probability of an event E by $P(E)$, which is read "P of E."

In this section, we will briefly discuss empirical probability. The emphasis in the remaining sections is on theoretical probability. Following is the formula for computing empirical probability, or relative frequency.

Empirical Probability (Relative Frequency)

$$P(E) = \frac{\text{number of times event } E \text{ has occurred}}{\text{total number of times the experiment has been performed}}$$

The probability of an event, whether empirical or theoretical, is always a number between 0 and 1, inclusive, and may be expressed as a decimal number or a fraction. An empirical probability of 0 indicates that the event has never occurred. An empirical probability of 1 indicates that the event has always occurred.

EXAMPLE 1 *Heads Up!*

In 100 tosses of a fair coin, 44 landed heads up. Find the empirical probability of the coin landing heads up.

SOLUTION: Let E be the event that the coin lands heads up. Then

$$P(E) = \frac{44}{100} = 0.44$$

EXAMPLE 2 *Weight Reduction*

A pharmaceutical company is testing a new drug that is supposed to help with weight reduction. The drug is given to 500 individuals with the following outcomes.

Weight reduced	Weight unchanged	Weight increased
279	92	129

If this drug is given to an individual, find the empirical probability that the person's weight is (a) reduced, (b) unchanged, (c) increased.

DID YOU KNOW

One Blue Eye, One Brown Eye

Seeing a dog or a person with different-colored eyes is unusual, but not that unusual. Our features, such as eye color, are determined genetically, as are certain illnesses. Below is a chart listing some genetic disorders and their rates of occurrence. Genetics, which has its foundation in probability, is used in explaining the cause of these illnesses, and it plays an important role in research to find cures for genetic disorders.

Disorder/Incidence

Cystic fibrosis:
1 out of 2500 Caucasian births

Down syndrome:
1 out of 800–1000 births

Duchenne muscular dystrophy:
1 out of 3300 male births

Fragile X syndrome:
1 out of 1500 male births
1 out of 2500 female births

Hemophilia A:
1 out of 8500 male births

Polycystic kidney disease:
1 out of 3000 births

Prader–Willi syndrome:
1 out of 12,000 births

Sickle-cell anemia:
1 out of 400–600 African-American births

Tay-Sachs disease:
1 out of 3600 Ashkenazi Jewish births

SOLUTION:

a) Let E be the event that the weight is reduced.

$$P(E) = \frac{279}{500} = 0.558$$

b) Let E be the event that the weight is unchanged.

$$P(E) = \frac{92}{500} = 0.184$$

c) Let E be the event that the weight is increased.

$$P(E) = \frac{129}{500} = 0.258$$

Empirical probability is used when probabilities cannot be theoretically calculated. For example, life insurance companies use empirical probabilities to determine the chance of an individual in a certain profession, with certain risk factors, living to age 65.

Empirical Probability in Genetics

Using empirical probability, Gregor Mendel (1822–1884) developed the laws of heredity by crossbreeding different types of "pure" pea plants and observing the relative frequencies of the resulting offspring. These laws became the foundation for the study of genetics. For example, when he crossbred a pure yellow pea plant and a pure green pea plant, the resulting offspring (the first generation) were always yellow; see Fig. 12.1(a). When he crossbred a pure round-seeded pea plant and a pure wrinkled-seeded pea plant, the resulting offspring (the first generation) were always round; see Fig. 12.1(b).

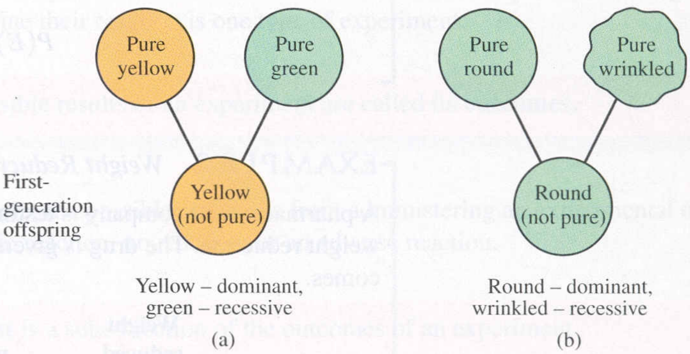


Figure 12.1

Mendel called traits such as yellow color and round seeds ***dominant*** because they overcame or "dominated" the other trait. He labeled the green and wrinkled traits ***recessive***.

Mendel then crossbred the offspring of the first generation. The resulting second generation offspring had both the dominant and the recessive traits of their grandparents; see Fig. 12.2(a) and (b). What's more, these traits always appeared in approximately a 3 to 1 ratio of dominant to recessive.

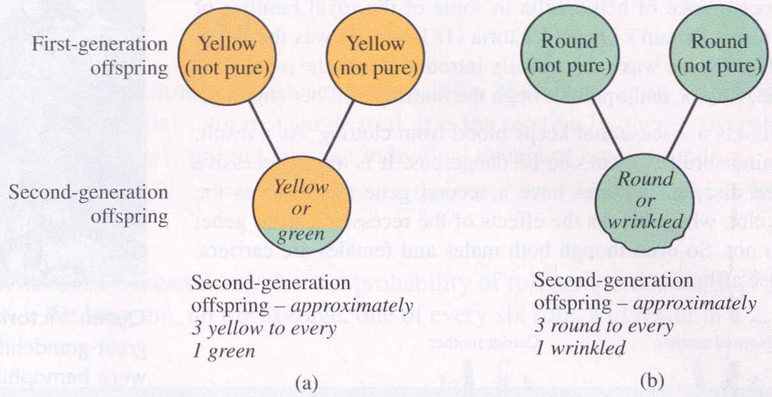


Figure 12.2

Table 12.1 lists some of the actual results of Mendel's experiments with pea plants. Note that the ratio of dominant trait to recessive trait in the second-generation offspring is about 3 to 1 for each of the experiments. The empirical probability of the dominant trait has also been calculated. How would you find the empirical probability of the recessive trait?

TABLE 12.1 Second-Generation Offspring

Dominant Trait	Number with Dominant Trait	Recessive Trait	Number with Recessive Trait	Ratio of Dominant to Recessive	*P* (Dominant Trait)
Yellow seeds	6022	Green seeds	2001	3.01 to 1	$\frac{6022}{8023} \approx 0.75$
Round seeds	5474	Wrinkled seeds	1850	2.96 to 1	$\frac{5474}{7324} \approx 0.75$

From his work Mendel concluded that the sex cells (now called gametes) of the pure yellow (dominant) pea plant carried some factor that caused the offspring to be yellow, and the gametes of the green variety had a variant factor that "induced the development of green plants." In 1909, Danish geneticist W. Johannsen called these factors "genes." Mendel's work led to the understanding that each pea plant contains two genes for color, one that comes from the mother and the other from the father. If the two genes are alike—for instance, both for yellow plants or both for green plants—the plant will be that color. If the genes for color are different, the plant will grow the color of the dominant gene. Thus, if one parent contributes a gene for the plant to be yellow (dominant) and the other parent contributes a gene for the plant to be green (recessive), the plant will be yellow.

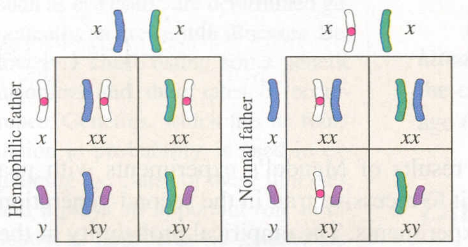
DID YOU KNOW

The Royal Disease

The effect of genetic inheritance is dramatically demonstrated by the occurrence of hemophilia in some of the royal families of Europe. Great Britain's Queen Victoria (1819–1901) was the initial carrier. The disease was subsequently introduced into the royal lines of Prussia, Russia, and Spain through the marriages of her children.

Hemophilia is a disease that keeps blood from clotting. As a result, even a minor bruise or cut can be dangerous. It is also a recessive sex-linked disease. Females have a second gene that enables the blood to clot, which blocks the effects of the recessive carrier gene; males do not. So even though both males and females are carriers, the disease afflicts only males.

Queen Victoria had 9 children, 26 grandchildren, and 34 great-grandchildren. Among them 1 son and 9 grandsons were hemophiliacs, and 2 daughters and 4 granddaughters were carriers of the gene for hemophilia. The genetic line of the present-day British royal family is free of the disease.



In humans, genes are located on 23 pairs of *chromosomes.* Each parent contributes one member of each pair to a child. The gene that affects blood clotting is carried on the *x* chromosome. Females have two *x* chromosomes; males have one *x* chromosome and one *y* chromosome. Thus, on the diagram on the left, *xx* represents a female and *xy* represents a male. A female with the defective gene, symbolized by , will be a carrier, whereas a male with the defective gene will have hemophilia.

The Law of Large Numbers

"The laws of probability, so true in general, so fallacious in particular."
Edward Gibbon, 1796

Most of us accept that if a "fair coin" is tossed many, many times, it will land heads up approximately half of the time. Intuitively, we can guess that the probability that a fair coin will land heads up is $\frac{1}{2}$. Does this mean that if a coin is tossed twice, it will land heads up exactly once? If a fair coin is tossed 10 times, will there necessarily be five heads? The answer is clearly no. What then does it mean when we state that the probability that a fair coin will land heads up is $\frac{1}{2}$? To answer this question, let's examine Table 12.2, which shows what may occur when a fair coin is tossed a given number of times.

TABLE 12.2

Number of Tosses	Expected Number of Heads	Actual Number of Heads Observed	Relative Frequency of Heads
10	5	4	$\frac{4}{10} = 0.4$
100	50	45	$\frac{45}{100} = 0.45$
1000	500	546	$\frac{546}{1,000} = 0.546$
10,000	5000	4852	$\frac{4852}{10,000} = 0.4852$
100,000	50,000	49,770	$\frac{49,770}{100,000} = 0.49770$

The last column of Table 12.2, the relative frequency of heads, is a ratio of the number of heads observed to the total number of tosses of the coin. The relative frequency is the empirical probability, as defined earlier. Note as the number of tosses increases, the relative frequency of heads gets closer and closer to $\frac{1}{2}$, or 0.5, which is what we expect.

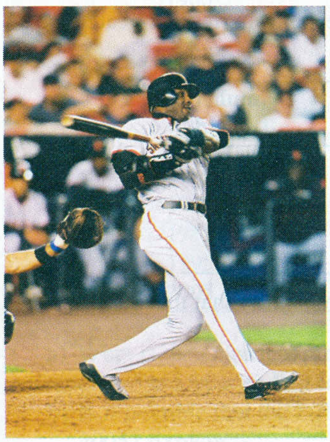
The nature of probability is summarized by the law of large numbers.

The **law of large numbers** states that probability statements apply in practice to a large number of trials, not to a single trial. It is the relative frequency over the long run that is accurately predictable, not individual events or precise totals.

What does it mean to say that the probability of rolling a 2 on a die is $\frac{1}{6}$? It means that over the long run, on the average, one of every six rolls will result in a 2.

DID YOU KNOW

Batting Averages

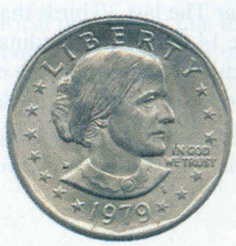


If Barry Bonds of the San Francisco Giants gets three hits in his first three at bats of the season, he is batting a thousand (1.000). But over the course of the 162 games of the season (with three or four at bats per game), however, his batting average will fall closer to .370 (his 2002 major league leading batting average). In 2002, out of 403 at bats Bonds had 149 hits, an average above all other players' but much less than 1.000. His batting average is a relative frequency (or empirical probability) of hits to at bats. It is only the long-term average that we take seriously because it is based on the law of large numbers.

SECTION 12.1 EXERCISES

Concept/Writing Exercises

1. What is an experiment?
2. **a)** What are outcomes of an experiment?
   b) What is an event?
3. What is empirical probability and how is empirical probability determined?
4. What are theoretical probabilities based on?
5. Explain in your own words the law of large numbers.
6. Explain in your own words why empirical probabilities are used in determining premiums for life insurance policies.
7. The theoretical probability of a coin landing heads up is $\frac{1}{2}$. Does this probability mean that if a coin is flipped two times, one flip will land heads up? If not, what does it mean?



8. The theoretical probability of rolling a 4 on a die is $\frac{1}{6}$. Does this probability mean that, if a die is rolled six times, one 4 will appear? If not, what does it mean?

9. To determine premiums, life insurance companies must compute the probable date of death. On the basis of a great deal of research Mr. Duncan, age 36, is expected to live another 43.21 years. Does this determination mean that Mr. Duncan will live until he is 79.21 years old? If not, what does it mean?

10. a) Explain how you would find the empirical probability of rolling a 5 on a die.
b) What do you believe is the empirical probability of rolling a 5?
c) Determine the empirical probability of rolling a 5 by rolling a die 40 times.

Practice the Skills

11. *Flip a Coin* Flip a coin 50 times and record the results. Determine the empirical probability of tossing
a) a head.
b) a tail.
c) Does the probability of tossing a head appear to be the same as tossing a tail?

12. *Roll a Die* Roll a die 50 times and record the results. Determine the empirical probability of rolling
a) a 1.
b) a 6.
c) Does the probability of rolling a 1 appear to be the same as the probability of rolling a 6? Explain.

13. *Pair of Dice* Roll a pair of dice 60 times and record the sums. Determine the empirical probability of rolling a sum of
a) 2.
b) 7.
c) Does the probability of rolling a sum of 2 appear to be the same as the probability of rolling a sum of 7?

14. *Two Coins* Toss two coins 50 times and record the number of times exactly one head was obtained. Determine the empirical probability of tossing exactly one head.

Problem Solving

15. *Birds at a Feeder* The last 30 birds that fed at the Haines' bird feeder were 14 finches, 10 cardinals, and 6 blue jays. Use this information to determine the empirical probability that the next bird to feed from the feeder is
a) a finch.
b) a cardinal.
c) a blue jay.

See Exercise 15

16. *Music Purchases* At the Virgin Music store in Times Square, 60 people entering the store were selected at random and were asked to state their favorite type of music. Of the 60, 24 selected rock, 16 selected country, 8 selected classical, and 12 said something other than rock, country, or classical. Determine the empirical probability that the next person entering the store favors
a) rock music.
b) country music.
c) something other than rock, country, or classical music.

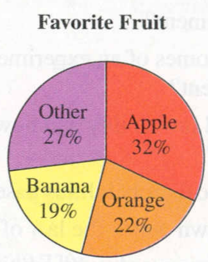
17. *Veterinarian* In a given week, a veterinarian treated the following animals.

Animal	Number Treated
Dog	40
Cat	35
Bird	15
Iguana	5

Determine the empirical probability that the next animal she treats is
a) a dog.
b) a cat.
c) an iguana.

18. *Prader–Willi Syndrome* In a sample of 50,000 first-born babies, 5 were found to have Prader–Willi syndrome. Find the empirical probability that a family's first child will be born with this syndrome.

19. *Favorite Fruit* At the produce department of a grocery store, 900 people were asked to name their favorite fruit. The following graph indicates their response.



a) Explain why this graph illustrates empirical probabilities.

If one person is selected at random from the 900 people sampled, determine the empirical probability that the person's favorite fruit is

b) an apple.

c) an orange.

d) a banana.

20. *Tax Returns* In 2001 about 40,244 million tax returns were filed electronically out of a total of about 131,100 million tax returns that were filed. Determine the empirical probability that a person selected at random who filed a tax return in 2001 filed it

a) electronically.

b) nonelectronically.

21. *Dow Jones Gains* The following table shows the gain made by the Dow Jones Industrial Average (DJIA) in each year that ends in a 5 since records have been kept.

Years Ending in 5	
Year	**DJIA Return**
1885	27.7%
1895	2.3%
1905	38.2%
1915	86.5%
1925	30.0%
1935	38.5%
1945	26.6%
1955	20.8%
1965	10.9%
1975	38.3%
1985	27.7%
1995	36.8%
2005	?

a) What is the empirical probability that the DJIA will increase in a year ending in 5?

b) Is it possible that the DJIA could have a loss in the year 2005? Explain.

22. *Grade Distribution* Mr. Doole's grade distribution over the past 3 years for a course in college algebra is shown in the chart below.

Grade	Number
A	43
B	182
C	260
D	90
F	62
I	8

If Sue Gilligan plans to take college algebra with Mr. Doole, determine the empirical probability she receives a grade of

a) A.

b) C.

c) D or higher.

23. *Election* In an election for student council president at Russell Sage College, a sample of 80 students were polled and asked for whom they planned to vote. The table shows the results of the poll.

Candidate	Votes
Allison	22
Emily	18
Kimberly	20
Johanna	14
Other	6

If one student from the sample was selected at random, determine the empirical probability the person planned to vote for

a) Allison.

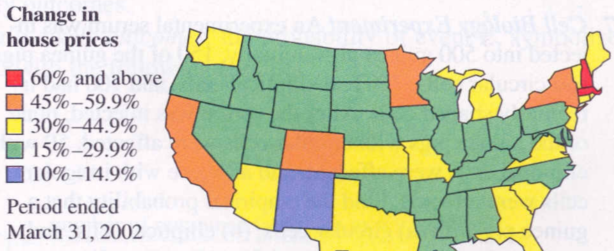
b) Emily.

c) Kimberly.

d) Johanna.

e) Someone other than the four people listed above.

24. *Housing Prices* The following illustration shows how house prices have changed in the United States, by state, for the period March 1, 1997, through March 1, 2002.*



Source: Office of Federal Housing Enterprise Oversight

If one state is selected at random, determine the empirical probability that the state's average house price increased by

a) 60% and above.

b) 45%–59.9%.

c) 30%–44.9%.

d) 15%–29.9%.

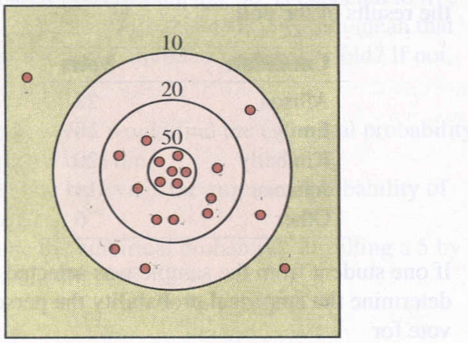
e) 10%–14.9%.

25. *Hitting a Bull's-Eye* The pattern of hits shown on the target resulted from a marksman firing 20 rounds. For a single shot

a) find the empirical probability that the marksman hits the 50-point bull's-eye (the center of the target).

*For information purposes, the District of Columbia's change in housing prices was 60% and above (do not consider this information in answering the exercises).

b) find the empirical probability that the marksman does not hit the bull's-eye.
c) find the empirical probability that the marksman scores at least 20 points.
d) find the empirical probability that the marksman does not score any points (the area outside the large circle).



26. *Rock Toss* Jim Handy finds an irregularly shaped five-sided rock. He labels each side and tosses the rock 100 times. The results of his tosses are shown in the table. Find the empirical probability that the rock will land on side 4 if tossed again.

Side	1	2	3	4	5
Frequency	32	18	15	13	22

27. *Cell Biology Experiment* An experimental serum was injected into 500 guinea pigs. Initially, 150 of the guinea pigs had circular cells, 250 had elliptical cells, and 100 had irregularly shaped cells. After the serum was injected, none of the guinea pigs with circular cells were affected, 50 with elliptical cells were affected, and all those with irregular cells were affected. Find the empirical probability that a guinea pig with (a) circular cells, (b) elliptical cells, and (c) irregular cells will be affected by injection of the serum.

28. *Baby Gender* In the United States, more male babies are born than female. In 2000, 2,076,969 males were born and 1,981,845 females were born. Find the empirical probability of an individual being born
a) male.
b) female.

29. *Mendel's Experiment* In one of Mendel's experiments (see pages 656–657), he crossbred nonpure purple flower pea plants. These purple pea plants had two traits for flowers, purple (dominant) and white (recessive). The result of this crossbreeding was 705 second-generation plants with purple flowers and 224 second-generation plants with white flowers. Find the empirical probability of a second-generation plant having
a) white flowers.
b) purple flowers.

30. *Second-Generation Offspring* In another experiment, Mendel crossbred nonpure tall pea plants. As a result, the second-generation offspring were 787 tall plants and 277 short plants. Find the empirical probability of a second-generation plant being
a) tall.
b) short.

Challenge Problems/Group Activities

31. a) *Design an Experiment* Which do you believe is used more frequently in a book or newspaper article, the word *a* or the word *the*?
b) Design an experiment to determine the empirical probabilities (or relative frequencies) of the words *a* and *the* appearing in a book or newspaper article.
c) Perform the experiment in part (b) and determine the empirical probabilities.
d) Which word appears to occur more frequently?

Recreational Mathematics

32. *Cola Preference* Can people selected at random distinguish Coke from Pepsi? Which do they prefer?
a) Design an experiment to determine the empirical probability that a person selected at random can select Coke when given samples of both Coke and Pepsi.
b) Perform the experiment in part (a) and determine the empirical probability.
c) Determine the empirical probability that a person selected at random will prefer Coke over Pepsi.

Internet/Research Activities

33. Write a paper on how insurance companies use empirical probabilities in determining insurance premiums. An insurance agent may be able to direct you to a source of information.

34. Write a paper on how Gregor Mendel's use of empirical probability led to the development of the science of genetics. You may want to check with a biology professor to determine references to use.

12.2 THEORETICAL PROBABILITY

Should you spend the 37 cents for a stamp to return a sweepstakes ticket? What are your chances of winning a lottery? If you go to a carnival, bazaar, or casino, which games provide the greatest chance of winning? These and similar questions can be answered once you have an understanding of theoretical probability. *In the remainder of this chapter, the word probability will refer to theoretical probability.*

TIMELY TIP To be able to do the problems in this section and the remainder of the chapter you must have a thorough understanding of fractions. If you have forgotten how to work with fractions, we strongly suggest that you review Section 5.3 before beginning this section.

Recall from Section 12.1 that the results of an experiment are called outcomes. When you roll a die and observe the number of points that face up, the possible outcomes are 1, 2, 3, 4, 5, and 6. It is equally likely that you will roll any one of the possible numbers.

If each outcome of an experiment has the same chance of occurring as any other outcome, they are said to be **equally likely outcomes.**

Can you think of a second set of equally likely outcomes when a die is rolled? An odd number is as likely to be rolled as an even number. Therefore, odd and even are another set of equally likely outcomes.

If an event E has *equally likely outcomes,* the probability of event E, symbolized by $P(E)$, may be calculated with the following formula.

Probability

$$P(E) = \frac{\text{number of outcomes favorable to } E}{\text{total number of possible outcomes}}$$

Example 1 illustrates how to use this formula.

EXAMPLE 1 *Finding Probabilities*

A die is rolled. Find the probability of rolling

a) a 2. b) an even number. c) a number greater than 4.

d) a 7. e) a number less than 7.

SOLUTION:

a) There are six possible equally likely outcomes: 1, 2, 3, 4, 5, and 6. The event of rolling a 2 can occur in only one way.

$$P(2) = \frac{\text{number of outcomes that will result in a 2}}{\text{total number of possible outcomes}} = \frac{1}{6}$$

b) The event of rolling an even number can occur in three ways: 2, 4, or 6.

$$P(\text{even number}) = \frac{\text{number of outcomes that result in an even number}}{\text{total number of possible outcomes}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

c) Two numbers are greater than 4, namely, 5 and 6.

$$P(\text{number greater than 4}) = \frac{2}{6} = \frac{1}{3}$$

d) No outcomes will result in a 7. Thus, the event cannot occur and the probability is 0.

$$P(7) = \frac{0}{6} = 0$$

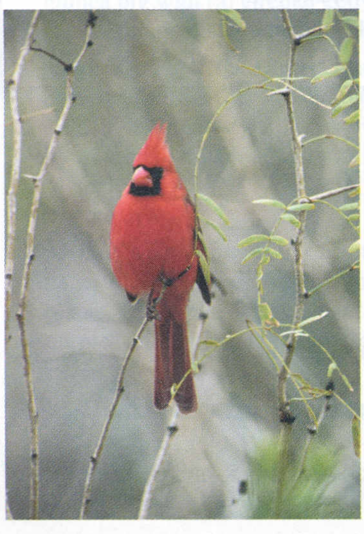
e) All the outcomes 1 through 6 are less than 7. Thus, the event must occur and the probability is 1.

$$P(\text{number less than 7}) = \frac{6}{6} = 1$$

Four important facts about probability follow.

Important Facts

1. The probability of an event that cannot occur is 0.
2. The probability of an event that must occur is 1.
3. Every probability is a number between 0 and 1 inclusive; that is, $0 \leq P(E) \leq 1$.
4. The sum of the probabilities of all possible outcomes of an experiment is 1.



Northern Cardinal

EXAMPLE 2 *Choosing One Bird from a List*

The names of 15 birds and their food preferences are listed in Table 12.3 on page 665. Each of the 15 birds' names is listed on a slip of paper, and the 15 slips are placed in a bag. One slip is to be selected at random from the bag. Find the probability that the slip contains the name of

a) a sparrow (any type listed).

b) a bird that has a high attractiveness to peanut kernels.

c) a bird that has a low attractiveness to peanut kernels, *and* a low attractiveness to cracked corn, *and* a high attractiveness to black striped sunflower seeds.

d) a bird that has a high attractiveness to either peanut kernels *or* cracked corn (or both).

TABLE 12.3 Birds and Their Food Preferences

Bird	Peanut Kernels	Cracked Corn	Black Striped Sunflower Seeds
American goldfinch	L	L	H
Blue jay	H	M	H
Chickadee	M	L	H
Common grackle	M	H	H
Evening grosbeak	L	L	H
House finch	M	L	H
House sparrow	L	M	M
Mourning dove	L	M	M
Northern cardinal	L	L	H
Purple finch	L	L	H
Scrub jay	H	L	H
Song sparrow	L	L	M
Tufted titmouse	H	L	H
White-crowned sparrow	H	M	H
White-throated sparrow	H	H	H

Note: H = high attractiveness; M = medium attractiveness; L = low attractiveness.

Source: How to Attract Birds (Ortho Books)

SOLUTION:

a) Four of the 15 birds listed are sparrows (house sparrow, song sparrow, white-crowned sparrow, and white-throated sparrow).

$$P(\text{sparrow}) = \frac{4}{15}$$

b) Five of the 15 birds listed have a high attractiveness to peanut kernels (blue jay, scrub jay, tufted titmouse, white-crowned sparrow, and white-throated sparrow).

$$P(\text{high attractiveness to peanut kernels}) = \frac{5}{15} = \frac{1}{3}$$

c) Reading across the rows reveals that 4 birds have a low attractiveness to peanut kernels and cracked corn and a high attractiveness to black striped sunflower seeds (American goldfinch, evening grosbeak, northern cardinal, and purple finch).

$$P\left(\begin{array}{c}\text{low attractiveness to peanuts, and low}\\ \text{to corn, and high to black sunflower seeds}\end{array}\right) = \frac{4}{15}$$

d) Six birds have a high attractiveness to either peanut kernels or to cracked corn (or both). They are the blue jay, common grackle, scrub jay, tufted titmouse, white-crowned sparrow, and white-throated sparrow.

$$P(\text{high attractiveness to peanut kernels or cracked corn}) = \frac{6}{15} = \frac{2}{5}$$

In any experiment, an event must either occur or not occur. *The sum of the probability that an event will occur and the probability that it will not occur is 1*. Thus, for any event A we conclude that

$$P(A) + P(\text{not } A) = 1$$
or
$$P(\text{not } A) = 1 - P(A)$$

For example, if the probability that event A will occur is $\frac{5}{12}$, the probability that event A will not occur is $1 - \frac{5}{12}$, or $\frac{7}{12}$. Similarly, if the probability that event A will not occur is 0.3, the probability that event A will occur is $1 - 0.3 = 0.7$ or $\frac{7}{10}$. We make use of this concept in Example 3.

EXAMPLE 3 *Selecting One Card from a Deck*

A standard deck of 52 playing cards is shown in Figure 12.3. The deck consists of four suits: hearts, clubs, diamonds, and spades. Each suit has 13 cards, including numbered cards ace (1) through 10 and three picture (or face) cards, the jack, the queen, and the king. Hearts and diamonds are red cards; clubs and spades are black cards. There are 12 picture cards, consisting of 4 jacks, 4 queens, and 4 kings. One card is to be selected at random from the deck of cards. Find the probability that the card selected is

a) a 9.
b) not a 9.
c) a diamond.
d) a jack *or* queen *or* king (a picture card).
e) a heart *and* a club.
f) a card greater than 6 *and* less than 9.

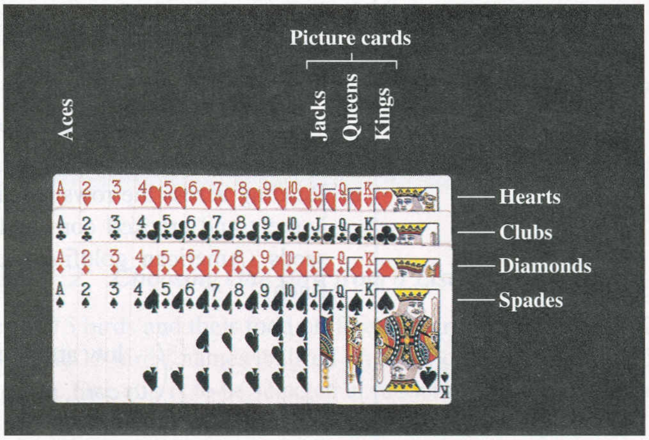


Figure 12.3

SOLUTION:

a) There are four 9's in a deck of 52 cards.

$$P(9) = \frac{4}{52} = \frac{1}{13}$$

b) $P(\text{not a } 9) = 1 - P(9) = 1 - \frac{1}{13} = \frac{12}{13}$

This probability could also have been found by noting that there are 48 cards that are not 9's in a deck of 52 cards.

$$P(\text{not a } 9) = \frac{48}{52} = \frac{12}{13}$$

c) There are 13 diamonds in the deck.

$$P(\text{diamond}) = \frac{13}{52} = \frac{1}{4}$$

d) There are 4 jacks, 4 queens, and 4 kings, or a total of 12 picture cards.

$$P(\text{jack } or \text{ queen } or \text{ king}) = \frac{12}{52} = \frac{3}{13}$$

e) The word *and* means that *both* events must occur. Since it is not possible to select one card that is both a heart and a club, the probability is 0.

$$P(\text{heart and club}) = \frac{0}{52} = 0$$

f) The cards that are both greater than 6 and less than 9 are 7's and 8's. There are four 7's and four 8's, or a total of eight cards.

$$P(\text{greater than 6 } and \text{ less than 9}) = \frac{8}{52} = \frac{2}{13}$$

SECTION 12.2 EXERCISES

Concept/Writing Exercises

1. What are equally likely outcomes?
2. Explain in your own words how to find the theoretical probability of an event.
3. State the relationship that exists for $P(A)$ and $P(\text{not } A)$.
4. If the probability that an event occurs is $\frac{4}{9}$, determine the probability that the event does not occur.
5. If the probability that an event occurs is 0.3, determine the probability that the event does not occur.
6. If the probability that an event does not occur is 0.25, determine the probability that the event occurs.
7. If the probability that an event does not occur is $\frac{5}{12}$, determine the probability that the event occurs.
8. How many of each of the following are there in a standard deck of cards?
   a) Total cards
   b) Hearts
   c) Red cards
   d) Fives
   e) Black cards
   f) Picture cards
   g) Aces
   h) Queens
9. Using the definition of probability, explain in your own words why the probability of an event that cannot occur is 0.
10. Using the definition of probability, explain in your own words why the probability of an event that must occur is 1.
11. Between what two numbers (inclusively) will all probabilities lie?
12. What is the sum of all the probabilities of all possible outcomes of an experiment?

Practice the Skills

13. ***Multiple-Choice Test*** A multiple-choice test has five possible answers for each question.
    a) If you guess at an answer, what is the probability that you select the correct answer for one particular question?

b) If you eliminate one of the five possible answers and guess from the remaining possibilities, what is the probability that you select the correct answer to that question?

14. ***Remote Control*** A TV remote has keys for channels 0 through 9. If you select one key at random,

a) what is the probability that you press channel 6?

b) what is the probability that you press a key for an even number?

c) what is the probability that you press a key for a number less than 7?

15. ***Raffle*** In a raffle where one number is chosen, determine the probability you would win if you have a choice of 50 numbers to choose from. Explain your answer.

16. ***Raffle*** In a raffle where one number is chosen, determine the probability you would win if you have a choice of 46 numbers to choose from. Explain your answer.

Select a Card *In Exercises 17–26, one card is selected at random from a deck of cards. Find the probability that the card selected is*

17. a 7.

18. a 7 or a 9.

19. not a 7.

20. the five of diamonds.

21. a black card.

22. a heart.

23. a red card or a black card.

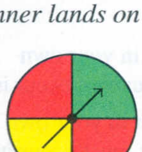
24. a red card and a black card.

25. a card greater than 4 and less than 9.

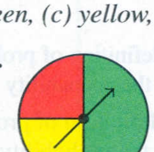
26. a jack and a heart.

Spin the Spinner *In Exercises 27–30, assume that the spinner cannot land on a line. Find the probability that the spinner lands on (a) red, (b) green, (c) yellow, (d) blue.*

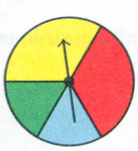
27.



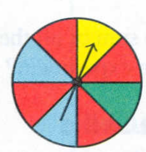
28.



29.



30.



Picnic *In Exercises 31–34, a bin at a picnic contains 100 cans of soda covered by ice. There are 30 cans of cola, 40 cans of orange soda, 10 cans of ginger ale, and 20 cans of root beer. The cans are all the same size and shape. If one can is selected at random from the bin, find the probability that the soda selected is*

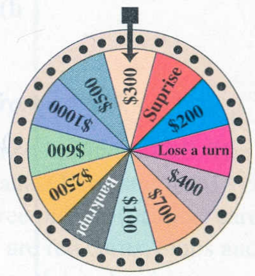
31. orange soda.

32. cola or orange soda.

33. cola, root beer, or orange soda.

34. ginger ale.

Wheel of Fortune *In Exercises 35–38, use the small replica of the Wheel of Fortune.*



If the wheel is spun at random, find the probability of the sector indicated stopping under the pointer.

35. $600

36. A number greater than $400

37. Lose a turn or Bankrupt

38. $2500 or Surprise

Tennis Balls *In Exercises 39–42, 50 tennis balls including 23 Wilson, 17 Penn, and 10 other brand-name balls are on a tennis court. Barry Wood closes his eyes and arbitrarily picks up a ball from the court. Determine the probability the ball selected is*

39. a Wilson.

40. a Penn.

41. not a Penn.

42. a Wilson or a Penn.

Traffic Light *In Exercises 43–46, a traffic light is red for 25 sec, yellow for 5 sec, and green for 55 sec. What is the probability that when you reach the light,*

43. the light is green.

44. the light is yellow.

45. the light is not red.

46. the light is not green.

Mississippi *In Exercises 47–52, each individual letter of the word Mississippi is placed on a piece of paper, and all 11 pieces of paper are placed in a hat. If one letter is selected at random from the hat, find the probability that*

47. the letter *s* is selected.

48. the letter *s* is not selected.

49. a vowel is selected.

50. the letter *i* or *p* is selected.

51. the letter *v* is not selected.

52. the letter *w* is selected.

Manatees *In Exercises 53–56, use the following chart, which shows information about manatee deaths from 1991 through 2001.*

Year	Boat Deaths	Totals Deaths*
1991	53	174
1992	38	163
1993	35	145
1994	49	193
1995	42	201
1996	60	415
1997	54	242
1998	66	231
1999	82	268
2000	78	273
2001	81	325

* Manatees are an endangered species. A major cause of manatee deaths is related to boating accidents. The year 2001 was the second worst year ever recorded for manatee deaths.

Source: St. Petersburg Times, January 5, 2002

If one year from 1991 up to and including 2001 is selected at random, determine the probability that in the year selected

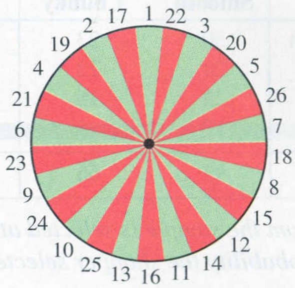
53. exactly 60 manatee deaths that year were caused by boating accidents.

54. the total number of manatee deaths exceeded 250.

55. the number of manatee deaths by boating accidents exceeded 50 and the number of total manatee deaths exceeded 250.

56. the number of manatee deaths by boating accidents was less than or equal to 40 but the total number of manatee deaths was greater than or equal to 163.

Dart Board *In Exercises 57–60, a dart is thrown randomly and sticks on the circular dart board with 26 partitions, as shown.*



Assuming that the dart cannot land on the black area or on a border between colors, find the probability that the dart lands on

57. the area marked 15.

58. an orange area.

59. an area marked with a number greater than or equal to 22.

60. an area marked with a number greater than 6 and less than or equal to 9.

Car Manufacturer *In Exercises 61–66, refer to the following table, which shows the results of a survey regarding the manufacturer of the cars driven by the people who were interviewed. In the table, GM represents General Motors.*

	GM, Ford, DaimlerChrysler	Other Manufacturer	Total
Men	260	85	345
Women	273	97	370
Total	533	182	715

If one person who completed the survey is selected at random, determine the probability that the person selected

61. is a man.

62. is a woman.

63. drives a car manufactured by GM, Ford or DaimlerChrysler.

64. drives a car manufactured by a company other than GM, Ford, or DaimlerChrysler.

65. is a woman who drives a car manufactured by a company other than GM, Ford, or DaimlerChrysler.

66. is a man who drives a car manufactured by GM, Ford, or DaimlerChrysler.

Peanut Butter Preference *In Exercises 67–72, refer to the following table, which contains information about a sample of shoppers selecting various brands of peanut butter at a grocery store. Assume that each shopper purchased exactly one jar of peanut butter.*

Brand	Smooth	Chunky	Total
Peter Pan	30	23	53
Jiff	28	22	50
Skippy	23	16	39
Other	12	5	17
Total	93	66	159

If one shopper from the sample is selected at random, determine the probability the shopper selected

67. Jif peanut butter.

68. Skippy peanut butter.

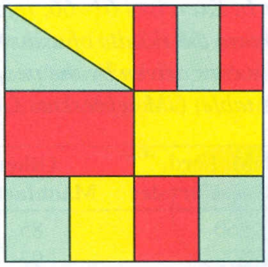
69. a chunky peanut butter.

70. a smooth peanut butter.

71. Peter Pan chunky peanut butter.

72. Jiff smooth peanut butter.

Bean Bag Toss *In Exercises 73–77, a bean bag is randomly thrown onto the square table and does not touch a line.*



Find the probability that the bean bag lands on

73. a red area.

74. a green area.

75. a yellow area.

76. a red or green area.

77. a yellow or green area.

78. a red or yellow area.

Challenge Problems/Group Activities

Before working Exercises 79 and 80, reread the material on genetics in Section 12.1.

79. **Genetics** Cystic fibrosis is an inherited disease that occurs in about 1 in every 2500 Caucasian births in North America and in about 1 in every 250,000 non-Caucasian births in North America. Let's denote the cystic fibrosis gene as c and a disease-free gene as C. Since the disease-free gene is dominant, only a person with cc genes will have the disease. A person who has Cc genes is a carrier of cystic fibrosis but does not actually have the disease. If one parent has CC genes and the other parent has cc genes, find the probability that

a) an offspring will inherit cystic fibrosis, that is, cc genes.

b) an offspring will be a carrier of cystic fibrosis but not contract the disease.

80. **Genetics** Sickle-cell anemia is an inherited disease that occurs in about 1 in every 500 African-American births and about 1 in every 160,000 non-African-American births. Unlike cystic fibrosis, in which the cystic fibrosis gene is recessive, sickle-cell anemia is *codominant.* In other words, a person inheriting two sickle-cell genes will have sickle-cell anemia, whereas a person inheriting only one of the sickle-cell genes will have a mild version of sickle-cell anemia, called *sickle-cell trait.* Let's call the disease-free genes s_1 and the sickle cell gene s_2. If both parents have s_1s_2 genes, determine the probability that

a) an offspring will have sickle-cell anemia.

b) an offspring will have the sickle-cell trait.

c) an offspring will have neither sickle-cell anemia nor the sickle cell trait.

In Exercises 81 and 82, the solutions involve material that we will discuss in later sections of the chapter. Try to solve them before reading ahead.

81. **Marbles** A bottle contains two red and two green marbles, and a second bottle also contains two red and two green marbles. If you select one marble at random from each bottle, find the probability (to be discussed in Section 12.6) that you obtain

a) two red marbles.

b) two green marbles.

c) a red marble from the first bottle and a green marble from the second bottle.

82. **Birds** Consider Table 12.3 on page 665. Suppose you are told that one bird's name was selected from the birds listed and the bird selected has a low attractiveness to peanut kernels. Find the probability (to be discussed in Section 12.7) that

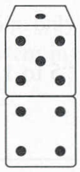
a) the bird is a sparrow.

b) the bird has a high attractiveness to cracked corn.

c) the bird has a high attractiveness to black striped sunflower seeds.

Recreational Exercise

83. *Dice* On a die, the sum of the dots on the opposite faces is seven. Two six-sided dice are placed together on top of one another, on a table, as shown in the figure below. The top and bottom faces of the bottom die, and the bottom face of the top die cannot be seen. If you walk around the table, what is the sum of all the dots on all the visible faces of the dice?



Internet/Research Activity

84. On page 654 we briefly discuss Jacob Bernoulli. The Bernoulli family produced several prominent mathematicians, including Jacob I, Johann I, and Daniel. Write a paper on the Bernoulli family, indicating some of the accomplishments of each of the three Bernoullis named and their relationship to each other. Indicate which Bernoulli the Bernoulli numbers are named after, which Bernoulli the Bernoulli theorem in statistics is named after, and which Bernoulli the Bernoulli theorem of fluid dynamics is named after.

12.3 ODDS

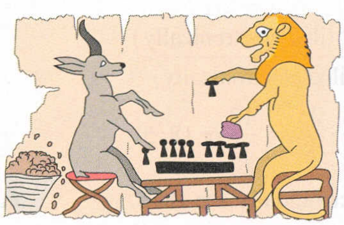
The odds against winning a lottery are 7 million to 1; the odds against being audited by the IRS this year are 47 to 1. We see the word *odds* daily in newspapers and magazines and often use it ourselves. Yet there is a widespread misunderstanding of its meaning. In this section, we will explain the meaning of odds.

The odds given at horse races, at craps, and at all gambling games in Las Vegas and other casinos throughout the world are always *odds against* unless they are otherwise specified. The *odds against* an event is a ratio of the probability that the event will fail to occur (failure) to the probability the event will occur (success). Thus, *to find odds you must first know or determine the probability of success and the probability of failure.*

$$\text{Odds against event} = \frac{P(\text{event fails to occur})}{P(\text{event occurs})} = \frac{P(\text{failure})}{P(\text{success})}$$

DID YOU KNOW

Gambling in Ancient Times



Archaeologists have found evidence of gambling in all cultures, from the Stone Age Australian aborigines to the ancient Egyptians, and across cultures touched by the Roman Empire. There is an equally long history of moral and legal opposition to gambling.

EXAMPLE 1 *Rolling a 4*

Find the odds against rolling a 4 on one roll of a die.

SOLUTION: Before we can determine the odds, we must first determine the probability of rolling a 4 (success) and the probability of not rolling a 4 (failure). When a die is rolled there are six possible outcomes: 1, 2, 3, 4, 5, and 6.

$$P(\text{rolling a } 4) = \frac{1}{6} \qquad P(\text{failure to roll a } 4) = \frac{5}{6}$$

Now that we know the probabilities of success and failure, we can determine the odds against rolling a 4.

$$\text{Odds against rolling a 4} = \frac{P(\text{failure to roll a 4})}{P(\text{rolling a 4})} = \frac{\frac{5}{6}}{\frac{1}{6}} = \frac{5}{\cancel{6}} \cdot \frac{\cancel{6}}{1} = \frac{5}{1}$$

The ratio $\frac{5}{1}$ is commonly written as 5:1 and is read "5 to 1." Thus, the odds against rolling a 4 are 5 to 1. ▲

TIMELY TIP The denominators of the probabilities in an odds problem will always divide out, as was shown in Example 1.

In Example 1, consider the possible outcomes of the die: 1, 2, 3, 4, 5, 6. Over the long run, one of every six rolls will result in a 4, and five of every six rolls will result in a number other than 4. Therefore, if a person was gambling, for each dollar bet in favor of the rolling of a 4, \$5 should be bet against the rolling of a 4 if the person is to break even. The person betting in favor of the rolling of a 4 will either lose \$1 (if a number other than a 4 is rolled) or win \$5 (if a 4 is rolled). The person betting against the rolling of a 4 will either win \$1 (if a number other than a 4 is rolled) or lose \$5 (if a 4 is rolled). If this game is played for a long enough period, each player theoretically will break even.

EXAMPLE 2 *Tax Returns*

In 2002, about 5 of every 23 tax returns are filed electronically. If one tax return that was filed is selected at random, what are the odds against that tax return being filed electronically?

SOLUTION: The probability that a tax return selected at random was filed electronically is $\frac{5}{23}$. Therefore, the probability that a tax return selected at random *is not* filed electronically is $1 - \frac{5}{23}$, or $\frac{18}{23}$.

$$\text{Odds against return being filed electronically} = \frac{P(\text{return not filed electronically})}{P(\text{return filed electronically})} = \frac{18/23}{5/23} = \frac{18}{\cancel{23}} \cdot \frac{\cancel{23}}{5} = \frac{18}{5} \text{ or } 18:5$$

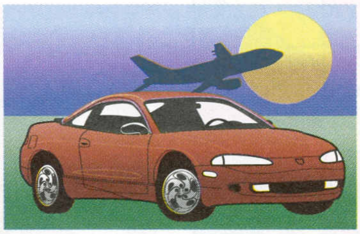
Thus, the odds against the tax return being filed electronically are 18:5. ▲

Although odds are generally given against an event, at times they may be given in favor of an event. The *odds in favor of* an event are expressed as a ratio of the probability that the event will occur to the probability that the event will fail to occur.

$$\text{Odds in favor of event} = \frac{P(\text{event occurs})}{P(\text{event fails to occur})} = \frac{P(\text{success})}{P(\text{failure})}$$

DID YOU KNOW

Flying Is Safer



Many people fear the risks of air travel, but the odds in favor of dying in an airplane crash are relatively low: approximately 1 to 354,319. The odds in favor of dying in a car accident are approximately 1 to 18,585. Thus, the odds in favor of dying in a car accident are much greater than the odds in favor of dying in an airplane accident.

DID YOU KNOW

What Were the Odds on This One?

In the year 1905, only two automobiles were registered in the entire state of Missouri.

In the year 1905, Missouri's only two registered automobiles were involved in a head-on collision!

If the odds *against* an event are $a{:}b$, the odds *in favor of* the event are $b{:}a$.

Example 3 involves a circle graph that contains percents; see Fig. 12.4. Before we discuss Example 3, let us briefly discuss percents. Recall that probabilities are numbers between 0 and 1, inclusive. We can change a percent between 0% and 100% to a probability by writing the percent as a fraction or a decimal number. In Fig. 12.4, we see 38% in one of the sectors (or areas) of the circle. To change 38% to a probability we can write $\frac{38}{100}$ or 0.38. Notice that both the fraction and the decimal number are numbers between 0 and 1, inclusive.

EXAMPLE 3 *Attending the Least Expensive Colleges*

The following circle graph shows that most college students in the United States attend the least expensive colleges.

Most Attend Least Expensive Schools
Percent of full-time undergraduates attending schools by cost of tuition and fees, 2002–2003: Numbers are rounded to the nearest percent.

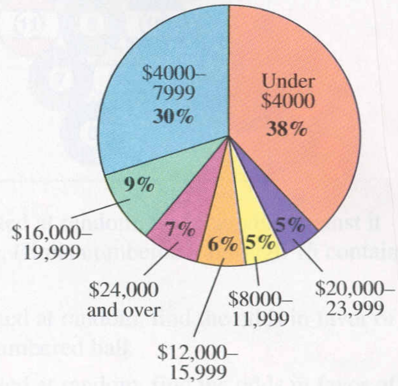


Figure 12.4 Source: The College Board.

Use the graph to determine

a) the odds against a college student selected at random attending a college that costs \$16,000–\$19,999.

b) the odds in favor of a college student selected at random attending a college that costs \$16,000–\$19,999.

SOLUTION:

a) The graph shows that 9%, or $\frac{9}{100}$, of all college students attend a college whose costs are \$16,000–\$19,999. Thus, the probability that a student attends a college whose costs are \$16,000–\$19,999 is $\frac{9}{100}$. The probability the student's college costs are not \$16,000–\$19,999 is therefore $1 - \frac{9}{100} = \frac{91}{100}$.

$$\text{Odds against college whose costs are \$16,000–\$19,999} = \frac{P(\text{college costs not \$16,000–\$19,999})}{P(\text{college costs \$16,000–\$19,999})}$$

$$= \frac{\frac{91}{100}}{\frac{9}{100}} = \frac{91}{\cancel{100}} \cdot \frac{\cancel{100}}{9} = \frac{91}{9} \text{ or } 91:9$$

Thus, the odds against a college student selected at random attending a college that costs \$16,000–\$19,999 are 91 : 9.

b) The odds in favor of a college student selected at random attending a college that costs \$16,000–\$19,999 are 9 : 91. ▲

Finding Probabilities from Odds

When odds are given, either in favor of or against a particular event, it is possible to determine the probability that the event occurs and the probability that the event does not occur. The denominators of the probabilities are found by adding the numbers in the odds statement. The numerators of the probabilities are the numbers given in the odds statements.

EXAMPLE 4 *Determining Probabilities from Odds*

The odds against Robin Murphy being admitted to the college of her choice are 9:2. Find the probability that (a) Robin is admitted and (b) Robin is not admitted.

SOLUTION:

a) We have been given odds against and have been asked to find probabilities.

$$\text{Odds against being admitted} = \frac{P(\text{fails to be admitted})}{P(\text{is admitted})}$$

Since the odds statement is 9:2, the denominators of both the probability of success and the probability of failure must be 9 + 2 or 11. To get the odds ratio of 9:2 the probabilities must be $\frac{9}{11}$ and $\frac{2}{11}$. Since odds against is a ratio of failure to success, the $\frac{9}{11}$ and $\frac{2}{11}$ represent the probabilities of failure and success, respectively. Thus, the probability that Robin is admitted (success) is $\frac{2}{11}$.

b) The probability that Robin is not admitted (failure) is $\frac{9}{11}$. ▲

Odds and probability statements are sometimes stated incorrectly. For example, consider the statement, "The odds of being selected to represent the district are 1 in 5." Odds are given using the word *to*, not *in*. Thus, there is a mistake in this statement. The correct statement might be, "The odds of being selected to represent the district are 1 to 5" or "The probability of being selected to represent the district is 1 in 5." Without additional information, it is not possible to tell which is the correct interpretation.

SECTION 12.3 EXERCISES

Concept/Writing Exercises

1. Explain in your own words how to determine the odds against an event.
2. Explain in your own words how to determine the odds in favor of an event.
3. Which odds are generally quoted, odds against or odds in favor?
4. Explain how to determine probabilities when you are given an odds statement.
5. The odds in favor of winning the door prize are 5 to 9. Find the odds against winning the door prize.
6. The odds against Thor's Lightning winning the horse race are 7:3. Find the odds in favor of Thor's Lightning winning.
7. If the odds against an event are 1:1, what is the probability the event will
   a) occur.
   b) fail to occur.
   Explain your answer.
8. If the probability an event will occur is $\frac{1}{2}$, determine
   a) the probability the event will fail to occur.
   b) the odds against the event occurring.
   c) the odds in favor of the event occurring.
   Explain your answer.

Practice the Skills/Problem Solving

9. *Dressing Up* Lalo Jaquez is going to wear a blue sportcoat and is trying to decide what tie he should wear to work. In his closet, he has 27 ties, 8 of which he feels go well with the sportcoat. If Lalo selects one tie at random, determine
   a) the probability it goes well with the sportcoat.
   b) the probability it does not go well with the sportcoat.
   c) the odds against it going well with the sportcoat.
   d) the odds in favor of it going well with the sportcoat.

10. *Making a Donation* In her wallet, Anne Kelly has 14 bills. Seven are $1 bills, two are $5 bills, four are $10 bills, and one is a $20 bill. She passes a volunteer seeking donations for the Salvation Army and decides to select one bill at random from her wallet and give it to the Salvation Army. Determine
    a) the probability she selects a $5 bill.
    b) the probability she does not select a $5 bill.
    c) the odds in favor of her selecting a $5 bill.
    d) the odds against her selecting a $5 bill.

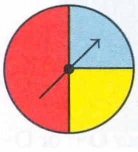
Toss a Die In Exercises 11–14, a die is tossed. Find the odds against rolling

11. a 3.
12. an even number.
13. a number less than 3.
14. a number greater than 4.

Deck of Cards In Exercises 15–18, a card is picked from a deck of cards. Find the odds against and the odds in favor of selecting

15. a queen.
16. a heart.
17. a picture card.
18. a card greater than 5 (ace is low).

Spin the Spinner In Exercises 19–22, assume that the spinner cannot land on a line. Find the odds against the spinner landing on the color red.

19.


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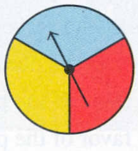

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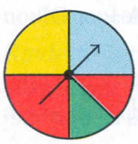

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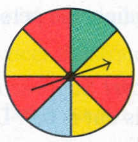
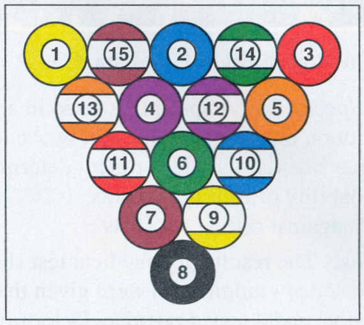


23. *Students* One person is selected at random from a class of 16 men and 14 women. Find the odds against selecting
    a) a woman.
    b) a man.

24. *Lottery* One million tickets are sold for a lottery where a single prize will be awarded.
    a) If you purchase a ticket, find your odds against winning.
    b) If you purchase 10 tickets, find your odds against winning.

Billiard Balls In Exercises 25–30, use the rack of 15 billiard balls shown.



25. If one ball is selected at random, find the odds against it containing a stripe. (Balls numbered 9 through 15 contain stripes.)
26. If one ball is selected at random, find the odds in favor of it being an even-numbered ball.
27. If one ball is selected at random, find the odds in favor of it being a ball other than the 8 ball.
28. If one ball is selected at random, find the odds against it containing any red coloring (solid or striped).
29. If one ball is selected at random, find the odds against it containing a number greater than or equal to 9.
30. If one ball is selected at random, find the odds in favor of it containing two digits.

31. *Bowl Games* The number of college football bowl games has grown in recent years. The table below indicates the nine bowl games with the greatest per-team payout in 2003.

Bowl Game	Per-team Payout (millions)
Tostitos Fiesta	13.5
Fed Ex Orange	13.5
Rose	13.5
Nokia Sugar	13.5
Capital One	5.125
SBC Cotton	3
Outback	2.55
Chick-Fil A Peach	2
Pacific Life Holiday	2

If one of the bowl games listed in the chart is selected at random, determine
    a) the probability the per-team payout is greater than $5 million.

b) the odds against the per-team payout being greater than $5 million.

The Rose Bowl in Pasadena, CA

32. *Rolling a Special Die* A special die used in a game contains one dot on one side, two dots on two sides, and three dots on three sides. If the die is rolled, determine
a) the probability of rolling two dots.
b) the odds against rolling two dots.

33. *Medical Tests* The results of a medical test show that of 76 people selected at random who were given the test, 72 tested negative and 4 tested positive. Determine the odds against a person selected at random testing negative on the test. Explain how you determined your answer.

34. *A Red Marble* A box contains 9 red and 2 blue marbles. If you select one marble at random from the box, determine the odds against selecting a red marble. Explain how you determined your answer.

35. *Teaching Award* The odds in favor of June White winning the teaching award are 8 : 5. Find the probability that
a) June wins.
b) June does not win.

36. *Hot Dog Contest* The odds in favor of Boris Penzed winning the hot dog eating contest are 2 : 7. Find the probability that Boris will
a) win the contest.
b) not win the contest.

37. *Getting Promoted* The odds against Jason Judd getting promoted are 4 : 11. Find the probability that Jason gets promoted.

38. *Winning a Race* The odds against Paul Phillips winning the 100 yard dash are 5 : 2. Find the probability that
a) Paul wins.
b) Paul loses.

Playing Bingo *When playing bingo, 75 balls are placed in a bin and balls are selected at random. Each ball is marked with a letter and number as indicated in the following chart.*

B	I	N	G	O
1–15	16–30	31–45	46–60	61–75

For example, there are balls marked B1, B2, up to B15; I16, I17, up to I30; and so on. In Exercises 39–44, assuming one bingo ball is selected at random, determine

39. the probability it contains the letter *G*.

40. the probability it does not contain the letter *G*.

41. the odds in favor of it containing the letter *G*.

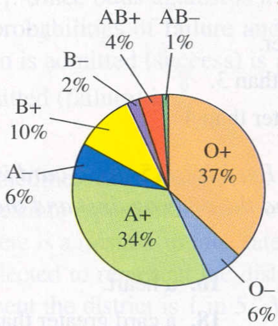
42. the odds against it containing the letter *G*.

43. the odds against it being *B*9.

44. the odds in favor of it being *B*9.

Blood Types *In Exercises 45–50, the following circle graph shows the percent of Americans with the various types of blood.*

Blood Types of Americans



Source: 2003 Time Almanac

If one American is selected at random, use the graph to determine

45. the probability the person has A+ blood.

46. the probability the person has B− blood.

47. the odds against the person having A+ blood.

48. the odds in favor of the person having B− blood.

49. the odds in favor of the person having either O+ or O− blood.

50. the odds against the person having either A+ or O+ blood.

51. *Rock Concert* Suppose that the probability that a rock concert sells out is 0.9. Determine the odds against the concert selling out.

52. *Working Overtime* Suppose that the probability that you are asked to work overtime this week is 3/8. Determine the odds in favor of your being asked to work overtime.

53. *Bookcase Assembly* Suppose that the probability that all the parts needed to assemble a bookcase are included in the carton is $\frac{7}{8}$. Determine the odds in favor of the carton including all the needed parts.

54. *IRS Audit* One in 42 individuals whose salaries range between \$10,000 and \$40,000 will be randomly selected to have his or her income tax returns audited. Mr. Frank is in this income tax range. Find

a) the probability that Mr. Frank will be audited.

b) the odds against Mr. Frank being audited.

55. *Arthritis* Gout, a form of arthritis, is much less common in women than in men. In general, 20 out of 21 people with gout are men.

a) If J. Douglas has gout, what is the probability that J. Douglas is a man?

b) If J. Douglas has gout, what are the odds against J. Douglas's being a woman?

Challenge Problems/Group Activities

56. *Odds Against* Find the odds against an even number or a number greater than 3 being rolled on a die.

57. *Horse Racing* Racetracks quote the approximate odds against each horse winning on a large board called a *tote board*. The odds quoted on a tote board for a race with five horses is as follows.

Horse Number	Odds
1	7:2
2	2:1
3	15:1
4	7:5
5	1:1

Find the probability of each horse winning the race. (Do not be concerned that the sum of the probabilities is not 1.)

58. *Roulette* Turn to the roulette wheel illustrated on page 687. If the wheel is spun, find

a) the probability that the ball lands on red.

b) the odds against the ball landing on red.

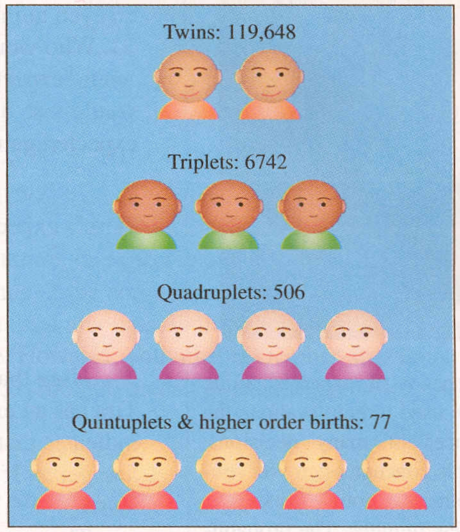
c) the probability that the ball lands on 0 or 00.

d) the odds in favor of the ball landing on 0 or 00.

Recreational Exercise

59. *Multiple Births* Multiple births make up about 3% of births a year in the United States. The following illustrates the number and type of multiple births in 2000.

Multiple Births in the United States in 2000



Source: National Center for Health Statistics

Using the above information, determine an estimate for the odds against a birth being a multiple birth in 2000.

Internet/Research Activities

60. *State Lottery* Determine whether your state has a lottery. If so, do research and write a paper indicating

a) the probability of winning the grand prize.

b) the odds against winning the grand prize.

c) Explain, using real objects such as pennies or table tennis balls, what these odds really mean.

61. *Casino Advantages* There are many types of games of chance to choose from at casinos. The house has the advantage in each game, but the advantages differ according to the game.

a) List the games available at a typical casino.

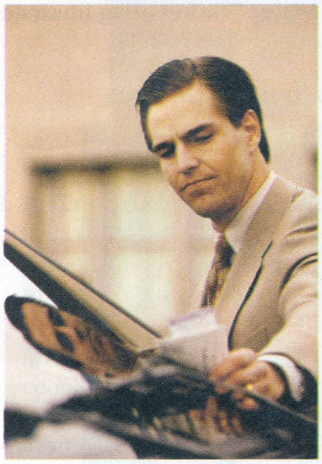
b) List those for which the house has the smallest advantage of winning.

c) List those for which the house has the greatest advantage of winning.

12.4 EXPECTED VALUE (EXPECTATION)

DID YOU KNOW

Tough Decision



The concept of expected value can be used to help evaluate the consequences of many decisions. You use this concept when you consider whether to double-park your car for a few minutes. The probability of being caught may be low, but the penalty, a parking ticket, may be high. You weigh these factors when you decide whether to spend time looking for a legal parking spot.

Expected value, also called ***expectation,*** is often used to determine the expected results of an experiment or business venture over the long run. Expectation is used to make important decisions in many different areas. In business, for example, expectation is used to predict future profits of a new product. In the insurance industry, expectation is used to determine how much each insurance policy should cost for the company to make an overall profit. Expectation is also used to predict the expected gain or loss in games of chance such as the lottery, roulette, craps, and slot machines.

Consider the following: Tim tells Barbara that he will give her \$1 if she can roll an even number on a single die. If she fails to roll an even number, she must give Tim \$1. Who would win money in the long run if this game were played many times? We would expect in the long run that half the time Tim would win \$1 and half the time he would lose \$1; therefore, Tim would break even. Mathematically, we could find Tim's expected gain or loss by the following procedure:

$$\text{Tim's expected gain or loss} = P\begin{pmatrix}\text{Tim}\\ \text{wins}\end{pmatrix} \cdot \begin{pmatrix}\text{amount}\\ \text{Tim wins}\end{pmatrix} + P\begin{pmatrix}\text{Tim}\\ \text{loses}\end{pmatrix} \cdot \begin{pmatrix}\text{amount}\\ \text{Tim loses}\end{pmatrix}$$

$$= \frac{1}{2}(\$1) + \frac{1}{2}(-\$1) = \$0$$

Note that the loss is written as a negative number. This procedure indicates that Tim has an expected gain or loss (or expected value) of \$0. The expected value of zero indicates that he would indeed break even, as we had anticipated. Thus, the game is a *fair game.* If his expected value were positive, it would indicate a gain; if negative, a loss.

The expected value, E, is calculated by multiplying the probability of an event occurring by the *net* amount gained or lost if the event occurs. If there are a number of different events and amounts to be considered, we use the following formula.

Expected Value

$$E = P_1 \cdot A_1 + P_2 \cdot A_2 + P_3 \cdot A_3 + \cdots + P_n \cdot A_n$$

The symbol P_1 represents the probability that the first event will occur, and A_1 represents the net amount won or lost if the first event occurs. P_2 is the probability of the second event, and A_2 is the net amount won or lost if the second event occurs, and so on. The sum of these products of the probabilities and their respective amounts is the expected value. The expected value is the average (or mean) result that would be obtained if the experiment were performed a great many times.

EXAMPLE 1 *A New Business Venture*

Southwest Airlines is considering adding a route to the city of Cedar Rapids, Iowa. Before the company makes their decision as to whether or not to service Cedar Rapids, it needs to consider many factors, including their potential profits and losses. Factors that may affect the company's profits and losses include the number

of competing airlines, the potential number of customers, the overhead costs, and fees they must pay. After considerable research, the company estimates that if it serves Cedar Rapids, there is a 60% chance of making an \$800,000 profit, a 10% chance of breaking even, and a 30% chance of losing \$1,200,000. How much can Southwest Airlines "expect" to make on this new route?

SOLUTION: The three amounts to be considered are a gain of \$800,000, breaking even at \$0, and a loss of \$1,200,000. The probability of gaining \$800,000 is 0.6, the probability of breaking even is 0.1, and the probability of losing \$1,200,000 is 0.3.

$$\begin{aligned}\text{Southwest's expectation} &= \overbrace{P_1 \cdot A_1}^{\text{Gain}} + \overbrace{P_2 \cdot A_2}^{\text{Break even}} + \overbrace{P_3 \cdot A_3}^{\text{Loss}} \\ &= (0.6)(\$800{,}000) + (0.1)(\$0) + (0.3)(-\$1{,}200{,}000) \\ &= \$480{,}000 + \$0 - \$360{,}000 \\ &= \$120{,}000\end{aligned}$$

Southwest Airlines has an expectation, or expected average gain, of \$120,000 for adding this particular service. Thus, if the company opened routes like this one, with these particular probabilities and amounts, in the long run it would have an average gain of \$120,000 per route. However, you must remember, that there is a 30% chance that Southwest will lose \$1,200,000 on this *particular* route (or any particular route with these probabilities and amounts.)

EXAMPLE 2 *Test-Taking Strategy*

Maria is taking a multiple-choice exam in which there are five possible answers for each question. The instructions indicate that she will be awarded 2 points for each correct response, that she will lose $\frac{1}{2}$ point for each incorrect response, and that no points will be added or subtracted for answers left blank.

a) If Maria does not know the correct answer to a question, is it to her advantage or disadvantage to guess at an answer?

b) If she can eliminate one of the possible choices, is it to her advantage or disadvantage to guess at the answer?

SOLUTION:

a) Let's determine the expected value if Maria guesses at an answer. Only one of five possible answers is correct.

$$P(\text{guesses correctly}) = \frac{1}{5} \qquad P(\text{guesses incorrectly}) = \frac{4}{5}$$

$$\begin{aligned}\text{Maria's expectation} &= \overbrace{P_1 \cdot A_1}^{\text{Guesses correctly}} + \overbrace{P_2 \cdot A_2}^{\text{Guesses incorrectly}} \\ &= \frac{1}{5}(2) + \frac{4}{5}\left(-\frac{1}{2}\right) \\ &= \frac{2}{5} - \frac{2}{5} = 0\end{aligned}$$

Thus, Maria's expectation is zero when she guesses. Therefore, over the long run she will neither gain nor lose points by guessing.

b) If Maria can eliminate one possible choice, one of four answers will be correct.

$$P(\text{guesses correctly}) = \frac{1}{4} \qquad P(\text{guesses incorrectly}) = \frac{3}{4}$$

$$\text{Maria's expectation} = \underbrace{P_1 \cdot A_1}_{\text{Guesses correctly}} + \underbrace{P_2 \cdot A_2}_{\text{Guesses incorrectly}}$$

$$= \frac{1}{4}(2) + \frac{3}{4}\left(-\frac{1}{2}\right)$$

$$= \frac{2}{4} - \frac{3}{8} = \frac{4}{8} - \frac{3}{8} = \frac{1}{8}$$

Since the expectation is a positive $\frac{1}{8}$, Maria will, on average, gain $\frac{1}{8}$ point each time she guesses when she can eliminate one possible choice. ▲

EXAMPLE 3 *Pothole Repairs*

A highway crew repairs 30 potholes a day in dry weather and 12 potholes a day in wet weather. If the weather in this region is wet 40% of the time, find the expected (average) number of potholes that can be repaired per day.

SOLUTION: The amounts in this problem are the number of potholes repaired. Since the weather is wet 40% of the time, it will be dry 100% − 40% = 60% of the time. When written as probabilities, 60% and 40% are 0.60 and 0.40, respectively.

$$E = P(\text{dry}) \cdot (\text{amount repaired}) + P(\text{wet}) \cdot (\text{amount repaired})$$

$$= 0.60(30) + 0.40(12) = 18.0 + 4.8 = 22.8$$

Thus, the average, or expected, number of potholes repaired per day is 22.8. ▲

When we gave the expectation formula we indicated that the amounts were the **net amounts,** which are the actual amounts gained or lost. Examples 4 and 5 illustrate how net amounts are used in two applications of expected value.

EXAMPLE 4 *Winning a Door Prize*

When Josh Rosenberg attends a charity event, he is given a free ticket for the $50 door prize. A total of 100 tickets will be given out. Determine his expectation of winning the door prize.

SOLUTION: The probability of winning the door prize is $\frac{1}{100}$ since Josh has 1 of 100 tickets. If he wins, his net or actual winnings will be $50 since he did not pay for the ticket. The probability that Josh loses is $\frac{99}{100}$. If Josh loses, the amount he loses is $0 because he did not pay for the ticket.

$$\text{Expectation} = P(\text{Josh wins}) \cdot (\text{amount won}) + P(\text{Josh loses}) \cdot (\text{amount lost})$$

$$= \frac{1}{100}(50) + \frac{99}{100}(0) = \frac{50}{100} = 0.50$$

Thus, Josh's expectation is $0.50, or 50 cents. ▲

DID YOU KNOW

It Pays to Be Original

Commonly Selected Numbers

7-14-21-28-35-42
1-2-3-4-5-6
5-10-15-20-25-30
3-8-13-18-23-28

The numbers you select when picking lottery numbers have no effect on your probability of winning. However, your expectation (expected winnings) varies greatly with the numbers you select. Because the jackpot is divided among all the winners, the fewer the number of winners the more each winner receives. There are some groups of numbers that are commonly selected. Some are illustrated above. In a lottery with a large jackpot, there may be as many as 10,000 people who select the numbers 7-14-21-28-35-42. If the jackpot was $40 million and these numbers were selected, each winner would receive about $4000. That is quite a difference from the $40 million a single winner would receive. Many people use birthdays or other dates when selecting lottery numbers. Therefore, there may be fewer people selecting numbers greater than 31.

TIMELY TIP In Example 4 and in all the previous examples of expectation, the sum of the probabilities of the events has always been 1. This should always be the case, that is, the sum of the probabilities in any expectation problem should always be 1.

Now we will consider a problem similar to Example 4, but this time we will assume that Josh must purchase the ticket for the door prize.

EXAMPLE 5 *Winning a Door Prize*

When Josh Rosenberg attends a charity event, he is given the opportunity to purchase a ticket for the $50 door prize. The cost of the ticket is $2, and 100 tickets will be sold. Determine Josh's expectation if he purchases one ticket.

SOLUTION: As in Example 4, Josh's probability of winning is $\frac{1}{100}$. However, if he does win, his actual or net winnings will be $48. The $48 is obtained by subtracting the cost of the ticket, $2, from the amount of the door prize, $50. There is also a probability of $\frac{99}{100}$ that Josh will not win the door prize. If he does not win the door prize, he has lost the $2 that he paid for the ticket. Therefore, there are two amounts that we must consider when we determine Josh's expectation, winning $48 and losing $2.

$$\begin{aligned}\text{Expectation} &= P(\text{Josh wins}) \cdot (\text{amount won}) + P(\text{Josh loses}) \cdot (\text{amount lost}) \\ &= \frac{1}{100}(48) + \frac{99}{100}(-2) \\ &= \frac{48}{100} - \frac{198}{100} = -\frac{150}{100} = -1.50\end{aligned}$$

Josh's expectation is $-\$1.50$ when he purchases one ticket. ▲

In Example 5, we determined that Josh's expectation was $-\$1.50$ when he purchased one ticket. If he purchased two tickets, his expectation would be $2(-\$1.50)$, or $-\$3.00$. We could also compute Josh's expectation if he purchased two tickets as follows

$$E = \frac{2}{100}(46) + \frac{98}{100}(-4) = -3.00$$

This answer, $-\$3.00$, checks with the answer obtained by multiplying the expectation for a single ticket by 2.

Let's look at one more example where a person must pay for a chance to win a prize. In the following example, there will be more than two amounts to consider.

EXAMPLE 6 *Raffle Tickets*

One thousand raffle tickets are sold for $1 each. One grand prize of $500 and two consolation prizes of $100 will be awarded. The tickets are placed in a bin. The winning tickets will be selected from the bin. Assuming that each ticket selected for a prize is returned to the bin before the next ticket is selected, determine

a) Irene Drew's expectation if she purchases one ticket.

b) Irene's expectation if she purchases five tickets.

SOLUTION:

a) Three amounts are to be considered: the net gain in winning the grand prize, the net gain in winning one of the consolation prizes, and the loss of the cost of the ticket. If Irene wins the grand prize, her net gain is \$499 (\$500 minus \$1 spent for the ticket). If Irene wins one of the consolation prizes, her net gain is \$99 (\$100 minus \$1). We are told to assume that each winning ticket is replaced in the bin after being selected. The probability that Irene wins the grand prize is $\frac{1}{1000}$. Since two consolation prizes will be awarded, the probability that she wins a consolation prize is $\frac{2}{1000}$. The probability that she does not win a prize is $1 - \frac{1}{1000} - \frac{2}{1000} = \frac{997}{1000}$.

$$E = P_1 \cdot A_1 + P_2 \cdot A_2 + P_3 \cdot A_3$$

$$= \frac{1}{1000}(\$499) + \frac{2}{1000}(\$99) + \frac{997}{1000}(-\$1)$$

$$= \frac{499}{1000} + \frac{198}{1000} - \frac{997}{1000} = -\frac{300}{1000} = -0.30$$

Thus, Irene's expectation is $-\$0.30$ per ticket purchased.

b) On average, Irene loses 30 cents on each ticket purchased. On five tickets her expectation is $(-\$0.30)(5)$, or $-\$1.50$.

In Example 5, we determined that Josh's expectation was $-\$1.50$. Now let's determine how to find out how much should have been charged for a ticket so that his expectation would be \$0. If Josh's expectation were to be \$0, he could be expected to break even over the long run. Suppose that Josh paid 50 cents, or \$0.50, for the ticket. His expectation, if paying \$0.50 for the ticket, would be calculated as shown below.

$$\text{Expectation} = P(\text{Josh wins}) \cdot (\text{amount won}) + P(\text{Josh loses}) \cdot (\text{amount lost})$$

$$= \frac{1}{100}(49.50) + \frac{99}{100}(-\$0.50)$$

$$= \frac{49.50}{100} - \frac{49.50}{100} = 0$$

Thus, if Josh paid 50 cents per ticket, his expectation would be \$0. The 50 cents, in this case, is called the fair price of the ticket. The ***fair price*** is the amount to be paid that will result in an expected value of \$0. The fair price may be found by adding the *cost to play* to the *expected value*.

Fair price = expected value + cost to play

In Example 5, the cost to play was \$2 and the expected value was determined to be $-\$1.50$. The fair price for a ticket in Example 5 may be found as follows.

$$\text{Fair price} = \text{expected value} + \text{cost to play}$$
$$= -1.50 + 2.00 = 0.50$$

We obtained a fair price of \$0.50. If the tickets were sold for the fair price of \$0.50 each, Josh's expectation would be \$0, as shown above. Can you now find the fair price that Irene would pay for a raffle ticket in Example 6? In Example 6, the cost of a ticket was \$1 and we determined that the expected value was $-\$0.30$.

$$\text{Fair price} = \text{expected value} + \text{cost to play}$$
$$= -\$0.30 + \$1.00 = \$0.70$$

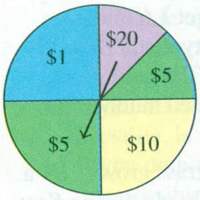
Thus, the fair price for a ticket in Example 6 is \$0.70, or 70 cents. Verify for yourself now that if the tickets were sold for \$0.70, the expectation would be \$0.00.

EXAMPLE 7 *Expectation and Fair Price*

Suppose that you are playing a game in which you spin the pointer shown in the figure in the margin, and you are awarded the amount shown under the pointer. If it costs \$8 to play the game, determine

a) the expectation of a person who plays the game.

b) the fair price to play the game.



SOLUTION:

a) There are four numbers on which the pointer can land: 1, 5, 10, and 20. The following chart shows the probability of the pointer landing on each number and the actual amount won or lost if the pointer lands on that number. The probabilities are obtained using the areas of the circle. The amounts won or lost are determined by subtracting the cost to play, \$8, from each indicated amount.

Amount Shown on Wheel	\$1	\$5	\$10	\$20
Probability	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
Amount Won or Lost	−\$7	−\$3	\$2	\$12

Notice that the sum of the probabilities is 1, which shows that all possible outcomes have been considered.

Now let's find the expectation. There are four amounts to consider.

$$\begin{aligned}\text{Expectation} &= P(\text{lands on \$1})\cdot(\text{amount}) + P(\text{lands on \$5})\cdot(\text{amount})\\ &\quad + P(\text{lands on \$10})\cdot(\text{amount}) + P(\text{lands on \$20})\cdot(\text{amount})\\ &= \frac{1}{4}(-7)+\frac{3}{8}(-3)+\frac{1}{4}(2)+\frac{1}{8}(12)\\ &= -\frac{7}{4}-\frac{9}{8}+\frac{2}{4}+\frac{12}{8}\\ &= -\frac{14}{8}-\frac{9}{8}+\frac{4}{8}+\frac{12}{8} = -\frac{7}{8} = -\$0.875\end{aligned}$$

Thus, the expectation is −\$0.875.

b) Fair price = expectation + cost to play

$$= -\$0.875 + \$8 = \$7.125$$

Thus, the fair price is about \$7.13.

SECTION 12.4 EXERCISES

Concept/Writing Exercises

1. What does the expected value of an experiment or business venture represent?
2. What does an expected value of 0 mean?
3. What is meant by the fair price of a game of chance?
4. Write the formula used to find the expected value of an experiment with
   a) two possible outcomes.
   b) three possible outcomes.

5. If the expected value and cost to play are known for a particular game of chance, explain how you can determine the fair price to pay to play that game of chance.
6. Is the fair price to pay for a game of chance the same as the expected value of that game of chance? Explain your answer.
7. If a particular game costs \$1.50 to play and the expectation for the game is −\$1.00, what is the fair price to pay to play the game? Explain how you determined your answer.
8. If a particular game cost \$3.00 to play and the expectation for the game is −\$2.00, what is the fair price to pay to play the game? Explain how you determined your answer.

Practice the Skills/Problem Solving

9. *Three Tickets* On a \$1 lottery ticket, Marty Smith's expected value is −\$0.40. What is Marty's expected value if he purchases three lottery tickets?

10. *Expected Value* If on a \$1 bet, Paul Goldstein's expected value is \$0.20. What is Paul's expected value on a \$5 bet?

11. *Expected Attendance* For a showing of a specific movie, an AMC theater estimates that 120 people will attend if it is not raining. If it is raining, the theater estimates that 200 people will attend. The meteorologist predicts a 70% chance of rain tomorrow. Determine the expected number of people who will attend the movie.

12. *A New Business* In a proposed business venture, Stephanie Morrison estimates that there is a 60% chance she will make \$80,000 and a 40% chance she will lose \$20,000. Determine Stephanie's expected value.

13. *Basketball* Diana Taurasi is a star player for the University of Connecticut Huskies women's basketball team. She has injured her ankle and it is doubtful if she will be able to play in an upcoming game. If she can play, the coach estimates that the Huskies will score 78 points. If she is not able to play, the coach estimates that they will score 62 points. The team doctor estimates that there is a 50% chance Diana will play. Determine the number of points the team can expect to score.

Shareese Grant (left) and Diana Taurasi

14. *Seminar Attendance* At an investment tax seminar, Judy Johnson estimates that 20 people will attend if it does not rain and 12 people will attend if it rains. The weather forecast indicates a 40% chance it will not rain and a 60% chance it will rain on the day of the seminar. Determine the expected number of people who will attend the seminar.

15. *TV Shows* The NBC television network is scheduling its fall lineup of shows. For the Thursday night 8 P.M. slot, NBC has selected the show *The West Wing*. If its rival network CBS schedules the show *CSI—Crime Scene Investigation* during the same time slot, NBC estimates that *The West Wing* will get 1.2 million viewers. However, if CBS schedules the show *Judging Amy* during that time slot, NBC estimates that *The West Wing* will get 1.6 million viewers. NBC believes that the probability that CBS will show *CSI* is 0.4 and the probability that CBS will show *Judging Amy* is 0.6. Determine the expected number of viewers for the show *The West Wing*.

16. *Seattle Greenery* In July in Seattle, the grass grows $\frac{1}{2}$ in. a day on a sunny day and $\frac{1}{4}$ in. a day on a cloudy day. In Seattle in July, 75% of the days are sunny and 25% are cloudy.
a) Find the expected amount of grass growth on a typical day in July in Seattle.
b) Find the expected total grass growth in the month of July in Seattle.

17. *Investment Club* The Triple L investment club is considering purchasing a certain stock. After considerable research the club members determine that there is a 60% chance of making \$10,000, a 10% chance of breaking even, and a 30% chance of losing \$7200. Find the expectation of this purchase.

18. *Clothing Sale* At a special clothing sale at the Crescent Oaks Country Club, after the cashier rings up your purchase, you select a slip of paper from a box. The slip of paper indicates the dollar amount, either \$5 or \$10, that is deducted from your purchase price. The probability of selecting a slip indicating \$5 is $\frac{7}{10}$ and the probability of selecting a slip indicating \$10 is $\frac{3}{10}$. If your original purchase before you select the slip of paper is \$100, determine
a) the expected dollar amount to be deducted from your purchase.
b) the expected dollar amount you will pay for your purchase.

19. *Fortune Cookies* At the Royal Dragon Chinese restaurant, a slip in the fortune cookies indicates a dollar amount that will be subtracted from your total bill. A bag of 10 fortune cookies is given to you from which you will select 1. If seven fortune cookies contain "\$1 off," two contain "\$2 off," and one contains "\$5 off," find the expectation of a selection.

20. *Pick a Card* Mike and Dave play the following game: Mike picks a card from a deck of cards. If he selects a heart, Dave gives him \$5. If not, he gives Dave \$2.
a) Find Mike's expectation.
b) Find Dave's expectation.

21. *Roll a Die* Cortney and Kelly play the following game. Kelly rolls a die. If she rolls a number greater than 4, Cortney gives Kelly \$8. If Kelly does not roll a number greater than 4, she gives Cortney \$5.
a) Determine Kelly's expectation.
b) Determine Cortney's expectation.

22. ***Blue Chips and Red Chips*** A bag contains 3 blue chips and 2 red chips. Chi and Dolly play the following game. Chi selects one chip at random from the bag. If Chi selects a blue chip, Dolly gives Chi \$5. If Chi selects a red chip, Chi gives Dolly \$8.
   a) Determine Chi's expectation.
   b) Determine Dolly's expectation.

23. ***Multiple-Choice Test*** A multiple-choice exam has five possible answers for each question. For each correct answer, you are awarded 5 points. For each incorrect answer, 1 point is subtracted from your score. For answers left blank, no points are added or subtracted.
   a) If you do not know the correct answer to a particular question, is it to your advantage to guess? Explain.
   b) If you do not know the correct answer but can eliminate one possible choice, is it to your advantage to guess? Explain.

24. ***Multiple-Choice Test*** A multiple-choice exam has four possible answers for each question. For each correct answer, you are awarded 5 points. For each incorrect answer, 2 points are subtracted from your score. For answers left blank, no points are added or subtracted.
   a) If you do not know the correct answer to a particular question, is it to your advantage to guess? Explain.
   b) If you do not know the correct answer but can eliminate one possible choice, is it to your advantage to guess? Explain.

Exercises 25–28 deal with raffle drawings. Assume that after each ticket is drawn, the ticket that was drawn is mixed in with the other tickets before the next selection. Therefore, the selections are being made with replacement.

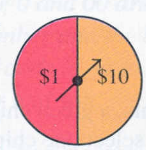
25. ***Raffle Tickets*** Five hundred raffle tickets are sold for \$2 each. One prize of \$400 is to be awarded.
   a) Raul Mondesi purchases one ticket. Find his expected value.
   b) Determine the fair price of a ticket.

26. ***Raffle Tickets*** One thousand raffle tickets are sold for \$1 each. One prize of \$800 is to be awarded.
   a) Rena Condos purchases one ticket. Find her expected value.
   b) Determine the fair price of a ticket.

27. ***Raffle Tickets*** Two thousand raffle tickets are sold for \$3.00 each. Three prizes will be awarded: one for \$1000 and two for \$500. Jeremy Sharp purchases one of these tickets.
   a) Determine his expected value.
   b) Determine the fair price of a ticket.

28. ***Raffle Tickets*** Ten thousand raffle tickets are sold for \$5 each. Four prizes will be awarded: one for \$10,000, one for \$5000, and two for \$1000. Sidhardt purchases one of these tickets.
   a) Determine his expected value.
   b) Determine the fair price of a ticket.

In Exercises 29 and 30, assume that a person spins the pointer and is awarded the amount indicated by the pointer. Determine the person's expectation.

29.


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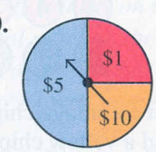
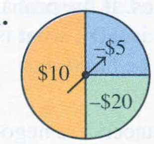
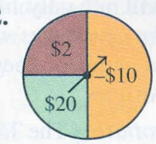
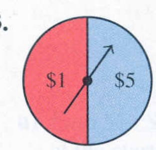


In Exercises 31 and 32, assume that a person spins the pointer and is awarded the amount indicated if the pointer points to a positive number but must pay the amount indicated if the pointer points to a negative number. Determine the person's expectation if the person plays the game.

31.


32.


In Exercises 33–36, assume that a person spins the pointer and is awarded the amount indicated by the pointer. If it costs \$2 to play the game, determine
a) the expectation of a person who plays the game.
b) the fair price to play the game.

33.


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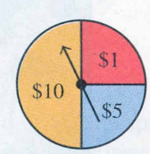

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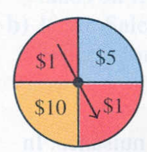
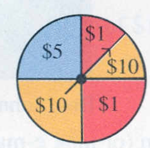


36.


37. ***Reaching Base Safely*** Based on past history, Jim Devias has a 17% chance of reaching first base safely, a 10% chance of hitting a double, a 2% chance of hitting a triple, an 8% chance of hitting a home run, and a 63% chance of making an out at his next at bat. Determine Jim's expected number of bases for his next at bat.

38. ***Life Insurance*** According to Bristol Mutual Life Insurance's mortality table, the probability that a 20-year-old woman will survive 1 year is 0.994 and the probability that she will die within 1 year is 0.006. If she buys a \$10,000 1-year policy for \$100, what is Bristol Mutual's expected gain or loss?

39. *Choosing a Colored Chip* In a box there are a total of 10 chips. The chips are orange, green, or yellow, as shown below.

If you select an orange chip you get 4 points, a green chip 3 points, and a yellow chip 1 point. If you select one chip at random, determine the expected number of points you will get.

40. *Choosing a Colored Chip* Repeat Exercise 39 but assume that an orange chip is worth 5 points, a green chip 2 points, and a yellow chip −3 points (3 points are taken away).

41. *Airline Hiring* American Airlines has requested new routes. If the new routes are granted, American will hire 850 new employees. If the new routes are not granted, American will hire only 140 new employees. If the probability that the new routes will be approved is 0.34, what is the expected number of new employees to be hired by American Airlines?

42. *Salary Negotiating* The Tampa Bay Buccaneers are negotiating salary with quarterback Brad Johnson. If the Buccaneers reach the Super Bowl, his salary will be \$2.3 million per year. If the Bucs do not reach the Super Bowl, his salary will be \$1.7 million per year. There is a 62% chance that the Bucs will reach the Super Bowl and a 38% chance that the Bucs will not reach the Super Bowl. Determine the expected salary for Brad Johnson.

Brad Johnson

43. *Spam* Spam (or junk e-mail) is a growing nuisance. In January 2003, about 11% of computer users spent about 10 minutes reviewing and deleting spam, 65% of computer users spent about 15 minutes reviewing and deleting spam, and 24% of computer users spent about 20 minutes reviewing and deleting spam. If a computer user is selected at random, determine the expected time he or she will spend reviewing and deleting spam.

44. *Rolltop Desk* The owner of an antique store estimates that there is a 40% chance she will make \$1000 when she sells an antique rolltop desk, a 50% chance she will make \$500 when she sells the desk, and a 10% chance she will break even when she sells the desk. Determine the expected amount she will make when she sells the desk.

45. *Rolling a Die* A die is rolled many times, and the points facing up are recorded. Find the expected (average) number of points facing up over the long run.

46. *Lawsuit* Don Vello is considering bringing a lawsuit against the Dummote Chemical Company. His lawyer estimates that there is a 70% chance Don will make \$40,000, a 10% chance Don will break even, and a 20% chance they will lose the case and Don will need to pay \$30,000 in legal fees. Estimate Don's expected gain or loss if he proceeds with the lawsuit.

47. *Road Service* On a clear day in Boston, the Automobile Association of American (AAA) makes an average of 110 service calls for motorist assistance, on a rainy day it makes an average of 160 service calls, and on a snowy day it makes an average of 210 service calls. If the weather in Boston is clear 200 days of the year, rainy 100 days of the year, and snowy 65 days of the year, find the expected number of service calls made by the AAA in a given day.

48. *Real Estate* The expenses for Jorge Estrada, a real estate agent, to list, advertise, and attempt to sell a house are \$1000. If Jorge succeeds in selling the house, he will receive a commission of 6% of the sales price. If an agent with a different company sells the house, Jorge still receives 3% of the sales price. If the house is unsold after 3 months, Jorge loses the listing and receives nothing. Suppose that the probability that he sells a \$100,000 house is 0.2, the probability that another agent sells the house is 0.5, and the probability that the house is unsold after 3 months is 0.3. Find Jorge's expectation if he accepts this house for listing. Should Jorge list the house? Explain.

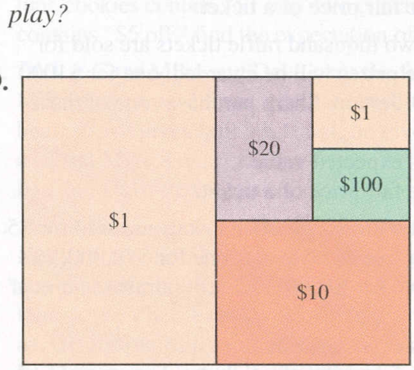
In Exercises 49 and 50, assume that you are blindfolded and throw a dart at the dart board shown. Assuming your dart sticks in the dart board,

a) *determine the probabilities that the dart lands on \$1, \$10, \$20, and \$100, respectively.*

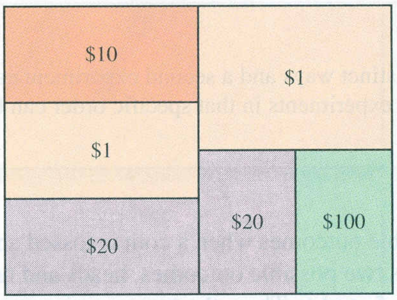
b) *If you win the amount of money indicated by the section of the board where the dart lands, find your expectation when you throw the dart.*

c) *If the game is to be fair, how much should you pay to play?*

49.



50.

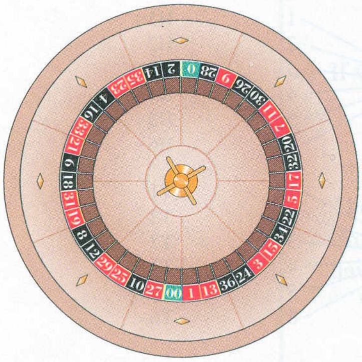


Challenge Problems/Group Activities

51. ***Term Life Insurance*** An insurance company will pay the face value of a term life insurance policy if the insured person dies during the term of the policy. For how much should an insurance company sell a 10-year term policy with a face value of $40,000 to a 30-year-old man for the company to make a profit? The probability of a 30-year-old man living to age 40 is 0.97. Explain your answer. Remember the customer pays for the insurance before the policy becomes effective.

52. ***Lottery Ticket*** Is it possible to determine your expectation when you purchase a lottery ticket? Explain.

Roulette *In Exercises 53 and 54, use the roulette wheel illustrated. A roulette wheel typically contains slots with numbers 1–36 and slots marked 0 and 00. A ball is spun on the wheel and comes to rest in one of the 38 slots. Eighteen numbers are colored red, and 18 numbers are colored black. The 0 and 00 are colored green. If you bet on one particular number and the ball lands on that number, the house pays off odds of 35 to 1. If you bet on a red number or black number and win, the house pays 1 to 1 (even money).*

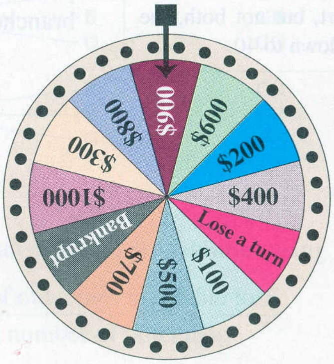


53. Find the expected value of betting $1 on a particular number.

54. Find the expected value of betting $1 on red.

Recreational Exercise

55. ***Wheel of Fortune*** The following is a miniature version of the Wheel of Fortune. When Dave Salem spins the wheel, he is awarded the amount on the wheel indicated by the pointer. If the wheel points to Bankrupt, he loses the total amount he has accumulated and also loses his turn. Assume that the wheel stops on a position at random and that each position is equally likely to occur.



a) Find Mr. Salem's expectation when he spins the wheel at the start of the game (he has no money to lose if he lands on Bankrupt).

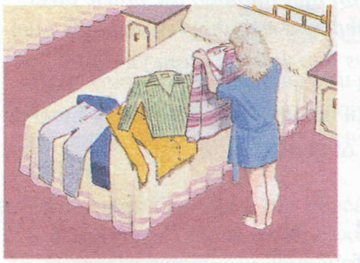
b) If Mr. Salem presently has a balance of $1800, find his expectation when he spins the wheel.

12.5 TREE DIAGRAMS

We stated earlier that the possible results of an experiment are called its outcomes. To solve more difficult probability problems we must first be able to determine all the possible outcomes of an experiment. The counting principle can be used to determine the number of outcomes of an experiment and is helpful in constructing tree diagrams.

DID YOU KNOW

Mix and Match



Fashion experts often suggest that you select your wardrobe from pieces of clothing that can be mixed and matched. If, for example, you have 7 shirts, 3 sweaters, and 4 pairs of pants, you actually have 84 possible outfits. If you wear only a sweater or a shirt, but not both, the possibilities go down to 40.

Counting Principle

If a first experiment can be performed in M distinct ways and a second experiment can be performed in N distinct ways, then the two experiments in that specific order can be performed in $M \cdot N$ distinct ways.

If we wanted to find the number of possible outcomes when a coin is tossed and a die is rolled, we could reason that the coin has two possible outcomes, heads and tails. The die has six possible outcomes: 1, 2, 3, 4, 5, and 6. Thus, the two experiments together have $2 \cdot 6$, or 12, possible outcomes.

A list of all the possible outcomes of an experiment is called a ***sample space***. Each individual outcome in the sample space is called a ***sample point***. ***Tree diagrams*** are helpful in determining sample spaces.

A tree diagram illustrating all the possible outcomes when a coin is tossed and a die is rolled (see Fig. 12.5) has two initial branches, one for each of the possible outcomes of the coin. Each of these branches will have six branches emerging from them, one for each of the possible outcomes of the die. That will give a total of 12 branches, the same number of possible outcomes found by using the counting principle. We can obtain the sample space by listing all the possible combinations of branches. Note that this sample space consists of 12 sample points.

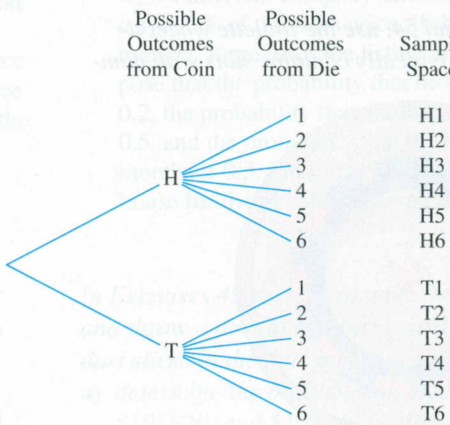


Figure 12.5

Example 1 uses the phrase "without replacement." This phrase tells us that once an item is selected, it cannot be selected again, making it impossible to select the same item twice.

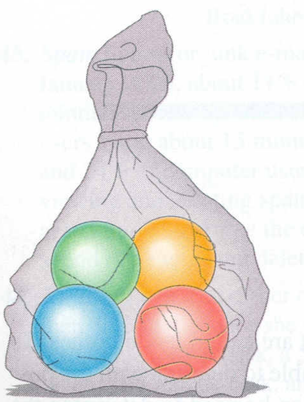


Figure 12.6

EXAMPLE 1 *Selecting Balls without Replacement*

Two balls are to be selected *without replacement* from a bag that contains one red, one blue, one green, and one orange ball (see Fig. 12.6).

a) Use the counting principle to determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
c) Find the probability that one green ball is selected.
d) Find the probability that an orange ball followed by a red ball is selected.

SOLUTION:

a) The first selection may be any one of the four balls. Once the first ball is selected, only three balls remain for the second selection. Thus, there are $4 \cdot 3$, or 12, sample points in the sample space.

b) The first ball selected can be red, blue, green, or orange. Since this experiment is done without replacement, the same colored ball cannot be selected twice. For example, if the first ball selected is red, the second ball selected must be either blue, green, or orange. The tree diagram and sample space are shown in Fig. 12.7. The sample space contains 12 points. That result checks with the answer obtained in part (a) using the counting principle.

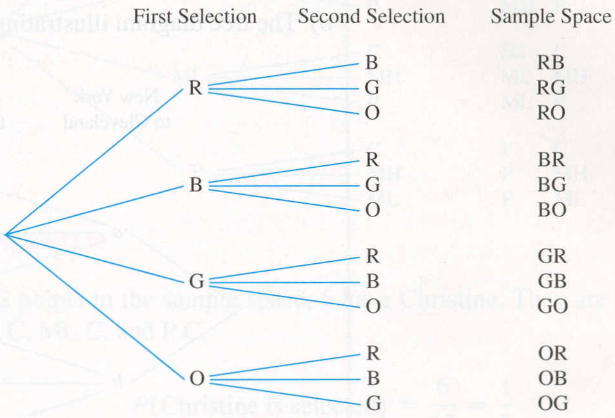


Figure 12.7

c) If we know the sample space, we can compute probabilities using the formula

$$P(E) = \frac{\text{number of outcomes favorable to } E}{\text{total number of outcomes}}$$

The total number of outcomes will be the number of points in the sample space. From Fig. 12.7 we determine that there are 12 possible outcomes. Six outcomes have one green ball: RG, BG, GR, GB, GO, and OG.

$$P(\text{one green ball is selected}) = \frac{6}{12} = \frac{1}{2}$$

d) One possible outcome meets the criteria of an orange ball followed by a red ball: OR.

$$P(\text{orange followed by red}) = \frac{1}{12}$$

The counting principle can be extended to any number of experiments, as illustrated in Example 2.

Figure 12.8

EXAMPLE 2 *Using the Counting Principle*

The Gilligans are driving from New York to San Francisco and wish to stop in Cleveland and Chicago. They are considering two highways from New York to Cleveland, three highways from Cleveland to Chicago, and two highways from Chicago to San Francisco, as illustrated in Fig. 12.8.

a) Use the counting principle to determine the number of different routes the Gilligans can take from New York to San Francisco.

b) Use a tree diagram to determine the routes.

c) If a route from New York to San Francisco is selected at random and all routes are considered equally likely, find the probability that both routes *a* and *g* are used.

d) Find the probability that neither routes *d* nor *f* are used.

SOLUTION:

a) Using the counting principle, we can determine that there are $2 \cdot 3 \cdot 2$, or 12, different routes the Gilligans can take from New York to San Francisco.

b) The tree diagram illustrating the 12 possibilities is given in Fig. 12.9.

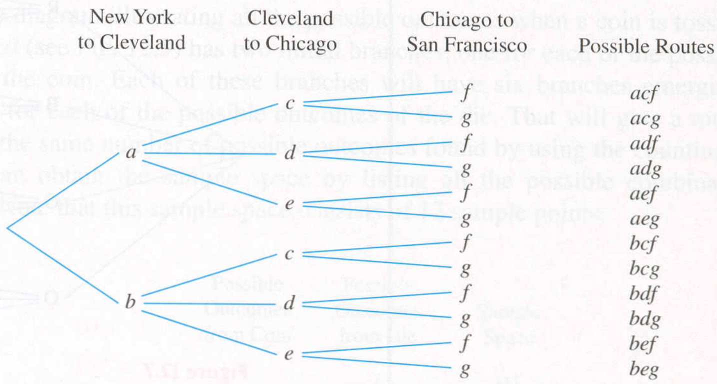


Figure 12.9

c) Of the 12 possible routes, 3 use both *a* and *g* (*acg*, *adg*, *aeg*).

$$P(\text{routes } a \text{ and } g \text{ are both used}) = \frac{3}{12} = \frac{1}{4}$$

d) Of the 12 possible routes, 4 use neither *d* nor *f* (*acg*, *aeg*, *bcg*, *beg*).

$$P(\text{neither } d \text{ nor } f \text{ is used}) = \frac{4}{12} = \frac{1}{3}$$

EXAMPLE 3 *Selecting Ticket Winners*

A radio station has two tickets to give away to a Britney Spears concert. It held a contest and narrowed the possible recipients down to four people: Christine (C), Mike Hammer (MH), Mike Levine (ML), and Phyllis (P). The names of two of these four people will be selected at random from a hat and the two people selected will be awarded the tickets.

a) Use the counting principle to determine the number of points in the sample space.

b) Construct a tree diagram and list the sample space.

c) Determine the probability that Christine is selected.

d) Determine the probability that neither Mike Hammer nor Mike Levine is selected.

e) Determine the probability that at least one Mike is selected.

SOLUTION:

a) The first selection may be any one of the four people; see Fig. 12.10. Once the first person is selected, only three people remain for the second selection. Thus, there are $4 \cdot 3$ or 12 sample points in the sample space.

b)

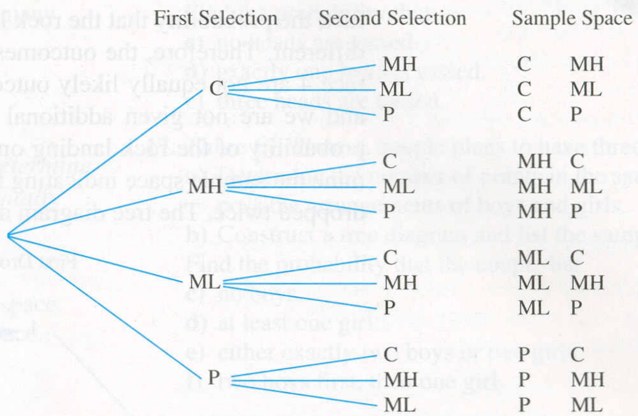


Figure 12.10

c) Of the 12 points in the sample space, 6 have Christine. They are C MH, C ML, C P, MH C, ML C, and P C.

$$P(\text{Christine is selected}) = \frac{6}{12} = \frac{1}{2}$$

d) Of the 12 points in the sample space, two have neither Mike. They are C P (Christine Phyllis) and P C (Phyllis Christine).

$$P(\text{neither Mike selected}) = \frac{2}{12} = \frac{1}{6}$$

e) At least one Mike means that one or more Mikes are selected. There are 10 points in the sample space with at least one Mike (all those except Christine Phyllis and Phyllis Christine).

$$P(\text{at least one Mike is selected}) = \frac{10}{12} = \frac{5}{6}$$ ▲

In Example 3, if you add the probability of no Mike being selected with the probability of at least one Mike being selected, you get $\frac{1}{6} + \frac{5}{6}$, or 1. In any probability problem, if E is a specific event, then either E happens at least one time or it does not happen at all. Thus, $P(E \text{ happening at least once}) + P(E \text{ does not happen}) = 1$. This leads to the following rule.

$$P(\text{event happening at least once}) = 1 - P(\text{event does not happen})$$

For example, suppose that the probability of not getting any red flowers from 3 seeds that are planted is $\frac{2}{7}$. Then the probability of getting at least one red flower from the 3 seeds that are planted is $1 - \frac{2}{7} = \frac{5}{7}$. We will use this rule in later sections.

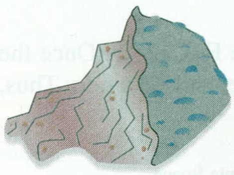


Figure 12.11

In all the tree diagrams in this section, the outcomes were always equally likely; that is, each outcome had the same probability of occurrence. Consider a rock that has 4 faces such that each face has a different surface area and the rock is not uniform in density (see Figure 12.11). When the rock is dropped, the probability that the rock lands on face 1 will not be the same as the probability that the rock lands on face 2. In fact, the probability that the rock lands on face 1, face 2, face 3, and face 4 may all be different. Therefore, the outcomes of the rock landing on face 1, face 2, face 3, and face 4 are not equally likely outcomes. Because the outcomes are not equally likely and we are not given additional information, we cannot determine the theoretical probability of the rock landing on each individual face. However, we can still determine the sample space indicating the faces that the rock may land on when the rock is dropped twice. The tree diagram and sample space is shown in Figure 12.12.

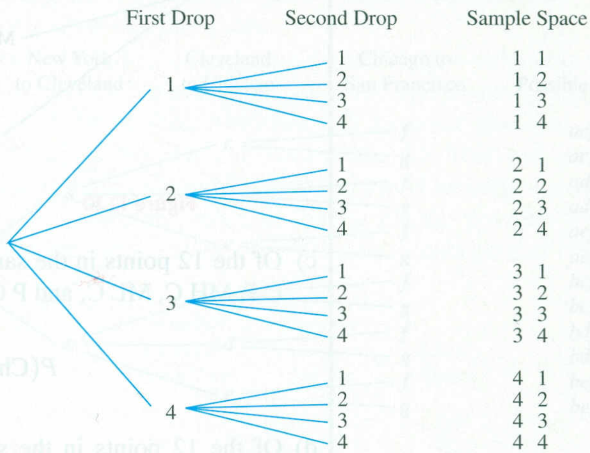


Figure 12.12

Since the outcomes are not equally likely, the probability of each of the sixteen sample points in the sample space occurring cannot be determined. If the outcomes were equally likely, then each of the sixteen points in the sample space would have a probability of $\frac{1}{16}$. See Exercises 29 and 30 which deal with outcomes that are not equally likely.

SECTION 12.5 EXERCISES

Concept/Writing Exercises

1. Explain the counting principle.
2. a) What is a sample space?
   b) What is a sample point?
3. If a first experiment can be performed in two distinct ways and a second experiment can be performed in seven distinct ways, how many possible ways can the two experiments be performed? Explain your answer.
4. In your own words, describe how to construct a tree diagram.
5. A problem states that two selections are made "without replacement." Explain what that means.
6. One experiment has five equally likely outcomes, and a second experiment has two equally likely outcomes. How many sample points will be in the sample space when the two experiments are performed one after the other?

Practice the Skills

7. ***Selecting States*** If two states are selected at random from the 50 states, use the counting principle to determine the number of possible outcomes if the states are selected
   a) with replacement.
   b) without replacement.
8. ***Selecting Dates*** If two dates are selected at random from the 365 days of the year, use the counting principle to determine the number of possible outcomes if the dates are selected
   a) with replacement.
   b) without replacement.
9. ***Selecting Batteries*** A bag contains six batteries, all of which are the same size and are equally likely to be selected. Each battery is a different brand. If you select three batteries at random, use the counting principle to deter-

mine how many points will be in the sample space if the batteries are selected

a) with replacement.
b) without replacement.

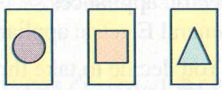
10. *Remote Control* Your television remote control has buttons for digits 0–9. If you press two buttons, how many numbers are possible if
a) the same button may be pressed twice.
b) the same button may not be pressed twice.

In Exercises 11–28, use the counting principle to determine the answer to part (a). Assume that each event is equally likely to occur.

11. *Coin Toss* Two coins are tossed.
a) Determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
Find the probability that
c) no heads are tossed.
d) exactly one head is tossed.
e) two heads are tossed.

12. *Boys and Girls* A couple plans to have two children.
a) Determine the number of points in the sample space of the possible arrangements of boys and girls.
b) Construct a tree diagram and list the sample space.
Find the probability that the couple has
c) two girls.
d) at least one girl.
e) a girl and then a boy.

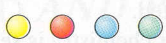
13. *Cards* A box contains three cards. On one card there is a circle. On another card there is a square, and on the third card there is a triangle.



Two cards are to be selected at random with replacement.
a) Determine the number of points in the sample space.
b) Construct a tree diagram and determine the sample space.
Find the probability that
c) two circles are selected.
d) a square and then a triangle are selected.
e) at least one circle is selected.

14. *Cards* Repeat Exercise 13 but assume the cards are drawn without replacement.

15. *Marble Selection* A hat contains four marbles: 1 yellow, 1 red, 1 blue, and 1 green. Two marbles are to be selected at random without replacement from the hat.



a) Determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
Find the probability of selecting
c) exactly 1 red marble.
d) at least 1 marble that is not red.
e) no green marbles.

16. *Three Coins* Three coins are tossed.
a) Determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
Find the probability that
c) no heads are tossed.
d) exactly one head is tossed.
e) three heads are tossed.

17. *Three Children* A couple plans to have three children.
a) Determine the number of points in the sample space of possible arrangements of boys and girls.
b) Construct a tree diagram and list the sample space.
Find the probability that the couple has
c) no boys.
d) at least one girl.
e) either exactly two boys or two girls.
f) two boys first, then one girl.

18. *Pet Shop* A pet shop is selling a calico cat, a Siamese cat, a Persian cat, and a Himalayan cat. The Chens are going to select two cats to bring home as pets.
a) Determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
Find the probability they select
c) the Persian cat.
d) the Persian cat and the calico cat.
e) cats other than the Persian cat.

Problem Solving

19. *Rolling Dice* Two dice are rolled.
a) Determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
Find the probability that
c) a double (a 1, 1 or 2, 2, etc.) is rolled.
d) a sum of 7 is rolled.
e) a sum of 2 is rolled.
f) Are you as likely to roll a sum of 2 as you are of rolling a sum of 7? Explain your answer.

20. *Voting* At a homeowners association meeting, a board member can vote yes, no, or abstain on a motion. There are three motions the board is voting on.
a) Determine the number of points in the sample space.
b) Construct a tree diagram and determine the sample space.
Find the probability that a board member votes
c) no, yes, no in that order.
d) yes on exactly two of the motions.
e) yes on at least one motion.

21. *Gift Certificates* Three different people are to be selected at random, and each will be given one gift certificate. There is one certificate from Home Depot, one from Sears, and one from Outback Steakhouse. The first person selected gets to choose one of the certificates. The second person

selected gets to choose between the two remaining certificates. The third person selected gets the third certificate.

a) Determine the number of points in the sample space.

b) Construct a tree diagram and determine the sample space.

Find the probability that

c) Sears is selected first.

d) Home Depot is selected first and Outback Steakhouse is selected last.

e) The certificates are selected in this order: Sears, Outback Steakhouse, Home Depot.

22. *Shopping* Susan Forman has to purchase a box of cereal, a bottle of soda, and a can of vegetables. The types of cereal, soda, and vegetables she is considering are shown below.

Cereal: Rice Krispies, Frosted Flakes, Honeycomb
Soda: orange, black cherry, ginger ale
Vegetables: peas, carrots

a) Determine the number of points in the sample space.

b) Construct a tree diagram and determine the sample space.

Find the probability that she selects

c) Honeycomb for the cereal.

d) Rice Krispies and ginger ale.

e) a soda other than black cherry.

23. *Vacationing in Florida* Peter and Carol Collinge are vacationing in the Orlando, Florida, area. They have listed what they consider the major attractions in the area in two groups: Disney attractions and non-Disney attractions. The four Disney attractions are the Magic Kingdom, Epcot Center, MGM Studios, and Animal Kingdom. The four non-Disney attractions are Sea World, Universal Studios, Islands of Adventure, and Busch Gardens (Tampa). They decided to first visit one Disney and then one non-Disney attraction. They will select the name of the one Disney attraction from one hat and the name of one non-Disney attraction from a second hat.

a) Determine the number of points in the sample space.

b) Construct a tree diagram and determine the sample space.

Find the probability that they select

c) the Magic Kingdom or Epcot Center.

d) MGM Studios or Universal Studios.

e) the Magic Kingdom and either Sea World or Busch Gardens.

Epcot Center

24. *A New Computer* You visit Computer City to purchase a new computer system. You are going to purchase a computer, printer, and monitor from among the following brands.

Computer	Printer	Monitor
Compaq	Hewlett-Packard	Omega
IBM	Epson	Toshiba
Apple		
Dell		

a) Determine the number of points in the sample space.

b) Construct a tree diagram and determine the sample space.

Find the probability of selecting

c) an Apple computer.

d) a Hewlett-Packard printer.

e) an Apple computer and a Hewlett-Packard printer.

25. *Buying Appliances* Mr. and Mrs. J. B. Davis just moved into a new home and need to purchase kitchen appliances. The chart below shows the brands of appliances they are considering.

Refrigerator	Stove	Dishwasher
General Electric	General Electric	General Electric
Kenmore	Frigidaire	KitchenAid
Maytag	Roper	Whirlpool

The Davises are going to purchase a refrigerator, a stove, and a dishwasher.

a) Determine the number of points in the sample space.

b) Construct a tree diagram and determine the sample space.

Find the probability that they select

c) General Electric for all three appliances.

d) no General Electric appliances.

e) at least one General Electric appliance.

26. *Summer School* You decide to take three courses during summer school: an English course, a mathematics course, and a science course. The available courses that you can take are illustrated below.

English	Mathematics	Science
English composition	College algebra	Biology
English literature	Statistics	Geology
	Calculus	Chemistry
		Physics

a) Determine the number of points in the sample space.

b) Construct a tree diagram and determine the sample space.

Find the probability that

c) geology is selected.

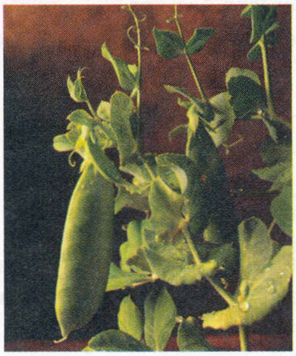
d) either geology or chemistry is selected.

e) calculus is not selected.

27. *Personal Characteristics* An individual can be classified as male or female with red, brown, black, or blonde hair and with brown, blue, or green eyes.

a) How many different classifications are possible (for example, male, red-headed, blue-eyed)?
b) Construct a tree diagram to determine the sample space.
c) If each outcome is equally likely, find the probability that the individual will be a male with black hair and blue eyes.
d) Find the probability that the individual will be a female with blonde hair.

28. *Mendel Revisited* A pea plant must have exactly one of each of the following pairs of traits: short (*s*) or tall (*t*); round (*r*) or wrinkled (*w*) seeds; yellow (*y*) or green (*g*) peas; and white (*wh*) or purple (*p*) flowers (for example, short, wrinkled, green pea with white flowers).



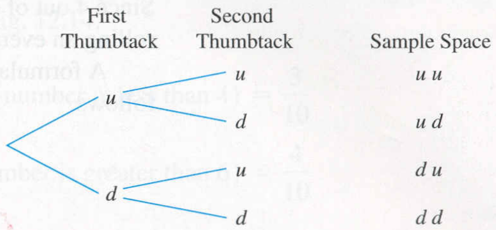
a) How many different classifications of pea plants are possible?
b) Use a tree diagram to determine the classifications possible.
c) If each characteristic is equally likely, find the probability that the pea plant will have round peas.
d) Find the probability that the pea plant will be short, have wrinkled seeds, have yellow seeds, and have purple flowers.

Challenge Problems/Group Activities

29. *Three Chips* Suppose that a bag contains one white chip and two red chips. Two chips are going to be selected at random from the bag *with replacement*.
a) What is the probability of selecting a white chip from the bag on the first selection?
b) What is the probability of selecting a red chip from the bag on the first selection?
c) Are the outcomes of selecting a white chip and selecting a red chip on the first selection equally likely? Explain.
d) The sample space when two chips are selected from the bag with replacement is ww, wr, rw, rr. Do you believe that the probability of selecting ww is greater than, equal to, or less than the probability of selecting rr? Explain.

30. A thumbtack is dropped on a concrete floor. Assume the thumbtack can only land point up (u) and point down (d), as shown in the figure below.

If two thumbtacks are dropped, one after the other, the tree diagram below can be used to show the possible outcomes.



a) Do you believe the outcomes of the thumbtack landing point up and the thumbtack landing point down are equally likely? Explain.
b) List the sample points in the sample space of this experiment.
c) Do you believe the probability that both thumbtacks land point up (uu) is the same as the probability that both thumbtacks land point down (dd)? Explain.
d) Can you compute the theoretical probability of the thumbtack landing point up and the theoretical probability of the thumbtack landing point down? Explain.
e) Obtain a box of thumbtacks and drop the thumbtacks out of the box with care. Determine the empirical probability of a thumbtack landing point up when dropped and the empirical probability of a thumbtack landing point down when dropped.

Recreational Exercise

31. *Ties* All my ties are red except two. All my ties are blue except two. All my ties are brown except two. How many ties do I have?

32. *Rock Faces* An experiment consists of 3 parts: flipping a coin, tossing a rock, and rolling a die. If the sample space consists of 60 sample points, determine the number of faces on the rock.

12.6 *OR* AND *AND* PROBLEMS

In Section 12.5, we showed how to work probability problems by constructing sample spaces. Often it is inconvenient or too time consuming to solve a problem by first constructing a sample space. For example, if an experiment consists of selecting two cards with replacement from a deck of 52 cards, there would be $52 \cdot 52$ or 2704 points in the sample space. Trying to list all these sample points could take hours. In this section, we learn how to solve **compound probability** problems that contain the words *and* or *or* without constructing a sample space.

Or Problems

The **or probability problem** requires obtaining a "successful" outcome for *at least one* of the given events. For example, suppose that we roll one die and we are interested in finding the probability of rolling an even number *or* a number greater than 4. For this situation rolling either a 2, 4, or 6 (an even number) or a 5 or 6 (a number greater than 4) would be considered successful. Note that the number 6 satisfies both criteria. Since 4 out of 6 of the numbers meet the criteria (the 2, 4, 5, and 6), the probability of rolling an even number *or* a number greater than 4 is $\frac{4}{6}$ or $\frac{2}{3}$.

A formula for finding the probability of event A or event B, symbolized $P(A \text{ or } B)$, follows.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Since we add (and subtract) probabilities to find $P(A \text{ or } B)$, this formula is sometimes referred to as the *addition formula*. We explain the use of the *or* formula in Example 1.

EXAMPLE 1 *Using the Addition Formula*

Each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 is written on a separate piece of paper. The 10 pieces of paper are then placed in a hat, and one piece is randomly selected. Find the probability that the piece of paper selected contains an even number or a number greater than 6.

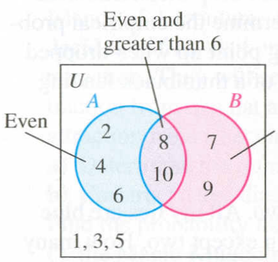


Figure 12.13

SOLUTION: We are asked to find the probability the number selected *is even* or *greater than 6*. Let's use set A to represent the statement, "the number is even," and set B to represent the statement, "the number is greater than 6." Figure 12.13 is a Venn diagram, as introduced in Chapter 2, with sets A (even) and B (greater than 6). There are a total of 10 numbers, of which five are even (2, 4, 6, 8, and 10). Thus, the probability of selecting an even number is $\frac{5}{10}$. Four numbers are greater than 6: the 7, 8, 9, and 10. Thus, the probability of selecting a number greater than 6 is $\frac{4}{10}$. Two numbers are both even and greater than 6: the 8 and 10. Thus, the probability of selecting a number that is both even and greater than 6 is $\frac{2}{10}$.

If we substitute the appropriate statements for A and B in the formula, we obtain

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P\left(\begin{matrix}\text{even or}\\ \text{greater than 6}\end{matrix}\right) = P(\text{even}) + P\left(\begin{matrix}\text{greater}\\ \text{than 6}\end{matrix}\right) - P\left(\begin{matrix}\text{even and}\\ \text{greater than 6}\end{matrix}\right)$$

$$= \frac{5}{10} + \frac{4}{10} - \frac{2}{10}$$

$$= \frac{7}{10}$$

Thus, the probability of selecting an even number or a number greater than 6 is $\frac{7}{10}$. ▲

Example 1 illustrates that when finding the probability of A or B, we add the probabilities of events A and B and then subtract the probability of both events occurring simultaneously.

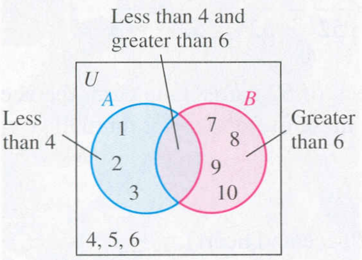


Figure 12.14

EXAMPLE 2 *Using the Addition Formula*

Consider the same sample space, the numbers 1 through 10, as in Example 1. If one piece of paper is selected, find the probability that it contains a number less than 4 or a number greater than 6.

SOLUTION: Let A represent the statement, "the number is less than 4," and let B represent the statement, "the number is greater than 6." A Venn diagram illustrating these statements is shown in Fig. 12.14.

$$P(\text{number is less than } 4) = \frac{3}{10}$$

$$P(\text{number is greater than } 6) = \frac{4}{10}$$

Since there are no numbers that are *both* less than 4 and greater than 6, $P(\text{number is less than 4 and greater than 6}) = 0$. Therefore,

$$P\begin{pmatrix}\text{number is}\\ \text{less than 4}\\ \text{or greater}\\ \text{than 6}\end{pmatrix} = P\begin{pmatrix}\text{number is}\\ \text{less than 4}\end{pmatrix} + P\begin{pmatrix}\text{number is}\\ \text{greater than 6}\end{pmatrix} - P\begin{pmatrix}\text{number is}\\ \text{less than 4}\\ \text{and greater}\\ \text{than 6}\end{pmatrix}$$

$$= \frac{3}{10} + \frac{4}{10} - 0 = \frac{7}{10}$$

Thus, the probability of selecting a number less than 4 or greater than 6 is $\frac{7}{10}$. ▲

In Example 2, it is impossible to select a number that is both less than 4 *and* greater than 6 when only one number is to be selected. Events such as these are said to be *mutually exclusive*.

Two events A and B are **mutually exclusive** if it is impossible for both events to occur simultaneously.

If events A and B are mutually exclusive, then $P(A \text{ and } B) = 0$, and the addition formula simplifies to $P(A \text{ or } B) = P(A) + P(B)$.

EXAMPLE 3 *Probability of A or B*

One card is selected from a standard deck of playing cards. Determine whether the following pairs of events are mutually exclusive, and find $P(A \text{ or } B)$.

a) A = an ace, B = a king

b) A = an ace, B = a heart

c) A = a red card, B = a black card

d) A = a picture card, B = a red card

The ace of hearts is both an ace and a heart.

SOLUTION:

a) There are four aces and four kings in a standard deck of 52 cards. It is impossible to select both an ace and a king when only one card is selected. Therefore, these events are mutually exclusive.

$$P(\text{ace or king}) = P(\text{ace}) + P(\text{king}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$$

b) There are 4 aces and 13 hearts in a standard deck of 52 cards. One card, the ace of hearts, is both an ace and a heart. Therefore, these events are not mutually exclusive.

$$P(\text{ace}) = \frac{4}{52} \qquad P(\text{heart}) = \frac{13}{52} \qquad P(\text{ace and heart}) = \frac{1}{52}$$

$$\begin{aligned} P(\text{ace or heart}) &= P(\text{ace}) + P(\text{heart}) - P(\text{ace and heart}) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \frac{16}{52} = \frac{4}{13} \end{aligned}$$

c) There are 26 red cards and 26 black cards in a standard deck of 52 cards. It is impossible to select one card that is both a red card and a black card. Therefore, the events are mutually exclusive.

$$\begin{aligned} P(\text{red or black}) &= P(\text{red}) + P(\text{black}) \\ &= \frac{26}{52} + \frac{26}{52} = \frac{52}{52} = 1 \end{aligned}$$

Therefore, a red card or a black card must be selected.

d) There are 12 picture cards in a standard deck of 52 cards. Six of the 12 picture cards are red (the jacks, queens, and kings of hearts and diamonds). Thus, selecting a picture card and a red card are not mutually exclusive.

$$\begin{aligned} P\begin{pmatrix}\text{picture card}\\ \text{or red card}\end{pmatrix} &= P\begin{pmatrix}\text{picture}\\ \text{card}\end{pmatrix} + P\begin{pmatrix}\text{red}\\ \text{card}\end{pmatrix} - P\begin{pmatrix}\text{picture card}\\ \text{and red card}\end{pmatrix} \\ &= \frac{12}{52} + \frac{26}{52} - \frac{6}{52} \\ &= \frac{32}{52} = \frac{8}{13} \end{aligned}$$

And Problems

A second type of probability problem is the ***and* probability problem**, which requires obtaining a favorable outcome in *each* of the given events. For example, suppose that *two* cards are to be selected from a deck of cards and we are interested in the probability of selecting two aces (one ace *and* then a second ace). Only if *both* cards selected are aces would this experiment be considered successful. A formula for finding the probability of events A and B, symbolized $P(A \text{ and } B)$, follows.

DID YOU KNOW

They're All Girls

The probability that a husband and wife have nine children where all are girls is $(\frac{1}{2})^9$ or $\frac{1}{512}$. The odds against a family having nine girls in a row is 511 to 1. Thus, for every 512 families with nine children, on average, one family will have all girls and 511 will not. However, should another child be born to that one family, the probability of the tenth child being a girl is still $\frac{1}{2}$.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Since we multiply to find $P(A$ and $B)$, this formula is sometimes referred to as the *multiplication formula*. When using the multiplication formula, we **always assume that event *A* has occurred when calculating *P*(*B*)** because we are determining the probability of obtaining a favorable outcome in both of the given events.*

Unless we specify otherwise, $P(A$ and $B)$ indicates that we are determining the probability that event *A* occurs *and then* event *B* occurs (in that order). Consider a bag that contains three chips: 1 red chip, 1 blue chip, and 1 green chip. Suppose that two chips are selected from the bag with replacement. The tree diagram and sample space for the experiment are shown in Fig. 12.15. There are nine possible outcomes for the two selections, as indicated in the sample space.

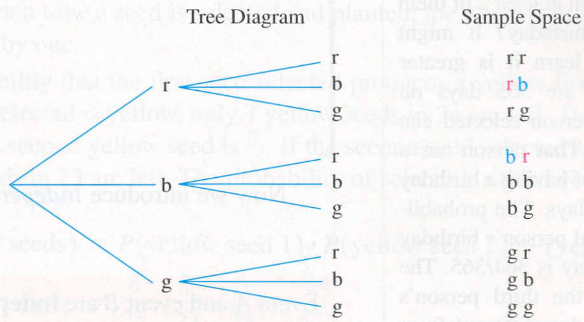


Figure 12.15

Notice that the probability of selecting a red chip followed by a blue chip (rb), indicated by P(red and blue), is $\frac{1}{9}$. The probability of selecting a red chip and a blue chip, in any order (rb or br), is $\frac{2}{9}$. In this section, when we ask for $P(A$ and $B)$, it means the probability of event *A* occurring *and then* event *B* occurring, in that order.

EXAMPLE 4 *An Experiment with Replacement*

Two cards are to be selected *with replacement* from a deck of cards. Find the probability that two aces will be selected.

SOLUTION: Since the deck of 52 cards contains four aces, the probability of selecting an ace on the first draw is $\frac{4}{52}$. The card selected is then returned to the deck. Therefore, the probability of selecting an ace on the second draw remains $\frac{4}{52}$.

If we let *A* represent the selection of the first ace and *B* represent the selection of the second ace, the formula may be written

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

$$P(2 \text{ aces}) = P(\text{ace 1 } \textit{and} \text{ ace 2}) = P(\text{ace 1}) \cdot P(\text{ace 2})$$

$$= \frac{4}{52} \cdot \frac{4}{52}$$

$$= \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

*$P(B)$, assuming that event *A* has occurred, may be denoted $P(B \mid A)$, which is read "the probability of *B*, given *A*." We will discuss this type of probability (conditional probability) further in Section 12.7.

DID YOU KNOW

The Birthday Problem

Among 24 people chosen at random, what would you guess is the probability that at least 2 of them have the same birthday? It might surprise you to learn it is greater than 0.5. There are 365 days on which the first person selected can have a birthday. That person has a 365/365 chance of having a birthday on one of those days. The probability that the second person's birthday is on any other day is 364/365. The probability that the third person's birthday is on a day different from the first two is 363/365, and so on. The probability that the 24th person has a birthday on any other day than the first 23 people is 342/365. Thus, the probability, P, that of 24 people, no 2 have the same birthday is $(365/365) \times (364/365) \times (363/365) \times \cdots \times (342/365) = 0.462$. Then the probability of at least 2 people of 24 having the same birthday is $1 - P = 1 - 0.462 = 0.538$, or slightly larger than $\frac{1}{2}$.

EXAMPLE 5 *An Experiment without Replacement*

Two cards are to be selected *without replacement* from a deck of cards. Find the probability that two aces will be selected.

SOLUTION: This example is similar to Example 4. However, this time we are doing the experiment without replacing the first card selected to the deck before selecting the second card.

The probability of selecting an ace on the first draw is $\frac{4}{52}$. When calculating the probability of selecting the second ace, we must assume that the first ace has been selected. Once this first ace has been selected, only 51 cards, including 3 aces, remain in the deck. The probability of selecting an ace on the second draw becomes $\frac{3}{51}$. The probability of selecting two aces without replacement is

$$P(2\text{ aces}) = P(\text{ace }1) \cdot P(\text{ace }2)$$
$$= \frac{4}{52} \cdot \frac{3}{51}$$
$$= \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$$

Now we introduce *independent events.*

Event A and event B are **independent events** if the occurrence of either event in no way affects the probability of occurrence of the other event.

Rolling dice and tossing coins are examples of independent events. In Example 4, the events are independent since the first card was returned to the deck. The probability of selecting an ace on the second draw was not affected by the first selection. The events in Example 5 are not independent since the probability of the selection of the second ace was affected by removing the first ace selected from the deck. Such events are called *dependent events. Experiments done with replacement will result in independent events, and those done without replacement will result in dependent events.*

EXAMPLE 6 *Independent or Dependent Events?*

One hundred people attended a dinner to recognize the accomplishments of a local citizen. Three people in attendance will be selected at random without replacement, and each will be awarded one door prize. Are the events of selecting the three people who will be awarded the door prize independent or dependent events?

SOLUTION: The events are dependent since each time one person is selected, it changes the probability of the next person being selected. In the first selection, the probability that a specific individual is selected is $\frac{1}{100}$. If that person is not selected first, the probability that the specific person is selected second changes to $\frac{1}{99}$. In general, in any experiment in which two or more items are selected *without replacement*, the events will be dependent.

The multiplication formula may be extended to more than two events, as illustrated in Example 7.

EXAMPLE 7 *Flower Probabilities*

A package of 25 zinnia seeds contain 12 seeds for red flowers, 5 seeds for white flowers, and 8 seeds for yellow flowers. Three seeds are randomly selected and planted. Find the probability of each of the following.

a) All three seeds will produce yellow flowers.

b) The first seed selected will produce a yellow flower, the second seed will produce a red flower, and the third seed will produce a yellow flower.

c) None of the seeds will produce yellow flowers.

d) At least one will produce yellow flowers.

SOLUTION: Each time a seed is selected and planted, the number of seeds remaining decreases by one.

a) The probability that the first seed selected produces a yellow flower is $\frac{8}{25}$. If the first seed selected is yellow, only 7 yellow seeds in 24 are left. The probability of selecting a second yellow seed is $\frac{7}{24}$. If the second seed selected is yellow, only 6 yellow seeds in 23 are left. The probability of selecting a third yellow seed is $\frac{6}{23}$.

$$P(3 \text{ yellow seeds}) = P(\text{yellow seed 1}) \cdot P(\text{yellow seed 2}) \cdot P(\text{yellow seed 3})$$
$$= \frac{8}{25} \cdot \frac{7}{24} \cdot \frac{6}{23} = \frac{14}{575}$$

b) The probability that the first seed selected produces a yellow flower is $\frac{8}{25}$. Once a seed for a yellow flower is selected, only 24 seeds are left. Twelve of the remaining 24 will produce red flowers. Thus, the probability that the second seed selected will produce a red flower is $\frac{12}{24}$. After the second seed has been selected, there are 23 seeds left, 7 of which will produce yellow flowers. The probability that the third seed produces a yellow flower is therefore $\frac{7}{23}$.

$$P\left(\begin{array}{c}\text{first yellow and second}\\ \text{red and third yellow}\end{array}\right) = P\left(\begin{array}{c}\text{first}\\ \text{yellow}\end{array}\right) \cdot P\left(\begin{array}{c}\text{second}\\ \text{red}\end{array}\right) \cdot P\left(\begin{array}{c}\text{third}\\ \text{yellow}\end{array}\right)$$
$$= \frac{8}{25} \cdot \frac{12}{24} \cdot \frac{7}{23} = \frac{28}{575}$$

c) If no flowers are to be yellow, they must either be red or white. Seventeen seeds will not produce yellow flowers (12 for red and 5 for white). The probability that the first seed does not produce a yellow flower is $\frac{17}{25}$. After the first seed has been selected, 16 of the remaining 24 seeds will not produce yellow flowers. After the second seed has been selected, 15 of the remaining 23 seeds will not produce yellow flowers.

$$P(\text{none yellow}) = P\left(\begin{array}{c}\text{first not}\\ \text{yellow}\end{array}\right) \cdot P\left(\begin{array}{c}\text{second not}\\ \text{yellow}\end{array}\right) \cdot P\left(\begin{array}{c}\text{third not}\\ \text{yellow}\end{array}\right)$$
$$= \frac{17}{25} \cdot \frac{16}{24} \cdot \frac{15}{23} = \frac{34}{115}$$

d) In Section 12.5, we learned that

$$P(\text{event happening at least once}) = 1 - P(\text{event does not happen})$$

In part (c), we found that the probability of selecting no yellow flowers is $\frac{34}{115}$. Therefore, the probability that at least one of the seeds will produce yellow flowers can be found as follows:

$$P(\text{at least one yellow flower}) = 1 - P(\text{no yellow flowers})$$
$$= 1 - \frac{34}{115} = \frac{115}{115} - \frac{34}{115} = \frac{81}{115}$$

▲

TIMELY TIP

Which formula to use

It is sometimes difficult to determine when to use the *or* formula and when to use the *and* formula. The following information may be helpful in deciding which formula to use.

***Or* formula**

Or problems will almost always contain the word *or* in the statement of the problem. For example, find the probability of selecting a heart *or* a 6. *Or* problems in this book generally involve only *one* selection. For example, "one card is selected" or "one die is rolled."

***And* formula**

And problems often do *not* use the word *and* in the statement of the problem. For example, "find the probability that both cards selected are red" or "find the probability that none of those selected is a banana" are both *and*-type problems. *And* problems in this book will generally involve *more than one* selection. For example, the problem may read "two cards are selected" or "three coins are flipped."

DID YOU KNOW

Slot Machines

You pull the handle, or push the button, to activate the slot machine. You are hoping that each of the three reels stops on Jackpot. The first reel shows Jackpot, the second reel shows Jackpot, and you hold your breath hoping the third reel shows Jackpot. When the third reel comes to rest, though, the Jackpot appears either one row above or one row below the row containing the other two Jackpots. You say to yourself, "I came so close" and try again. In reality, however, you probably did not come close to winning the Jackpot at all. The instant you pull the slot machine's arm, or push the button, the outcome is decided by a computer inside the slot machine. The computer uses stop motors to turn each reel and stop it at predetermined points. A random number generator within the computer is used to determine the outcomes. Although each reel generally has 22 stops, each stop is not equally likely. The computer assigns only a few random numbers to the stop that places the Jackpots on all three reels at the same time and many more random numbers to stop the reels on stops that are not the Jackpot. If you do happen to hit the Jackpot, it is simply because the random number generator happened to generate the right sequence of numbers the instant you activated the machine. For more detailed information check *www.howstuffworks.com*. Also see the Did You Know on page 714.

SECTION 12.6 EXERCISES

Concept/Writing Exercises

1. a) In $P(A \text{ or } B)$, what does the word *or* indicate?
   b) In $P(A \text{ and } B)$, what does the word *and* indicate?
2. a) Give the formula for $P(A \text{ or } B)$.
   b) In your own words, explain how to determine $P(A \text{ or } B)$ with the formula.
3. a) What are mutually exclusive events? Give an example.
   b) How do you calculate $P(A \text{ or } B)$ when A and B are mutually exclusive?
4. a) Give the formula for $P(A \text{ and } B)$.
   b) In your own words, explain how to determine $P(A \text{ and } B)$ with the formula.
5. When finding $P(B)$ using the formula $P(A \text{ and } B)$, what do we always assume?
6. What are independent events? Give an example.
7. What are dependent events? Give an example.
8. A family is selected at random. Let event A be the mother likes classical music. Let event B be the daughter likes classical music.
   a) Are events A and B mutually exclusive? Explain.
   b) Are they independent events? Explain.
9. A family is selected at random. Let event A be the father likes to swim. Let event B be the mother likes chocolate chip cookies.
   a) Are events A and B mutually exclusive? Explain.
   b) Are they independent events? Explain.
10. An individual is selected at random. Let event A be the individual owns a computer. Let event B be the individual owns a digital camera.
    a) Are events A and B mutually exclusive? Explain.
    b) Are they independent events? Explain.
11. If events A and B are mutually exclusive, explain why the formula $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ can be simplified to $P(A \text{ or } B) = P(A) + P(B)$.
12. a) Write a problem that you would use the *or formula* to solve. Solve the problem and give the answer.
    b) Write a problem that you would use the *and formula* to solve. Solve the problem and give the answer.

Practice the Skills

In Exercises 13–16, find the indicated probability.

13. If $P(A) = 0.6$, $P(B) = 0.4$, and $P(A \text{ and } B) = 0.3$, find $P(A \text{ or } B)$.
14. If $P(A \text{ or } B) = 0.9$, $P(A) = 0.5$, and $P(B) = 0.6$, find $P(A \text{ and } B)$.
15. If $P(A \text{ or } B) = 0.8$, $P(A) = 0.4$, and $P(A \text{ and } B) = 0.1$, find $P(B)$.
16. If $P(A \text{ or } B) = 0.6$, $P(B) = 0.3$, and $P(A \text{ and } B) = 0.1$, find $P(A)$.

Roll a Die *In Exercises 17–20, a single die is rolled one time. Find the probability of rolling*

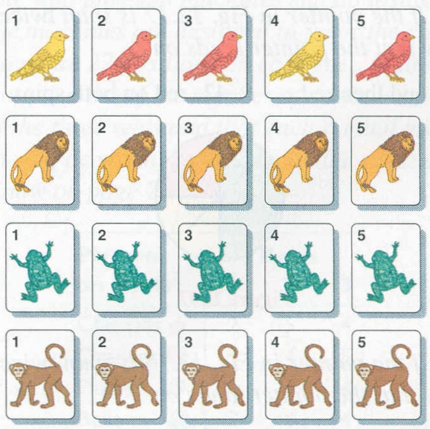
17. a 2 or 5.
18. an odd number or a number greater than 2.
19. a number greater than 4 or less than 2.
20. a number greater than 3 or less than 5.

Select One Card *In Exercises 21–26, one card is selected from a deck of playing cards. Find the probability of selecting*

21. an ace or a 2.
22. a jack or a diamond.
23. a picture card or a red card.
24. a club or a red card.
25. a card less than 7 or a club. (*Note:* The ace is considered a low card.)
26. a card greater than 9 or a black card.

Problem Solving

Select Two Cards *In Exercises 27–34, a board game uses the deck of 20 cards shown.*



Two cards are selected at random from this deck. Find the probability of the following
   a) *with replacement.*
   b) *without replacement.*

27. They both show frogs.
28. They both show the number 3.
29. The first shows a lion, and the second shows a bird.

30. The first shows a 2, and the second shows a 4.

31. The first shows a red bird, and the second shows a monkey.

32. They both show even numbers.

33. Neither shows an even number.

34. The first shows a lion, and the second shows a red bird.

Select One Card *In the deck of cards used in Exercises 27–34, if one card is drawn, find the probability that the card shows*

35. a monkey or an even number.

36. a yellow bird or a number greater than 4.

37. a lion or a 2.

38. a red bird or an even number.

Two Spins *In Exercises 39–48, assume that the pointer cannot land on the line and that each spin is independent.*

If the pointer in Fig. 12.16 is spun twice, find the probability that the pointer lands on

39. red on both spins.

40. red and then yellow.

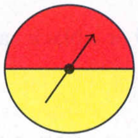


Figure 12.16

Two Spins *If the pointer in Fig. 12.17 is spun twice, find the probability that the pointer lands on*

41. green and then red.

42. red on both spins.

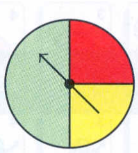


Figure 12.17

Two Spins *If the pointer in Fig. 12.18 is spun twice, find the probability that the pointer lands on*

43. red on both spins.

44. a color other than yellow on both spins.

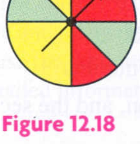


Figure 12.18

Two Spins *In Exercises 45–48, assume that the pointer in Fig. 12.16 is spun and then the pointer in Fig. 12.17 is spun. Find the probability of the pointers landing on*

45. red on both spins.

46. red on the first spin and yellow on the second spin.

47. a color other than yellow on both spins.

48. yellow on the first spin and a color other than yellow on the second spin.

Having a Family *In Exercises 49–52, a couple has three children. Assuming independence and the probability of a boy is $\frac{1}{2}$, find the probability that*

49. all three children are girls.

50. all three children are boys.

51. the youngest child is a boy and the older children are girls.

52. the youngest child is a girl, the middle child is a boy, and the oldest child is a girl.

53. a) **Five Children** The Martinos plan to have four children. Find the probability that all their children will be boys. (Assume that $P(\text{boy}) = \frac{1}{2}$, and assume independence.)

b) If their first four children are boys and Mrs. Martino is expecting another child, what is the probability that the fifth child will be a boy?

54. a) **The Probability of a Girl** The Bronsons plan to have seven children. Find the probability that all their children will be girls. (Assume that $P(\text{girl}) = \frac{1}{2}$, and assume independence.)

b) If their first seven children are girls and Mrs. Bronson is expecting another child, what is the probability that the eighth child will be a girl?

Golf Balls *Angel Sanchez has seven golf balls in one pocket of his golf bag: 4 Titleist balls, 2 Top Flite balls, and 1 Pinnacle ball. In Exercises 55–58, two balls will be selected at random. Find the probability of selecting each of the following*

a) *with replacement.*

b) *without replacement.*

55. a Titleist ball and then a Pinnacle ball

56. no Top Flite balls

57. at least one Top Flite ball

58. two Pinnacle balls

Health Insurance *A sample of 40 people yielded the following information about their health insurance.*

Number of People	Type of Insurance
22	Managed care plan
14	Traditional insurance
4	No insurance

Two people who provided information for the table were selected at random, without replacement. Find the probability that

59. neither had traditional insurance.

60. they both had a managed care plan.

61. at least one had traditional insurance.

62. the first had traditional insurance and the second had a managed care plan.

Home Builder *A sample of 30 women who recently had a home built yielded the following information about their builder.*

Number of Women	Would You Recommend Your Home Builder to a Friend
19	Yes
6	No
5	Not sure

Three women who provided information for the table were selected at random. Find the probability that

63. they would all recommend their home builder.

64. the first would not recommend her home builder but the second and third would recommend their home builder.

65. the first two would not recommend their home builder and the third is not sure if she would recommend her home builder.

66. the first would recommend her home builder but the second and third would not recommend their home builder.

An Experimental Drug *In Exercises 67–70, an experimental drug was given to a sample of 100 hospital patients with an unknown sickness. Of the total, 70 patients reacted favorably, 10 reacted unfavorably, and 20 were unaffected by the drug. Assume that this sample is representative of the entire population. If this drug is given to Mr. and Mrs. Rivera and their son Carlos, what is the probability of each of the following? (Assume independence.)*

67. Mrs. Rivera reacts favorably.

68. Mr. and Mrs. Rivera react favorably, and Carlos is unaffected.

69. All three react favorably.

70. None reacts favorably.

Multiple-Choice Exam *In Exercises 71–76, each question of a five-question multiple-choice exam has four possible answers. Gurshawn Salk picks an answer at random for each question. Find the probability that he selects the correct answer on*

71. any one question.

72. only the first question.

73. only the third and fourth questions.

74. all five questions.

75. none of the questions.

76. at least one of the questions.

A Slot Machine *In Exercises 77–80, consider a slot machine.*

Most people who play slot machines end up losing money because the machines are designed to favor the casino (the house). There are 22 positions on each reel. Assume that the following is a list of the number of symbols of each type on each of the three reels and that each symbol has the same chance of occurring (which is not the case; see the Did You Know on page 702).

Pictures on Reels	Reels 1	2	3
Cherries	2	5	4
Oranges	5	4	5
Plums	6	4	4
Bells	3	4	4
Melons	3	2	3
Bars	2	2	1
7s	1	1	1

For this slot machine, assuming that the wheels are independent, find the probability of obtaining

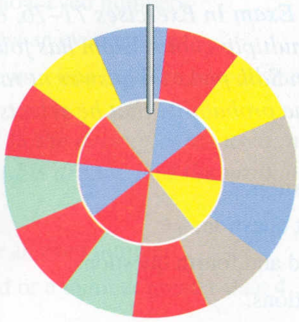
77. a bell on the first reel.

78. oranges on all three reels.

79. no bars.

80. three 7's.

Two Wheels *In Exercises 81–84, the following double wheel is spun.*



Assuming that the wheels are independent and each outcome is equally likely, find the probability of obtaining

81. yellow on both wheels.

82. red on the outer wheel and blue on the inner wheel.

83. red on neither wheel.

84. red on at least one wheel.

Hitting a Target *In Exercises 85–88, the probability that a heat-seeking torpedo will hit its target is 0.4. If the first torpedo hits its target, the probability that the second torpedo will hit the target increases to 0.9 because of the extra heat generated by the first explosion. If two heat-seeking torpedoes are fired at a target, find the probability that*

85. neither hits the target.

86. the first hits the target and the second misses the target.

87. both hit the target.

88. the first misses the target and the second hits the target.

89. ***Polygenetic Afflictions*** Certain birth defects and syndromes are *polygenetic* in nature. Typically, the chance that an offspring will be born with a polygenetic affliction is small. However, once an offspring is born with the affliction, the probability that future offspring of the same parents will be born with the same affliction increases. Let's assume that the probability of a child being born with affliction A is 0.001. If a child is born with this affliction, the probability of a future child being born with the same affliction becomes 0.04.

a) Are the events of the births of two children in the same family with affliction A independent? Explain.

b) A couple plans to have one child. Find the probability that the child will be born with this affliction.

A couple plans to have two children. Use the information provided to determine the probability that

c) both children will be born with the affliction.

d) the first will be born with the affliction and the second will not.

e) the first will not be born with the affliction and the second will.

f) neither will be born with the affliction.

90. ***Lottery Ticket*** In a bin are an equal number of balls marked with the digits 0, 1, 2, 3, . . . , 9. Three balls are to be selected from the bin, one after the other, at random with replacement to make the winning three digit lottery number. Ms. Jones has a lottery ticket with a three-digit number in the range 000 to 999. Find the probability that Ms. Jones's number is the winning number.

Chance of an Audit *In Exercises 91–94, assume that 32 in every 1000 people in the \$14,000–\$56,800 income bracket are audited yearly. Assuming that the returns to be audited are selected at random and that each year's selections are independent of the previous year's selections, find the probability that a person in this income bracket will be audited*

91. this year.

92. the next two years in succession.

93. this year but not next year.

94. neither this year nor next year.

Challenge Problems/Group Activities

95. ***Picking Chips*** A bag contains five red chips, three blue chips, and two yellow chips. Two chips are selected from the bag without replacement. Find the probability that two chips of the same color are selected.

96. ***Ten Yen Coins*** Ron has ten coins from Japan: three 1-yen coins, one 10-yen coin, two 20-yen coins, one 50-yen coin, and three 100-yen coins. He selects two coins at random without replacement. Assuming that each coin is equally likely to be selected, find the probability that Ron selects at least one 1-yen coin.

97. ***A Fair Game?*** Two playing cards are dealt to you from a well-shuffled standard deck of 52 cards. If either card is a diamond or if both are diamonds, you win; otherwise, you lose. Determine whether this game favors you, is fair, or favors the dealer. Explain your answer.

98. ***Picture Card Probability*** You have three cards: an ace, a king, and a queen. A friend shuffles the cards, selects two of them at random, and discards the third. You ask your friend to show you a picture card, and she turns over the king. What is the probability that she also has the queen?

Recreational Exercises

A Different Die *For Exercises 99–102, consider a die that has 1 dot on one side, 2 dots on two sides, and 3 dots on three sides.*

If the die is rolled twice, find the probability of rolling

99. two 2's. **100.** two 3's.

If the die is rolled only once, determine the probability of rolling

101. an even number or a number less than 3.

102. an odd number or a number greater than 1.

Internet/Research Activity

103. Girolamo Cardano (1501–1576) wrote *Liber de Ludo Aleae,* which is considered to be the first book on probability. Cardano had a number of different vocations. Do research and write a paper on the life and accomplishments of Girolamo Cardano.

12.7 CONDITIONAL PROBABILITY

In Section 12.6, we indicated that events are independent when the outcome of either event has no effect on the outcome of the other event. For example, selecting two cards from the deck of cards *with replacement* represents independent events. However, not all events with two selections are independent events. Consider this problem: Find the probability of selecting two aces from a deck of cards *without replacement.* The probability of selecting the first ace is $\frac{4}{52}$. The probability that the second ace is selected becomes $\frac{3}{51}$, since we assume that an ace was removed from the deck with the first selection. Since the probability of the second ace being selected is affected by the first ace being selected, these two events are *dependent.* Probability problems involving dependent events can be solved by using conditional probability.

Conditional Probability

In general, the probability of event E_2 occurring, given that an event E_1 has happened (or will happen; the time relationship does not matter) is called a **conditional probability** and is written $P(E_2 \mid E_1)$.

The symbol $P(E_2 \mid E_1)$, read "the probability of E_2, given E_1," represents the probability of E_2 occurring, assuming that E_1 has already occurred (or will occur).

DID YOU KNOW

Dice-Y Music

During the eighteenth century, composers were fascinated with the idea of creating compositions by rolling dice. A musical piece attributed by some to Wolfgang Amadeus Mozart, titled "Musical Dice Game," consists of 176 numbered musical fragments of three-quarter beats each. Two charts show the performer how to order the fragments given a particular roll of two dice. Modern composers can create music by using the same random technique and a computer; as in Mozart's time, however, the music created by random selection of notes or bars is still considered by musicologists to be aesthetically inferior to the work of a creative composer.

EXAMPLE 1 *Using Conditional Probability*

A single card is selected from a deck of cards. Find the probability it is a heart, given that it is red.

SOLUTION: We are told that the card is red. Thus, only 26 cards are possible, of which 13 are hearts. Therefore,

$$P(\text{heart} \mid \text{red}) \text{ or } P(\text{H} \mid \text{R}) = \frac{13}{26} = \frac{1}{2}$$

EXAMPLE 2 *Girls in a Family*

Given a family with two children, and assuming that boys and girls are equally likely, find the probability that the family has

a) two girls.

b) two girls if you know that at least one of the children is a girl.

c) two girls given that the older child is a girl.

DID YOU KNOW

Chance of Showers

At one time or another, you probably have been caught in a downpour on a day the weather forecaster had predicted sunny skies. Short-term (24-hour) weather forecasts are correct nearly 85% of the time, a level of accuracy achieved through the use of conditional probability. Computer models are used to analyze data taken on the ground and in the air and make predictions of atmospheric pressures at some future time, say 10 minutes ahead. Based on these predicted conditions, another forecast is then computed. This process is repeated until a weather map has been generated for the next 12, 24, 36, and 48 hours. Since each new prediction relies on the previous prediction being correct, the margin of error increases as the forecast extends farther into the future.

SOLUTION:

a) To find the probability that the family has two girls we can determine the sample space of a family with two children. Then, from the sample space we can determine the probability that both children are girls. The sample space of two children can be determined by a tree diagram (see Fig. 12.19).

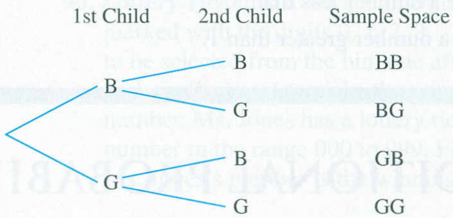


Figure 12.19

There are four possible equally likely outcomes, BB, BG, GB, and GG. Only one of the outcomes has two girls, GG. Thus,

$$P(\text{2 girls}) = \frac{1}{4}$$

b) We are given that at least one of the children is a girl. Therefore, for this problem the sample space is BG, GB, GG. Since there are three possibilities, of which only one has two girls, GG,

$$P(\text{both girls} \mid \text{at least one is a girl}) = \frac{1}{3}$$

c) If the older child is a girl, the sample space reduces to GB, GG. Thus,

$$P(\text{both girls} \mid \text{older child is a girl}) = \frac{1}{2}$$

There are a number of formulas that can be used to find conditional probabilities. The one we will use follows.

> **Conditional Probability**
>
> For any two events, E_1 and E_2,
>
> $$P(E_2 \mid E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)}$$

In the formula, $n(E_1 \text{ and } E_2)$ represents the number of sample points common to both event 1 and event 2, and $n(E_1)$ is the number of sample points in event E_1, the given event. Since the intersection of E_1 and E_2, symbolized $E_1 \cap E_2$, represents the sample points common to both E_1 and E_2, the formula can also be expressed as

$$P(E_2 \mid E_1) = \frac{n(E_1 \cap E_2)}{n(E_1)}$$

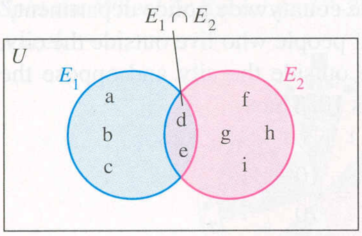


Figure 12.20

Figure 12.20 is helpful in explaining conditional probability.

Here, the number of elements in E_1 is five, the number of elements in E_2 is six, and the number of elements in both E_1 and E_2, or $E_1 \cap E_2$, is two.

$$P(E_2 \mid E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)} = \frac{2}{5}$$

Thus, for this situation, the probability of selecting an element from E_2, given that the element is in E_1, is $\frac{2}{5}$.

EXAMPLE 3 *Using the Conditional Probability Formula*

Each person in a sample of 200 residents in Lincoln County was asked whether he or she favored having a single countywide police department. The county consists of one large city and a number of small townships. The response of those sampled, with their place of residence specified, is given in the following table.

Residence	Favor	Oppose	Total
Live in city	80	50	130
Live outside city	60	10	70
Total	140	60	200

If one person from the sample is selected at random, find the probability that the person

a) favors a single countywide police department.

b) favors a single countywide police department, given that the person lives in the city.

c) opposes a single countywide police department, given that the person lives outside the city.

d) lives outside the city, given that the person opposes a single countywide police department.

SOLUTION:

a) The total number of respondents is 200, of which 140 favor a single countywide police department.

$$P(\text{favors a single countywide police department}) = \frac{140}{200} = \frac{7}{10}$$

b) We are given that the person lives in the city. Thus, this is a conditional probability problem. Let E_1 be the given information "the person lives in the city." Let E_2 be "the person favors a single countywide police department." We are being asked to find $P(E_2 \mid E_1)$. The number of people who live in the city, $n(E_1)$, is 130. The number of people who live in the city and favor a single countywide police department, $n(E_1 \text{ and } E_2)$, is 80. Thus,

$$P(E_2 \mid E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)} = \frac{80}{130} = \frac{8}{13}$$

c) We are given that the person lives outside the city. Thus, this is a conditional probability problem. Let E_1 be the given information "the person lives outside the city." Let E_2 be "the person opposes a single countywide police department." We are asked to find $P(E_2 \mid E_1)$. The number of people who live outside the city, $n(E_1)$, is 70. The number of people who live outside the city and oppose the countywide police department, $n(E_1 \text{ and } E_2)$, is 10. Thus,

$$P(E_2 \mid E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)} = \frac{10}{70} = \frac{1}{7}$$

d) We are given that the person opposes a single countywide police department. Thus, this is a conditional probability problem. Let E_1 be the given information "the person opposes a single countywide police department." Let E_2 be "the person lives outside the city." We are asked to find $P(E_2 \mid E_1)$. The number of people who oppose a single countywide police department, $n(E_1)$, is 60. The number of people who oppose a single countywide police department and live outside the city, $n(E_1 \text{ and } E_2)$, is 10. Thus,

$$P(E_2 \mid E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)} = \frac{10}{60} = \frac{1}{6}$$

In many of the examples, we used the words *given that.* Other words may be used instead. For example, in Example 3(b), the question could have been worded, "favors a single countywide police department *if* the person lives in the city."

SECTION 12.7 EXERCISES

Concept/Writing Exercises

1. What does the notation $P(E_2 \mid E_1)$ mean?
2. Give the formula for $P(E_2 \mid E_1)$.
3. If $n(E_1 \cap E_2) = 4$ and $n(E_1) = 12$, find $P(E_2 \mid E_1)$.
4. If $n(E_1 \cap E_2) = 5$ and $n(E_1) = 22$, find $P(E_2 \mid E_1)$.

Practice the Skills

Select a Circle *In Exercises 5–10, consider the circles shown.*

Assume that one circle is selected at random and each circle is equally likely to be selected. Find the probability of selecting

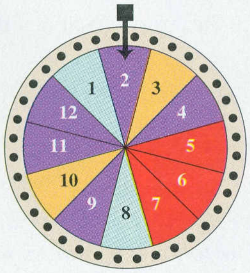
5. a 5, given that the circle is orange.
6. a 3, given that the circle is yellow.
7. an even number, given that the circle is not orange.
8. a number less than 2, given that the number is less than 5.
9. a red number, given that the circle is orange.
10. a number greater than 3, given that the circle is yellow.

Select a Number *In Exercises 11–16, consider the following figures.*

Assume that one number from 1 to 7 is equally likely to be selected at random. Each number corresponds to one of the seven figures shown. Find the probability of selecting

11. a circle, given that an odd number is selected.
12. a circle, given that a number greater than or equal to 5 is selected.
13. a red figure, given that an even number is selected.
14. a red or blue figure, given that an even number is selected.
15. a circle or square, given that a number less than 4 is selected.
16. a circle, given that an even number is selected.

Spin the Wheel *In Exercises 17–24, consider the following wheel.*



If the wheel is spun and each section is equally likely to stop under the pointer, find the probability that the pointer lands on

17. a five, given that the color is red.

18. an even number, given that the color is red.

19. purple, given that the number is odd.

20. a number greater than 4, given that the color is red.

21. a number greater than 4, given that the color is purple.

22. an even number, given that the color is red or purple.

23. gold, given that the number is greater than 5.

24. gold, given that the number is greater than 10.

Money from a Hat *In Exercises 25–28, assume that a hat contains four bills: a \$1 bill, a \$5 bill, a \$10 bill, and a \$20 bill. Two bills are to be selected at random with replacement. Construct a sample space as was done in Example 2 and find the probability that*

25. both bills are \$1 bills.

26. both bills are \$1 bills if the first selected is a \$1 bill.

27. both bills are \$5 bills if at least one of the bills is a \$5 bill.

28. both bills have a value greater than a \$5 bill if the second bill is a \$10 bill.

Two Dice *In Exercises 29–34, two dice are rolled one after the other. Construct a sample space and find the probability that the sum of the dots on the dice total*

29. 6.

30. 6 if the first die is a 1.

31. 6 if the first die is a 3.

32. an even number if the second die is a 2.

33. a number greater than 7 if the second die is a 5.

34. a 7 or 11 if the first die is a 5.

Problem Solving

Taste Test *In Exercises 35–40, use the following results of a taste test given at a local mall.*

	Prefers Coke	Prefers Pepsi	Total
Men	60	45	105
Women	50	62	112
Total	110	107	217

If one person from the sample is selected at random, find the probability the person selected

35. prefers Pepsi.

36. is a woman.

37. prefers Coke, given that a woman is selected.

38. prefers Pepsi, given that a man was selected.

39. is a man, given that the person prefers Coke.

40. is a woman, given that the person prefers Pepsi.

Lion or Elephant *Use the following information for Exercises 41–46. At a zoo, a sampling of children was asked if the zoo were to get one additional animal, would they prefer a lion or an elephant. The results of the survey follow.*

	Lion	Elephant	Total
Boys	90	110	200
Girls	75	85	160
Total	165	195	360

If one child who was in the survey is selected at random, find the probability that

41. the child is a girl.

42. the child selected the lion.

43. the child selected the elephant, given that the child was a boy.

44. the child selected the lion, given that the child was a girl.

45. the child selected is a boy, given that the child preferred the elephant.

46. the child selected is a girl, given that the child preferred the lion.

Videotapes and DVDs Use the following information for Exercises 47–52. At a Blockbuster video store, 300 people were surveyed to determine whether they rented DVDs, videotapes, or both. The results of the survey follow.

Age	Only DVDs	Only Videotapes	Both	Total
Under 30	60	39	21	120
30 or Older	64	94	22	180
Total	124	133	43	300

If one person from this survey is selected at random, determine the probability that the person

47. rented only videotapes.

48. was 30 or older.

49. rented only DVDs, given that the person was under 30.

50. rented both videotapes and DVDs, given that the person was 30 or older.

51. was under 30, given that the person rented both DVDs and videotapes.

52. was 30 or older, given that the person rented only videotapes.

Military Trials Use the following information concerning military trials for Exercises 53–58. Use the following data for the years indicated.

	Convictions	Acquittals	Total Cases
Air Force (1992–2001)	8166	667	8833
Army (1997–2001)	5024	434	5458
Navy/Marine Corps (1997–2001)	12,866	473	13,339
Total cases	26,056	1574	27,630

Source: Department of Defense.
Note: Statistics for the Navy and Marine Corps are maintained jointly.

Assuming that one person who was on trial was selected at random, determine the probability (as a decimal number rounded to four decimal places) that

53. the person was from the Air Force.

54. the person was acquitted.

55. the person was acquitted, given that the person was from the Army.

56. the person was convicted, given that the person was from the Navy/Marine Corps.

57. the person was in the Army, given that the person was convicted.

58. the person was in the Air Force, given that the person was acquitted.

Quality Control In Exercises 59–64, Sally Horsefall, a quality control inspector, is checking a sample of light bulbs for defects. The following table summarizes her findings.

Wattage	Good	Defective	Total
20	80	15	95
50	100	5	105
100	120	10	130
Total	300	30	330

If one of these light bulbs is selected at random, find the probability that the light bulb is

59. good.

60. good, given that it is 50 watts.

61. defective, given that it is 20 watts.

62. good, given that it is 100 watts.

63. good, given that it is 50 or 100 watts.

64. defective, given that it is not 50 watts.

News Survey In Exercises 65–70, 270 individuals are asked which evening news they watch most often. The results are summarized as follows.

Viewers	ABC	NBC	CBS	Other	Total
Men	30	20	40	55	145
Women	50	10	20	45	125
Total	80	30	60	100	270

If one of these individuals is selected at random, find the probability that the person watches

65. ABC or NBC.

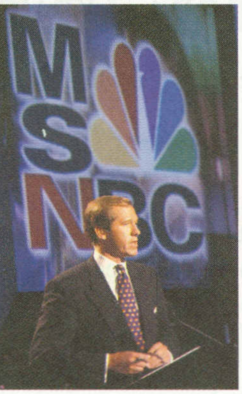
66. ABC, given that the individual is a woman.

67. ABC or NBC, given that the individual is a man.

68. a station other than CBS, given that the individual is a woman.

69. a station other than ABC, NBC, or CBS, given that the individual is a man.

70. NBC or CBS, given that the individual is a woman.



Brian Williams, MSNBC. See Exercises 65–70 on page 712.

Challenge Problems/Group Activities

Mutual Fund Holdings *Use the following information in Exercises 71–74. Mutual funds often hold many stocks. Each stock may be classified as a value stock, a growth stock, or a blend of the two. The stock may also be categorized by how large the company is. It may be classified as a large company stock, medium company stock, or small company stock. A selected mutual fund contains 200 stocks as illustrated in the following chart.*

	Value	Blend	Growth
Large	28	23	42
Medium	19	15	18
Small	26	12	17

Equity Investment Style

If one stock is selected at random from the mutual fund, find the probability that it is

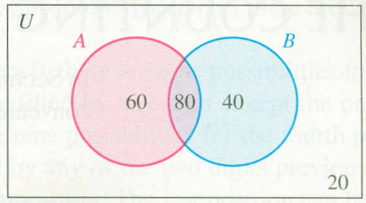
71. a large company stock.

72. a value stock.

73. a blend, given that it is a medium company stock.

74. a large company stock, given that it is a blend stock.

75. Consider the Venn diagram above and to the right. The numbers in the regions of the circle indicate the number of items that belong to that region. For example, 60 items are in set A but not in set B.



Find **a)** $n(A)$ **b)** $n(B)$ **c)** $P(A)$ **d)** $P(B)$

Use the formula on page 708 to find:

e) $P(A \mid B)$ **f)** $P(B \mid A)$

g) Explain why $P(A \mid B) \neq P(A) \cdot P(B)$.

76. A formula we gave for conditional probability is

$$P(E_2 \mid E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)}$$

This formula may be derived from the formula

$$P(E_2 \mid E_1) = \frac{P(E_1 \text{ and } E_2)}{P(E_1)}$$

Can you explain why? [*Hint:* Consider what happens to the denominators of $P(E_1 \text{ and } E_2)$ and $P(E_1)$ when they are expressed as fractions and the fractions are divided out.]

77. Given that $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \text{ and } B) = 0.12$, use the formula

$$P(E_2 \mid E_1) = \frac{P(E_1 \text{ and } E_2)}{P(E_1)}$$

to find

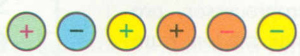
a) $P(A \mid B)$.

b) $P(B \mid A)$.

c) Are A and B independent? Explain.

Recreational Exercises

In Exercises 78–83, suppose that each circle is equally likely to be selected. One circle is selected at random.



Find the probability indicated.

78. P(green circle | + obtained)

79. P(+ | orange circle obtained)

80. P(yellow circle | − obtained)

81. P(green + | + obtained)

82. P(green or orange circle | green + obtained)

83. P(orange circle with green + | + obtained)

12.8 THE COUNTING PRINCIPLE AND PERMUTATIONS

In Section 12.5, we introduced the counting principle, which is repeated here for your convenience.

Counting Principle

If a first experiment can be performed in M distinct ways and a second experiment can be performed in N distinct ways, then the two experiments in that specific order can be performed in $M \cdot N$ distinct ways.

The counting principle is illustrated in Examples 1 and 2.

EXAMPLE 1 *Counting Principle: Passwords*

A password used to gain access to a computer account is to consist of two lowercase letters followed by four digits. Determine how many different passwords are possible if

a) repetition of letter and digits is permitted.
b) repetition of letters and digits is not permitted.
c) the first letter must be a vowel (*a*, *e*, *i*, *o*, *u*) and the first digit cannot be a 0, and repetition of letters and digits is not permitted.

SOLUTION: There are 26 letters and 10 digits (0–9). We have six positions to fill, as indicated.

$$\overline{L}\ \overline{L}\ \overline{D}\ \overline{D}\ \overline{D}\ \overline{D}$$

a) Since repetition is permitted, there are 26 possible choices for both the first and second positions. There are 10 possible choices for the third, fourth, fifth, and sixth positions.

$$\underset{L}{\underline{26}}\ \underset{L}{\underline{26}}\ \underset{D}{\underline{10}}\ \underset{D}{\underline{10}}\ \underset{D}{\underline{10}}\ \underset{D}{\underline{10}}$$

Since $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6{,}760{,}000$, there are 6,760,000 different possible arrangements.

b) There are 26 possibilities for the first position. Since repetition of letters is not permitted, there are only 25 possibilities for the second position. The same reasoning is used when determining the number of digits for positions 3 through 6.

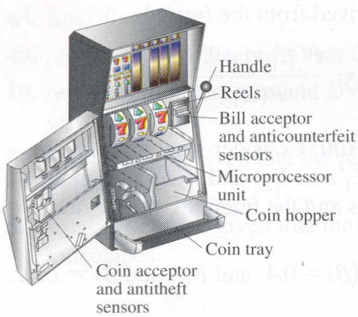
$$\underset{L}{\underline{26}}\ \underset{L}{\underline{25}}\ \underset{D}{\underline{10}}\ \underset{D}{\underline{9}}\ \underset{D}{\underline{8}}\ \underset{D}{\underline{7}}$$

Since $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3{,}276{,}000$, there are 3,276,000 different possible arrangements.

c) Since the first letter must be an *a*, *e*, *i*, *o*, or *u*, there are five possible choices for the first position. The second position can be filled by any of the letters except

DID YOU KNOW

Slot Machines



Source: Scientific American

Slot machines may qualify as the world's most user-friendly computer. Nowadays, slot machines are a conglomeration of microprocessors and memory chips. Each of three reels can stop at 22 positions (each one displaying various symbols or a blank space) for a total of 10,648 different arrangements ($22 \times 22 \times 22 = 10{,}648$). The heart of a contemporary slot machine is a microprocessor programmed to generate random numbers that can be assigned to any arrangement of the reels. In effect, the microprocessor generates random numbers that dictates the machine's display and payout. Because many or few random numbers can be assigned to any given arrangement, makers of slot machines can vary the odds as desired. For the top jackpot, the odds of obtaining the single correct arrangement may be set at 10 million to 1.

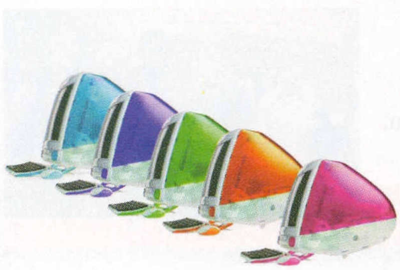
for the vowel selected for the first position. Therefore, there are 25 possibilities for the second position.

Since the first digit cannot be a 0, there are nine possibilities for the third position. The fourth position can be filled by any digit except the one selected for the third position. Thus, there are nine possibilities for the fourth position. Since the fifth position cannot be filled by any of the two digits previously used, there are eight possibilities for the fifth position. The last position can be filled by any of the seven remaining digits.

$$\frac{5}{L}\ \frac{25}{L}\ \frac{9}{D}\ \frac{9}{D}\ \frac{8}{D}\ \frac{7}{D}$$

Since $5 \cdot 25 \cdot 9 \cdot 9 \cdot 8 \cdot 7 = 567{,}000$, there are 567,000 different arrangements that meet the conditions specified. ▲

EXAMPLE 2 *Counting Principle: Computer Colors*



At Best Buy, they have just received a supply of Apple iMac computers. The computers come in the following colors: tangerine, strawberry, blueberry, grape, and lime. Sadie Bragg, the floor manager, decides to display one of each color computer.

a) In how many different ways can she display the five different color computers on a shelf?

b) If she wants to place the strawberry computer in the middle, in how many different ways can she arrange the computers?

c) If Sadie wants the blueberry computer to be the first computer and the strawberry computer to be the last computer, in how many different ways can she arrange the computers?

SOLUTION:

a) There are five positions to fill, using the five colors. In the first position, on the left, she can use any one of the five colors. In the second position, she can use any of the four remaining colors. In the third position, she can use any of the three remaining colors, and so on. The number of distinct possible arrangements is

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

b) We begin by satisfying the specified requirements stated. In this case, the strawberry computer must be placed in the middle. Therefore, there is only one possibility for the middle position.

$$\underline{\ \ }\ \underline{\ \ }\ \underline{1}\ \underline{\ \ }\ \underline{\ \ }$$

For the first position, there are now four possibilities. For the second position, there will be three possibilities. For the fourth position, there will be two possibilities. Finally, in the last position, there is only one possibility.

$$\underline{4} \cdot \underline{3} \cdot \underline{1} \cdot \underline{2} \cdot \underline{1} = 24$$

Thus, under the condition stated, there are 24 different possible arrangements.

c) For the first computer, there is only 1 possible computer, the blueberry computer. For the last computer, there is only 1 possible computer, the strawberry computer. This gives

$$\underline{1}\ \_\ \_\ \_\ \underline{1}$$

The second position can be filled by any of the three remaining computers. The third position can be filled by any of the two remaining computers. There is only one computer left for the fourth position. Thus, the number of possible arrangements is

$$\underline{1} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{1} = 6$$

There are only six possible arrangements that satisfy the given conditions. ▲

Permutations

Now we introduce the definition of a permutation.

A **permutation** is any *ordered arrangement* of a given set of objects.

Curly Moe Larry

"Larry, Curly, Moe" and "Curly, Moe, Larry" represent two different ordered arrangements or two different permutations of the same three names. In Example 2(a), there are 120 different ordered arrangements, or permutations, of the five colored computers. In Example 2(b), there are 24 different ordered arrangements, or permutations possible, if the strawberry computer must be displayed in the middle.

When determining the number of permutations possible, we assume that repetition of an item is not permitted. To help you understand and visualize permutations, we illustrate the various permutations possible when a triangle, rectangle, and circle are to be placed in a line, see Figure 12.21.

Six Permutations

Figure 12.21

Note for this set of three shapes that six different arrangements, or six permutations, are possible. We can obtain the number of permutations by using the counting principle. For the first position, there are three choices. There are then two choices for the second position, and only one choice is left for the third position.

$$\text{Number of permutations} = 3 \cdot 2 \cdot 1 = 6$$

The product $3 \cdot 2 \cdot 1$ is referred to as 3 factorial, and is written 3!. Thus,

$$3! = 3 \cdot 2 \cdot 1 = 6$$

Number of Permutations

The number of permutations of n distinct items is n factorial, symbolized $n!$, where

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

It is important to note that 0! is defined to be 1. Many calculators have the ability to determine factorials. Often to determine factorials you need to press the 2nd or INV key. Read your calculator manual to determine how to find factorials on your calculator.

EXAMPLE 3 *Children in Line*

In how many different ways can seven children be arranged in a line?

SOLUTION: Since there are seven children, the number of permutations is 7!

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

The seven children can be arranged in 5040 different ways. ▲

Example 4 illustrates how to use the counting principle to determine the number of permutations possible when only a part of the total number of items is to be selected and arranged.

Some of the many permutations of 3 of the 5 letters

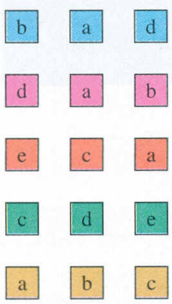


Figure 12.22

EXAMPLE 4 *Permutations of Three Out of Five Letters*

Consider the five letters a, b, c, d, e. In how many distinct ways can three letters be selected and arranged if repetition is not allowed?

SOLUTION: We are asked to select and arrange only three of the five possible letters. Figure 12.22 shows some possibilities. Using the counting principle, we find that there are five possible letters for the first choice, four possible letters for the second choice, and three possible letters for the third choice:

$$5 \cdot 4 \cdot 3 = 60$$

Thus, there are 60 different possible ordered arrangements, or permutations. ▲

In Example 4, we determined the number of different ways in which we could select and arrange three of the five items. We can indicate that by using the notation ${}_5P_3$. The notation ${}_5P_3$ is read "the number of permutations of five items taken three at a time." The notation ${}_nP_r$ is read "the number of permutations of n items taken r at a time."

We use the counting principle below to evaluate ${}_8P_4$, ${}_9P_3$, and ${}_{10}P_5$. Notice the relationship between the number preceding the P, the number following the P, and the last number in the product.

$${}_8P_4 = 8 \cdot 7 \cdot 6 \cdot \underbrace{5}_{\text{One more than } 8-4}$$

$${}_9P_3 = 9 \cdot 8 \cdot \underbrace{7}_{\text{One more than } 9-3}$$

$${}_{10}P_5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot \underbrace{6}_{\text{One more than } 10-5}$$

Notice that to evaluate ${}_nP_r$ we begin with n and form a product of r consecutive descending factors. For example, to evaluate ${}_{10}P_5$, we start with 10 and form a product of 5 consecutive descending factors (see the preceding illustration).

In general, the number of permutations of n items taken r at a time, ${}_nP_r$, may be found by the formula

$${}_nP_r = n(n-1)(n-2)\cdots\overbrace{(n-r+1)}^{\text{One more than } n-r}$$

Therefore, when evaluating ${}_{20}P_{15}$, we would find the product of consecutive decreasing integers from 20 to $(20 - 15 + 1)$ or 6, which is written as $20 \cdot 19 \cdot 18 \cdot 17 \cdot \cdots \cdot 6$.

Now let's develop an alternative formula that we can use to find the number of permutations possible when r objects are selected from n objects:

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

Now multiply the expression on the right side of the equals sign by $\frac{(n-r)!}{(n-r)!}$ which is equivalent to multiplying the expression by 1.

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)!}{(n-r)!}$$

For example,

$${}_{10}P_5 = 10 \cdot 9 \cdot \cdots \cdot 6 \times \frac{5!}{5!}$$

or

$${}_{10}P_5 = \frac{10 \cdot 9 \cdot \cdots \cdot 6 \times 5!}{5!}$$

Since $(n-r)!$ means $(n-r)(n-r-1)\cdots(3)(2)(1)$, the expression for ${}_nP_r$ can be rewritten as

$${}_nP_r = \frac{n(n-1)(n-2)\cdots(n-r+1)\overbrace{(n-r)(n-r-1)\cdots(3)(2)(1)}^{(n-r)!}}{(n-r)!}$$

Since the numerator of this expression is $n!$, we can write

$${}_nP_r = \frac{n!}{(n-r)!}$$

For example,

$${}_{10}P_5 = \frac{10!}{(10-5)!}$$

The number of permutations possible when r objects are selected from n objects is found by the **permutation formula**

$${}_nP_r = \frac{n!}{(n-r)!}$$

In Example 4, we found that when selecting three of five letters, there were 60 permutations. We can obtain the same result using the permutation formula:

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot 4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{\cancel{2}\cdot \cancel{1}} = 60$$

EXAMPLE 5 *Using the Permutation Formula*

You are among nine people forming a hiking club. Collectively, you decide to put each person's name in a hat and to randomly select a president, a vice president, and a secretary. How many different arrangements or permutations of officers are possible?

SOLUTION: Since there are nine people, $n = 9$, of which three are to be selected; thus $r = 3$.

$${}_9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = \frac{9\cdot 8\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 504$$

Thus, with nine people there can be 504 different arrangements for president, vice president, and secretary.

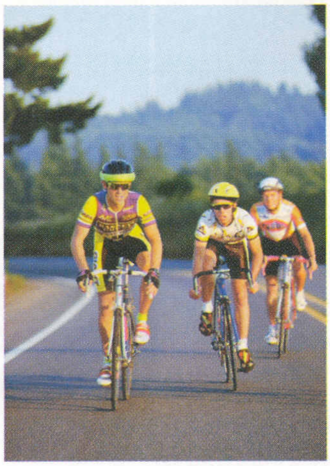
In Example 5, the fraction

$$\frac{9\cdot 8\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 504$$

can be also expressed as

$$\frac{9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}} = 504$$

The solution to Example 5, like other permutation problems, can also be obtained using the counting principle.



EXAMPLE 6

The Prince George County bicycle club has 10 different routes members wish to travel exactly once, but they only have 6 specific dates for their trips. In how many ways can the different routes be assigned to the dates scheduled for their trips?

SOLUTION: There are 10 possibile routes but only 6 specific dates scheduled for the trips. Since traveling route A on day 1 and traveling route B on day 2 is different than traveling route B on day 1 and traveling route A on day 2, we have a permutation problem. There are 10 possible routes; thus, $n = 10$. There are 6 routes that are going to be selected and assigned to different days; thus, $r = 6$. Now we calculate the number of different permutations of selecting and arranging the dates for 6 out of 10 possible routes.

$$_{10}P_6 = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = 151{,}200$$

There are 151,200 different ways that 6 routes can be selected and scheduled from the 10 possible routes. ▲

Example 6 could also be worked using the counting principle because we are discussing an *ordered arrangement* (a permutation) that is done *without replacement*. For the first date scheduled, there are 10 possible outcomes. For the second date selected, there are 9 possible outcomes. By continuing this process we would determine that the number of possible outcomes for the 6 different trips is $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151{,}200$.

We have worked permutation problems (selecting and arranging, without replacement, r items out of n *distinct* items) by using the counting principle and using the permutation formula. When you are given a permutation problem, unless specified by your instructor, you may use either technique to determine its solution.

Permutations of Duplicate Items

So far, all the examples we have discussed in this section have involved arrangements with distinct items. Now we will consider permutation problems in which some of the items to be arranged are duplicates. For example, the name BOB contains three letters, of which the two Bs are duplicates. How many permutations of the letters in the name BOB are possible? If the two Bs were distinguishable (one red and the other blue), there would be six permutations.

BOB BBO OBB

BOB BBO OBB

However, if the Bs are not distinguishable (replacing all colored Bs with black print), we see there are only three permutations.

BOB BBO OBB

The number of permutations of the letters in BOB can be computed as

$$\frac{3!}{2!} = \frac{3 \cdot \cancel{2 \cdot 1}}{\cancel{2 \cdot 1}} = 3$$

where 3! represents the number of permutations of three letters, assuming that none are duplicates, and 2! represents the number of ways the two items that are duplicates can be arranged (BB or BB). In general, we have the following rule.

Permutations of Duplicate Objects

The number of distinct permutations of n objects where n_1 of the objects are identical, n_2 of the objects are identical, . . . , n_r of the objects are identical is found by the formula

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

EXAMPLE 7 *Duplicate Letters*

In how many different ways can the letters of the word "MISSISSIPPI" be arranged?

SOLUTION: Of the 11 letters, four are I's, four are S's, and two are P's. The number of possible arrangements is

$$\frac{11!}{4!4!2!} = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1\cdot2\cdot1} = 11\cdot10\cdot9\cdot7\cdot5 = 34{,}650$$

There are 34,650 different possible arrangements of the letters in the word "MISSISSIPPI."

SECTION 12.8 EXERCISES

Concept/Writing Exercises

1. In your own words, state the counting principle.
2. In your own words, describe a permutation.
3. Give the formula for the number of permutations of n distinct items.
4. In your own words, explain how to find $n!$ for any whole number n.
5. How do you read ${}_nP_r$? When you evaluate ${}_nP_r$, what does the outcome represent?
6. Give the formula for the number of permutations of n objects when $n_1, n_2, \ldots, n_r$ of the objects are identical.
7. Give the formula for the number of permutations when r objects are selected from n objects.
8. Does ${}_1P_1 = {}_1P_0$? Explain.

Practice the Skills

In Exercises 9–20, evaluate the expression.

9. 6!
10. 8!
11. ${}_6P_2$
12. ${}_5P_2$
13. 0!
14. ${}_6P_4$
15. ${}_8P_0$
16. ${}_5P_0$
17. ${}_9P_4$
18. ${}_4P_4$
19. ${}_8P_5$
20. ${}_{10}P_6$

Problem solving

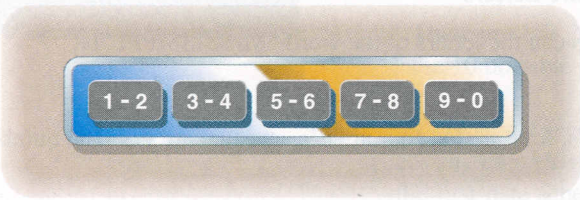
21. *ATM Codes* To use an automated teller machine, you generally must enter a four-digit code, using the digits 0–9. How many four-digit codes are possible if repetition of digits is permitted?
22. *Daily Double* The daily double at most racetracks consists of selecting the winning horse in both the first and second races. If the first race has seven entries and the second race

has eight entries, how many daily double tickets must you purchase to guarantee a win?

23. *Passwords* Assume that a password to log onto a computer account is to consist of three letters followed by three digits. Determine the number of possible passwords if
a) repetition is not permitted.
b) repetition is permitted.

24. *Passwords* Assume that a password to log onto a computer account is to consist of any four digits or letters (repetition is permitted). Determine the number of passwords possible if
a) the letters are not case sensitive (that is, a lowercase letter is treated the same as an uppercase letter).
b) the letters are case sensitive (that is, an uppercase letter is considered different than the same lowercase letter).

25. *Car Door Locks* Some doors on cars can be opened by pressing the correct sequence of buttons. A display of the five buttons by the door handle of a car follows.*



The correct sequence of five buttons must be pressed to unlock the door. If the same button may be pressed consecutively,
a) how many possible ways can the five buttons be pressed (repetition is permitted)?
b) If five buttons are pressed at random, find the probability that a sequence that unlocks the door will be entered.

26. *Social Security Numbers* A social security number consists of nine digits. How many different social security numbers are possible if repetition of digits is permitted?

27. *License Plate* A license plate is to have five uppercase letters or digits. Determine the number of license plates possible if repetition is permitted and if any position can contain either a letter or digit with the exception that the first position cannot contain the letter *O* or the number 0.

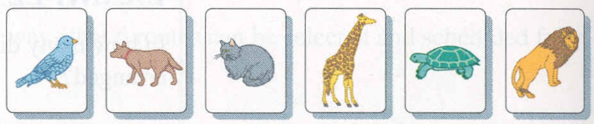
28. *Winning the Trifecta* The trifecta at most racetracks consists of selecting the first-, second-, and third-place finishers in a particular race in their proper order. If there are seven entries in the trifecta race, how many tickets must you purchase to guarantee a win?

29. *Advertising* The operator of the Sound Great Stereo store is planning a grand opening. He wishes to advertise that the store has many different sound systems available. It stocks 8 different CD players, 10 different receivers, and 9 different sets of speakers. Assuming that a sound system will consist of one CD player, one receiver, and one set of speakers and that all pieces are compatible, how many different sound systems can they advertise?

30. *Geometric Shapes* Consider the five figures shown.

In how many different ways can the figures be arranged
a) from left to right?
b) from top to bottom if placed one under the other?
c) from left to right if the triangle is to be placed on the far right?
d) from left to right if the circle is to be placed on the far left and the triangle is to be placed on the far right?

31. *Arranging Pictures* The six pictures shown are to be placed side by side along a wall.



In how many ways can they be arranged from left to right if
a) they can be arranged in any order?
b) the bird must be on the far left?
c) the bird must be on the far left and the giraffe must be next to the bird?
d) a four-legged animal must be on the far right?

32. *Car Rental* At a local car rental agency 8 identical midsize cars are available and 3 customers want midsize cars. If each customer can choose his or her own car, how many different ways can the cars be selected?

33. *Club Officers* If a club consists of ten members, how many different arrangements of president, vice-president, and secretary are possible?

34. *ISBN Codes* Each book registered in the Library of Congress must have an ISBN code number. For an ISBN number of the form D-DD-DDDDDD-D, where D represents a digit from 0–9, how many different ISBN numbers are possible if repetition of digits is allowed? (See research activity Exercise 70.)

35. *Wedding Reception* At the reception line of a wedding, the bride, the groom, the best man, the maid of honor, the four ushers, and the four bridesmaids must line up to receive the guests.

*On most cars, although each key lists two numbers, the key acts as a single number. Therefore, if your code is 1, 6, 8, 5, 3, the code 2, 5, 7, 6, 4 will also open the lock.

a) If these individuals can line up in any order, how many arrangements are possible?
b) If the groom must be the last in line and the bride must be next to the groom, and the others can line up in any order, how many arrangements are possible?
c) If the groom is to be last in line, the bride next to the groom, and males and females are to alternate, how many arrangements are possible?

36. *Disc Jockey* A disc jockey has 8 songs to play. Three are slow songs and 5 are fast songs. Each song is to be played only once. In how many ways can the disc jockey play the 8 songs if
a) the songs can be played in any order.
b) the first song must be a slow song and the last song must be a slow song.
c) the first two songs must be fast songs.

Letter Codes In Exercises 37–40, an identification code is to consist of two letters followed by four digits. How many different codes are possible if

37. repetition is not permitted?

38. repetition is permitted?

39. the first two entries must both be the same letter and repetition of the digits is not permitted?

40. the first letter must be a *A*, *B*, *C*, or *D* and repetition is not permitted?

License Plates In Exercises 41–44, a license plate is to consist of three digits followed by two uppercase letters. Determine the number of different license plates possible if

41. repetition of numbers and letters is permitted.

42. repetition of numbers and letters is not permitted.

43. the first and second digits must be odd, and repetition is not permitted.

44. the first digit cannot be zero, and repetition is not permitted.

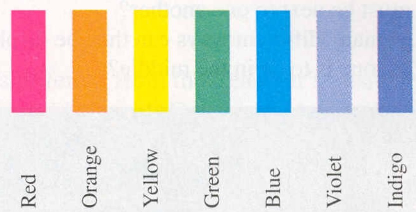
45. *Possible Phone Numbers* A telephone number consists of seven digits with the restriction that the first digit cannot be 0 or 1.
a) How many distinct telephone numbers are possible?
b) How many distinct telephone numbers are possible with three-digit area codes preceding the seven-digit number, where the first digit of the area code is not 0 or 1?
c) With the increasing use of cellular phones and paging systems, our society is beginning to run out of usable phone numbers. Various phone companies are developing phone numbers that use 11 digits instead of 7. How many distinct phone numbers can be made with 11 digits, assuming that the area code remains three digits and the first digit of the area code and the phone number cannot be 0 or 1?

46. *Selecting Cereal* Mrs. Williams and her 3 children go shopping at a local grocery store. Each of the children will be allowed to select one box of cereal for their own. On the store's shelf there are only 12 boxes of cereal, and each box contains a different type of cereal. In how many ways can the selections be made?

47. *Swimming Event* A swimming meet has 15 participants for the 100 meter freestyle event. The 6 participants with the fastest speeds will be listed, in the order of their speed, on the tote board. How many different ways are there for the names to be listed?

48. *History Test* In one question of a history test the student is asked to match 10 dates with 10 events; each date can only be matched with 1 event. In how many different ways can this question be answered?

49. *Color Permutations* Find the number of permutations of the colors in the spectrum that follows.



50. *Drive-Through at a Bank* A bank has three drive-through stations. Assuming that each is equally likely to be selected by customers, how many different ways can the next six drivers select a station?

51. *Computer Systems* At a computer store, a customer is considering 5 different computers, 4 different monitors, 7 different printers, and 2 different scanners. Assuming that each of the components is compatible with one another and that one of each is to be selected, determine the number of different computer systems possible.

52. *Selecting Furniture* The Johnsons just moved into their new home and are selecting furniture for the family room. They are considering 5 different sofas, 2 different chairs, and 6 different tables. They plan to select one item from each category. Determine the number of different ways they can select the furniture.

53. Determine the number of permutations of the letters of the word "EDUCATION."

54. Determine the number of permutations of the letters of the word "GOVERNMENT."

55. In how many ways can the letters in the word "INDEPENDENCE" be arranged?

56. In how many ways can the letters in the word "TENNESSEE" be arranged?

57. In how many ways can the digits in the number 1,324,324 be arranged?

58. In how many ways can the digits of the number 2,142,332 be arranged?

59. *Flag Messages* Five different colored flags will be placed on a pole, one beneath another. The arrangement of the colors indicates the message. How many messages are possible if five flags are to be selected from eight different colored flags?

60. *Multiple-Choice Test* Keri Kershaw is taking a 10-question multiple-choice exam. Each question has three possible answers, (a), (b), and (c). In how many possible ways can Keri answer the questions?

61. *Batting Order* In how many ways can the manager of a National League baseball team arrange his batting order of nine players if

a) the pitcher must bat last?

b) there are no restrictions?

62. *Painting Exhibit* Five Monet paintings are to be displayed in a museum.

a) In how many different ways can they be arranged if they must be next to one another?

b) In how many different ways can they be displayed if a specific one is to be in the middle?

The Water Lily Pond, Pink Harmony, 1900 by Claude Monet

Challenge Problems/Group Activities

63. *Car Keys* Door keys for a certain automobile are made from a blank key on which five cuts are made. Each cut may be one of five different depths.

a) How many different keys can be made?

b) If 400,000 of these automobiles are made such that each of the keys determined in part a) opens the same number of cars, find how many cars can be opened by a specific key?

c) If one of these cars is selected at random, what is the probability that the key selected at random will unlock the door?

64. *Voting* On a ballot, each committee member is asked to rank three of seven candidates for recommendation for promotion, giving their first, second, and third choices (no ties). What is the minimum number of ballots that must be cast to guarantee that at least two ballots are the same?

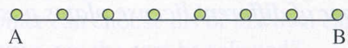
65. *Scrabble* Nancy Lin, who is playing Scrabble with Dale Grey, has seven different letters. She decides to test each five-letter permutation before her next move. If each permutation takes 5 sec, how long will it take Nancy to check all the permutations?

66. *Scrabble* In Exercise 65, assume, of Nancy's seven letters, that three are identical and two are identical. How long will it take Nancy to try all different permutations of her seven letters?

67. Does ${}_nP_r = {}_nP_{(n-r)}$ for all whole numbers, where $n \geq r$? Explain.

Recreational Exercises

68. *Stations* There are eight bus stations from town A to town B. How many different single tickets must be printed so that a passenger may purchase a ticket from any station to any other station?



69. *Bus Loop* How many tickets with different points of origin and destination can be sold on a bus line that travels a loop with 25 stops?

Internet/Research Activity

70. When a book is published it is assigned a 10-digit code number called the International Standard Book Number (ISBN). Do research and write a report on how this coding system works.

12.9 COMBINATIONS

When the order of the selection of the items is important to the final outcome, the problem is a permutation problem. When the order of the selection of the items is unimportant to the final outcome, the problem is a *combination* problem.

Recall from Section 12.8 that permutations are *ordered* arrangements. Thus, for example, a, b, c and b, c, a are two different permutations because the ordering of the three letters is different. The letters a, b, c and b, c, a represent the same combination of letters because the *same letters* are used in each set. However, the letters *a*, *b*, *c* and *a*, *b*, *d* represent two different combinations of letters because the letters contained in each set are different.

A **combination** is a distinct group (or set) of objects without regard to their arrangement.

EXAMPLE 1 *Permutation or Combination*

Determine whether the situation represents a permutation or combination problem.

a) A group of five friends, Arline, Inez, Judy, Dan, and Eunice, are forming a club. The group will elect a president and a treasurer. In how many different ways can the president and treasurer be selected?

b) Of the five individuals named, two will be attending a meeting together. In how many different ways can they do so?

SOLUTION:

a) Since the president's position is different from the treasurer's position, we have a permutation problem. Judy as president with Dan as treasurer is different from Dan as president with Judy as treasurer. The order of the selection is important.

b) Since the order in which the two individuals selected to attend the meeting is not important, we have a combination problem. There is no difference if Judy is selected and then Dan is selected, or if Dan is selected and then Judy is selected.

In Section 12.8, you learned that ${}_nP_r$ represents the number of permutations when r items are selected from n distinct items. *Similarly, ${}_nC_r$ represents the number of combinations when r items are selected from n distinct items.*

Consider the set of elements $\{a, b, c, d, e\}$. The number of permutations of two letters from the set is represented as ${}_5P_2$, and the number of combinations of two letters from the set is represented as ${}_5C_2$. Twenty permutations of two letters and 10 combinations of two letters are possible from these five letters. Thus, ${}_5P_2 = 20$ and ${}_5C_2 = 10$, as shown.

$$\textbf{Permutations} \qquad \left.\begin{matrix} ab, ba, ac, ca, ad, da, ae, ea, bc, cb, \\ bd, db, be, eb, cd, dc, ce, ec, de, ed \end{matrix}\right\} 20 \qquad \textbf{Combinations} \qquad \left.\begin{matrix} ab, ac, ad, ae, bc, \\ bd, be, cd, ce, de \end{matrix}\right\} 10$$

When discussing both combination and permutation problems, we always assume that the experiment is performed without replacement. That is why duplicate letters such as *aa* or *bb* are not included in the preceding example.

Note that from one combination of two letters two permutations can be formed. For example, the combination *ab* gives the permutations *ab* and *ba*, or twice as many permutations as combinations. Thus, for this example we may write

$${}_5P_2 = 2 \cdot ({}_5C_2)$$

Since $2 = 2!$ we may write

$${}_5P_2 = 2!({}_5C_2)$$

DID YOU KNOW

Powerball

On Thursday, December 26, 2002, Andrew "Jack" Whittaker Jr., 55, of Scott Depot, South Carolina, won the largest Powerball jackpot in history, a $314.9 million prize. It was the largest jackpot won worldwide by one person up until that date. Rather than taking 30 annual payments of about $10.5 million or about $6.9 million after taxes, Whittaker chose to take a smaller lump sum of $170.5 million, which comes to about $111.6 million after taxes.

As of this writing Powerball tickets are sold in 23 states, the District of Columbia, and the U.S. Virgin Islands. Winning the Powerball requires selecting 5 matching numbers of 53 as well as the Powerball number from 1 to 42. The winning numbers were 5-14-16-29-53, and the Powerball number was 7. The probability of winning is about 1 in 120,526,770. Verify this probability yourself.

The largest lottery jackpot in U.S. history was a Big Game prize of $363 million, won in 2000 by two ticket holders in Michigan and Illinois. The largest lottery in the world is Spain's annual contest known as *El Gordo*, or "the Fat One," which in 2002 paid out a total of $1.7 billion. Each year El Gordo has about 10,000 winners, ranging from $20 to $200,000 each.

DID YOU KNOW

Poker versus Bridge

A nice bridge hand

In popular card games, there is such a variety of possible combinations of cards that a player rarely gets the same hand twice. The total number of different 5-card poker hands using a standard deck of 52 cards is 2,598,960, and for 13-card bridge hands this number increases to 635,013,559,600.

If we repeated this same process for comparing the number of permutations in ${}_nP_r$ with the number of combinations in ${}_nC_r$, we would find that

$${}_nP_r = r!({}_nC_r)$$

Dividing both sides of the equation by $r!$ gives

$${}_nC_r = \frac{{}_nP_r}{r!}$$

Since ${}_nP_r = \dfrac{n!}{(n-r)!}$, the combination formula may be expressed as

$${}_nC_r = \frac{n!/(n-r)!}{r!} = \frac{n!}{(n-r)!r!}$$

The number of combinations possible when r objects are selected from n objects is found by the **combination formula**

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

EXAMPLE 2 *Exam Question Selection*

An exam consists of six questions. Any four may be selected for answering. In how many ways can this selection be made?

SOLUTION: This problem is a combination problem because the order in which the four questions are answered does not matter.

$${}_6C_4 = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!} = \frac{\overset{3}{\cancel{6}} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot 1 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 15$$

There are 15 different ways that four of the six questions can be selected. ▲

EXAMPLE 3 *Airline Seat Availability*

A commercial airline is seating passengers. With a few minutes to go before take-off, there are 9 passengers who are flying standby and wish to get seats, but only 5 seats are available. Determine the number of different ways a group of 5 people can be selected from the 9 to board the plane.

SOLUTION: This problem is a combination problem because the order in which the five people are selected is unimportant. There are a total of nine people, so $n = 9$. Five are to be selected, so $r = 5$.

$${}_9C_5 = \frac{9!}{(9-5)!5!} = \frac{9!}{4!5!} = \frac{9 \cdot \overset{2}{\cancel{8}} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 126$$

Thus, 126 different combinations are possible when five from a group of nine are to be selected. ▲

EXAMPLE 4 *Dinner Combinations*

At the Royal Dynasty Chinese restaurant, dinner for eight people consists of 3 items from column A, 4 items from column B and 3 items from column C. If columns A, B, and C have 5, 7, and 6 items, respectively, how many different dinner combinations are possible?

SOLUTION: For column A, 3 of 5 items must be selected, which can be represented as ${}_5C_3$. For column B, 4 of 7 items must be selected, which can be represented as ${}_7C_4$. For column C, 3 of 6 items must be selected, or ${}_6C_3$.

$$_5C_3 = 10 \quad _7C_4 = 35 \quad \text{and} \quad _6C_3 = 20$$

Using the counting principle, we can determine the total number of dinner combinations by multiplying the number of choices from columns A, B, and C:

$$\text{Total number of dinner choices} = {}_5C_3 \cdot {}_7C_4 \cdot {}_6C_3 = 10 \cdot 35 \cdot 20 = 7000$$

Therefore, 7000 different combinations are possible under these conditions. ▲

We have presented various counting methods, including the counting principle, permutations, and combinations. You often need to decide which method to use to solve a problem. Table 12.4 may help you in selecting the procedure to use.

TABLE 12.4 Summary of Counting Methods

	Determining the number of ways of selecting *r* items from *n* items. Repetition not permitted.	
	Permutations	**Combinations**
Counting Principle: If a first experiment can be performed in *M* distinct ways and a second experiment can be performed in *N* distinct ways, then the two experiments in that specific order can be performed in $M \cdot N$ distinct ways. The counting principle may be used with or without repetition of items. It is used when determining the number of different ways that two or more experiments can occur. It is also used when there are specific placement requirements, such as the first digit must be a 0 or 1.	Permutations are used when order is important. For example, *a, b, c* and *b, c, a* are two different permutations of the same three letters. $_nP_r = \dfrac{n!}{(n-r)!}$ Problems solved with the permutation formula may also be solved by using the counting principle.	Combinations are used when order is not important. For example, *a, b, c* and *b, c, a* are the same combination of three letters. But *a, b, c*, and *a, b, d* are two different combinations of three letters. $_nC_r = \dfrac{n!}{(n-r)!r!}$

SECTION 12.9 EXERCISES

Concept/Writing Exercises

1. In your own words, explain what is meant by a combination.
2. What does $_nC_r$ mean?
3. Give the formula for finding $_nC_r$.
4. What is the relationship between $_nC_r$ and $_nP_r$?
5. In your own words, explain the difference between a permutation and a combination.
6. Assume that you have six different objects and that you are going to select four without replacement. Will there be more combinations or more permutations of the four items? Explain.

Practice the Skills

In Exercises 7–20, evaluate the expression.

7. ${}_5C_3$ **8.** ${}_7C_2$

9. a) ${}_6C_4$ **b)** ${}_6P_4$ **10. a)** ${}_8C_2$ **b)** ${}_8P_2$

11. a) ${}_8C_0$ **b)** ${}_8P_0$ **12. a)** ${}_{12}C_8$ **b)** ${}_{12}P_8$

13. a) ${}_{10}C_3$ **b)** ${}_{10}P_3$ **14. a)** ${}_5C_5$ **b)** ${}_5P_5$

15. $\frac{{}_5C_3}{{}_5P_3}$ **16.** $\frac{{}_6C_2}{{}_6P_2}$ **17.** $\frac{{}_8C_5}{{}_8C_2}$

18. $\frac{{}_6C_6}{{}_8C_0}$ **19.** $\frac{{}_9P_5}{{}_{10}C_4}$ **20.** $\frac{{}_7P_0}{{}_7C_0}$

Problem Solving

21. *Car Rental* At a car rental agency, the agent has 9 identical midsize cars on her lot and 6 people have reserved midsize cars. In how many different ways can the 6 cars to be used be selected?

22. *Banana Split* An ice cream parlor has 20 different flavors. Cynthia orders a banana split and has to select 3 different flavors. How many different selections are possible?

23. *Test Essays* A student must select and answer four of five essay questions on a test. In how many ways can she do so?

24. *Book Selection* A textbook search committee is considering eight books for possible adoption. The committee has decided to select three of the eight for further consideration. In how many ways can it do so?

25. *Dinner Party* Cliff Michael is having a dinner party. He has 10 different bottles of wine on his wine rack and wants to select and bring out 4 bottles. In how many ways can he do so?

26. *Attending Plays* While visiting New York City, the Nygents want to attend 3 plays out of 10 plays they would like to see. In how many ways can they do so?

Theater district, New York City

27. *Taxi Ride* A group of 9 people wants to use taxis to go to a local restaurant. When the first taxi arrives, the group decides that 5 people should get into the taxi. In how many ways can this be done?

28. *Plants* Mary Robinson purchased a package of 24 plants, but she only needed 20 plants for planting. In how many ways can she select the 20 plants from the package to be planted?

29. *Entertainers* Ruth Eckerd Hall must select 8 of 12 possible entertainers for their summer schedule. In how many ways can that be done?

30. *CD Purchase* Neo Anderson wants to purchase six different CDs but only has enough money to purchase four. In how many ways can he select four of six CDs for purchase?

31. *Painting* Paula Dunst has 10 framed paintings she would like to mount on her wall in a straight line. However, her wall is only wide enough to hold 8 of her paintings. In how many ways can Paula select the 8 paintings to mount on the wall?

32. *Printers and Keyboards* Office Depot has nine different printers in stock and six different cordless keyboards in stock. The manager wants to place three of the nine printers and two of the six cordless keyboards on sale. In how many ways can the manager select the items to be listed as sale items?

33. *Quinella Bet* A quinella bet consists of selecting the first- and second-place winners, in any order, in a particular event. For example, suppose you select a 2–5 quinella. If 2 wins and 5 finishes second, or if 5 wins and 2 finishes second, you win. Mr. Smith goes to a jai alai match. In the match, 8 jai alai teams compete. How many quinella tickets must Mr. Smith purchase to guarantee a win?

Jai alai game

34. *Test Question* On an English test, Tito Ramirez must write an essay for three of the five questions in Part 1 and four of the six questions in Part 2. How many different combinations of questions can he answer?

35. *Big-Screen TVs* A television/stereo store has 12 different big-screen televisions and 8 stereo systems in stock. The store's manager wishes to place 3 big-screen TVs and 2 stereo systems on sale. In how many ways can that be done?

36. *Medical Research* At a medical research center an experimental drug is to be given to 12 people, 6 men and 6 women. If 10 men and 9 women have volunteered to be given the drug, in how many ways can the researcher choose the 12 people to be given the drug?

37. *An Editor's Choice* An editor has eight manuscripts for mathematics books and five manuscripts for computer science books. If he is to select five mathematics and three computer science manuscripts for publication, how many different choices does he have?

38. *Selecting Soda* Michael Miller is sent to the store to get 5 different bottles of regular soda and 3 different bottles of diet soda. If there are 10 different types of regular sodas and 7 different types of diet sodas to choose from, how many different choices does Michael have?

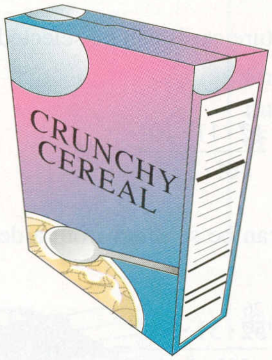
39. *Forming a Committee* How many different committees can be formed from 6 teachers and 50 students if the committee is to consist of 2 teachers and 3 students?

40. *Constructing a Test* A teacher is constructing a mathematics test consisting of 10 questions. She has a pool of 28 questions, which are classified by level of difficulty as follows: 6 difficult questions, 10 average questions, and 12 easy questions. How many different 10-question tests can she construct from the pool of 28 questions if her test is to have 3 difficult, 4 average, and 3 easy questions?

41. *Selecting Mutual Funds* George Holloway recently graduated from college and now has a job that provides a retirement investment plan. George wants to diversify his investments, so he wants to invest in three stock mutual funds and two bond mutual funds. If he has a choice of eight stock mutual funds and five bond mutual funds, how many different selections of mutual funds does he have?

42. *Door Prize* As part of a door prize, Mary McCarty won three tickets to a baseball game and three tickets to a theater performance. She decided to give all the tickets to friends. For the baseball game she is considering six different friends, and for the theater she is considering eight different friends. In how many ways can she distribute the tickets?

43. *New Breakfast Cereals* General Mills is testing 6 oat cereals, 5 wheat cereals, and 4 rice cereals. If it plans to market 3 of the oat cereals, 2 of the wheat cereals, and 2 of the rice cereals, how many different combinations are possible?



44. *Catering Service* A catering service is making up trays of hors d'oeuvres. The hors d'oeuvres are categorized as inexpensive, average, and expensive. If the client must select three of the seven inexpensive, five of the eight average, and two of the four expensive hors d'oeuvres, how many different choices are possible?

Challenge Problems/Group Activities

45. *Test Answers* Consider a 10-question test in which each question can be answered either correctly or incorrectly.

a) How many different ways are there to answer the questions so that eight are correct and two are incorrect?

b) How many different ways are there to answer the questions so that at least eight are correct?

46. a) *A Dinner Toast* Four people at dinner make a toast. If each person is to tap glasses with each other person one at a time, how many taps will take place?

b) Repeat part (a) with five people.

c) How many taps will there be if there are n people at the dinner table?

47. *Pascal's Triangle* The notation ${}_nC_r$ may be written $\binom{n}{r}$.

a) Use this notation to evaluate each of the combinations in the following array. Form a triangle of the results, similar to the one given, by placing the answer to each combination in the same relative position in the triangle.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

b) Using the number pattern in part (a), find the next row of numbers of the triangle (known as *Pascal's triangle*).

48. *Lottery Combinations* Determine the number of combinations possible in a state lottery where you must select

a) 6 of 46 numbers.

b) 6 of 47 numbers.

c) 6 of 48 numbers.

d) 6 of 49 numbers.

e) Does the number of combinations increase by the same amount going from part (a) to part (b) as from part (b) to part (c)?

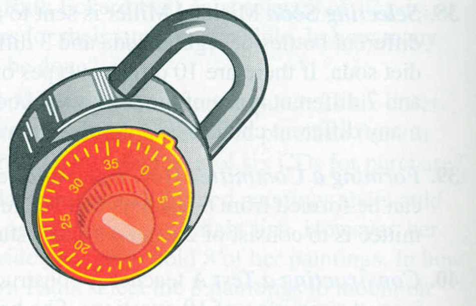
49. a) *Table Seating Arrangements* How many distinct ways can four people be seated in a row?

b) How many distinct ways can four people be seated at a circular table?

50. Show that ${}_nC_r = {}_nC_{(n-r)}$.

Recreational Exercise

51. a) *Combination Lock* To open a combination lock, you must know the lock's three-number sequence in its proper order. Repetition of numbers is permitted. Why is this lock more like a permutation lock than a combination lock? Why is it not a true permutation problem?
    b) Assuming that a combination lock has 40 numbers, determine how many different three-number arrangements are possible if repetition of numbers is allowed.
    c) Answer the question in part (b) if repetition is not allowed.



12.10 SOLVING PROBABILITY PROBLEMS BY USING COMBINATIONS

In Section 12.9, we discussed combination problems. Now we will use combinations to solve probability problems.

Suppose that we want to find the probability of selecting two picture cards (jacks, queens, or kings) when two cards are selected, without replacement, from a standard deck of 52 cards. Using the *and* probability formula discussed in Section 12.6, we could reason as follows.

$$P(\text{2 picture cards}) = P(\text{1st picture card}) \cdot P(\text{2nd picture card})$$
$$= \frac{12}{52} \cdot \frac{11}{51} = \frac{132}{2652}, \quad \text{or} \quad \frac{11}{221}$$

Since the order of the two picture cards selected is not important to the final answer, this problem can be considered a combination probability problem.

We can also find the probability of selecting two picture cards, using combinations, by finding the number of possible successful outcomes (selecting two picture cards) and dividing that answer by the total number of possible outcomes (selecting any two cards).

The number of ways in which two picture cards can be selected from the 12 picture cards in a deck is ${}_{12}C_2$, or

$${}_{12}C_2 = \frac{12!}{(12-2)!2!} = \frac{\overset{6}{\cancel{12}} \cdot 11 \cdot \cancel{10!}}{\cancel{10!} \cdot \cancel{2} \cdot 1} = 66$$

The number of ways in which two cards can be selected from a deck of 52 cards is ${}_{52}C_2$, or

$${}_{52}C_2 = \frac{52!}{(52-2)!2!} = \frac{\overset{26}{\cancel{52}} \cdot 51 \cdot \cancel{50!}}{\cancel{50!} \cdot \cancel{2} \cdot 1} = 1326$$

Thus,

$$P(\text{selecting 2 picture cards}) = \frac{{}_{12}C_2}{{}_{52}C_2} = \frac{66}{1326} = \frac{11}{221}$$

Note that the same answer is obtained with either method. To give you more exposure to counting techniques, we will work the problems in this section using combinations.

EXAMPLE 1 *Committee of Three Women*

A club consists of four men and five women. Three members are to be selected at random to form a committee. What is the probability that the committee will consist of three women?

SOLUTION: The order in which the three members are selected is not important. Therefore, we may work this problem using combinations.

$$P\begin{pmatrix}\text{committee consists}\\ \text{of 3 women}\end{pmatrix} = \frac{\text{number of possible committees with 3 women}}{\text{total number of possible 3-member committees}}$$

Since there are a total of 5 women, the number of possible committees with three women is ${}_5C_3 = 10$. Since there are a total of 9 people, the total number of possible three-member committees is ${}_9C_3 = 84$.

$$P(\text{committee consists of 3 women}) = \frac{10}{84} = \frac{5}{42}$$

The probability of randomly selecting a committee with three women is $\frac{5}{42}$.

EXAMPLE 2 *A Heart Flush*

A flush in the game of poker is five cards of the same suit (5 hearts, 5 diamonds, 5 clubs, or 5 spades). If you are dealt a five-card hand, find the probability that you will be dealt a heart flush.

SOLUTION: The order in which the five hearts are dealt is not important. Therefore, we may work this problem using combinations.

$$P(\text{heart flush}) = \frac{\text{number of possible 5-card heart flushes}}{\text{total number of possible 5-card hands}}$$

Since there are 13 hearts in a deck of cards, the number of possible five-card heart flush hands is ${}_{13}C_5 = 1287$. The total number of possible five-card hands in a deck of 52 cards is ${}_{52}C_5 = 2{,}598{,}960$.

$$P(\text{heart flush}) = \frac{{}_{13}C_5}{{}_{52}C_5} = \frac{1287}{2{,}598{,}960} = \frac{33}{66{,}640}$$

The probability of being dealt a heart flush is $\frac{33}{66{,}640}$, or ≈ 0.000495.

EXAMPLE 3 *Employment Assignments*

A temporary employment agency has six men and five women who wish to be assigned for the day. One employer has requested four employees for security positions, and the second employer has requested three employees for moving furniture

DID YOU KNOW

The Dead Man's Hand

James Butler "Wild Bill" Hickok (1837–1876), known as "the fastest gun in the West," was a famous scout and federal marshal. Hickok was shot in the back and mortally wounded by Jack McCall while playing poker in a saloon in Deadwood, Dakota Territory (now South Dakota). McCall was hanged for his deed. The cards Hickok was holding when shot are shown here. Since that time, the two pair, aces and eights, have become known as "the dead man's hand" (see Exercise 43). Hickok is buried next to Martha Canary, better known as "Calamity Jane," in a cemetery overlooking the town of Deadwood.

in an office building. If we assume that each of the potential employees has the same chance of being selected and being assigned at random and that only seven employees will be assigned, find the probability that

a) three men will be selected for moving furniture.

b) three men will be selected for moving furniture and four women will be selected for security positions.

SOLUTION:

$$\text{a) } P\begin{pmatrix}\text{3 men selected}\\ \text{for moving furniture}\end{pmatrix} = \frac{\begin{pmatrix}\text{number of possible combinations}\\ \text{of 3 men selected}\end{pmatrix}}{\begin{pmatrix}\text{total number of possible combinations}\\ \text{for selecting 3 people}\end{pmatrix}}$$

The number of possible combinations with 3 men is ${}_6C_3$. The total number of possible selections of three people is ${}_{11}C_3$.

$$P\begin{pmatrix}\text{3 men selected}\\ \text{for moving furniture}\end{pmatrix} = \frac{{}_6C_3}{{}_{11}C_3} = \frac{20}{165} = \frac{4}{33}$$

Thus, the probability that 3 men are selected is $\frac{4}{33}$.

b) The number of ways of selecting 3 men out of 6 is ${}_6C_3$ and the number of ways of selecting 4 women out of 5 is ${}_5C_4$. The total number of possible selections when 7 people are selected from 11 is ${}_{11}C_7$. Since both the 3 men *and* the 4 women must be selected, the probability is calculated as follows:

$$P\begin{pmatrix}\text{3 men and}\\ \text{4 women selected}\end{pmatrix} = \frac{\begin{pmatrix}\text{number of combinations}\\ \text{of 3 men selected}\end{pmatrix}\cdot\begin{pmatrix}\text{number of combinations}\\ \text{of 4 women selected}\end{pmatrix}}{\text{total number of possible combinations of 7 people}}$$

$$= \frac{{}_6C_3 \cdot {}_5C_4}{{}_{11}C_7} = \frac{20 \cdot 5}{330} = \frac{100}{330} = \frac{10}{33}$$

Thus, the probability is $\frac{10}{33}$. ▲

EXAMPLE 4 *New Breakfast Cereals*

Kellogg's is testing 12 new cereals for possible production. They are testing 3 oat cereals, 4 wheat cereals, and 5 rice cereals. If we assume that each of the 12 cereals has the same chance of being selected and that 4 new cereals will be produced, find the probability that

a) no wheat cereals are selected.

b) at least 1 wheat cereal is selected.

c) 2 wheat cereals and 2 rice cereals are selected.

SOLUTION:

a) If no wheat cereals are to be selected, then only oat and rice cereals must be selected. A total of 8 cereals are oat or rice. Thus, the number of ways that 4 oat or rice cereals may be selected from the 8 possible oat or rice cereals is ${}_8C_4$. The total number of possible selections is ${}_{12}C_4$.

$$P(\text{no wheat cereals}) = \frac{{}_8C_4}{{}_{12}C_4} = \frac{70}{495} = \frac{14}{99}$$

b) When 4 cereals are selected, the choice must contain either no wheat cereal or at least 1 wheat cereal. Since one of these outcomes must occur, the sum of the probabilities must be 1, or

$$P(\text{no wheat cereal}) + P(\text{at least 1 wheat cereal}) = 1$$

Therefore,

$$P(\text{at least 1 wheat cereal}) = 1 - P(\text{no wheat cereal})$$
$$= 1 - \frac{14}{99} = \frac{99}{99} - \frac{14}{99} = \frac{85}{99}$$

Note that the probability of selecting no wheat cereals, $\frac{14}{99}$, was found in part (a).

c) The number of ways of selecting 2 wheat cereals out of 4 wheat cereals is ${}_4C_2$, which equals 6. The number of ways of selecting 2 rice cereals out of 5 rice cereals is ${}_5C_2$, which equals 10. The total number of possible selections when 4 cereals are selected from the 12 choices is ${}_{12}C_4$. Since both the 2 wheat *and* the 2 rice cereals must be selected, the probability is calculated as follows.

$$P(\text{2 wheat and 2 rice}) = \frac{{}_4C_2 \cdot {}_5C_2}{{}_{12}C_4} = \frac{6 \cdot 10}{495} = \frac{60}{495} = \frac{4}{33}$$ ▲

EXAMPLE 5 *Baseball Cards*

Honus Wagner (left) generally considered the most valuable baseball card. The Mickey Mantle rookie card is on the right.

Derek Brock has 18 valuable baseball cards, including 6 Mickey Mantle cards, 4 Ken Griffey Jr. cards, and 5 Cal Ripken Jr. cards. He plans to sell 7 of his cards to finance part of his college education. If he selects the cards at random, what is the probability that 3 Mickey Mantle cards, 2 Ken Griffey Jr. cards, and 2 Cal Ripken Jr. cards are selected?

SOLUTION: The number of ways that Derek can select 3 out of 6 Mickey Mantle cards is ${}_6C_3$. The number of ways he can select 2 out of 4 Ken Griffey Jr. cards is ${}_4C_2$. The number of ways he can select 2 out of 5 Cal Ripken Jr. cards is ${}_5C_2$. He will select 7 from a total of 18 baseball cards. The number of ways he can do so is ${}_{18}C_7$. The probability that Derek selects 3 Mantle, 2 Griffey Jr., and 2 Ripken Jr. cards is calculated as follows.

$$P(\text{3 Mantle, 2 Griffey Jr., and 2 Ripken Jr.}) = \frac{{}_6C_3 \cdot {}_4C_2 \cdot {}_5C_2}{{}_{18}C_7}$$
$$= \frac{20 \cdot 6 \cdot 10}{31{,}824} = \frac{1200}{31{,}824} = \frac{25}{663}$$ ▲

SECTION 12.10 EXERCISES

Concept/Writing Exercises

In Exercises 1–8, set up the problem as if it were to be solved, but do not solve. Assume that each problem is to be done without replacement. Explain why you set up the exercises as you did.

1. Six red balls and four blue balls are in a bag. If four balls from the bag are to be selected at random, determine the probability of selecting four red balls.
2. A class consists of 19 girls and 15 boys. If 12 of the students are to be selected at random, determine the probability that they are all girls.

3. Three letters are to be selected at random from the English alphabet of 26 letters. Determine the probability that 3 vowels (a, e, i, o, u) are selected.

4. Determine the probability of being dealt 3 aces from a standard deck of 52 cards when 3 cards are dealt.

5. On a horse farm there are 18 horses, of which 10 are palaminos. If 5 horses are selected at random, determine the probability they are all palaminos.

6. Of 80 people attending a dance 28 have a college degree. If 4 people at the dance are selected at random, find the probability that each of the 4 has a college degree.

7. In a small area of a forest there are 30 trees, of which 16 are oak trees. Nine trees are to be selected at random. Find the probability that *none* of those selected are oak trees.

8. A class of 16 people contains 4 people whose birthday is in October. If 3 people from the class are selected at random, find the probability that *none* of those selected has an October birthday.

Practice the Skills/Problem Solving

In Exercises 9–18, the problems are to be done without replacement. Use combinations to determine probabilities.

9. *Green and Red Balls* A bag contains four red balls and five green balls. You plan to draw three balls at random. Find the probability of selecting three green balls.

10. *Drawing from a Hat* Each of the numbers 1–6 is written on a piece of paper, and the six pieces of paper are placed in a hat. If two numbers are selected at random, find the probability that both numbers selected are even.

11. *Flu Serum* A doctor has five doses of flu protection serum left. He has six women and eight men who want the medication. If the names of five of these people are selected at random, find the probability that five men's names are selected.

12. *Bills of Four Denominations* Duc Tran's wallet contains 8 bills of the following denominations: four \$5 bills, two \$10 bills, one \$20 bill, and one \$50 bill. If he selects two bills at random, determine the probability that he selects two \$5 bills.

13. *Selecting Digits* Each of the digits 0–9 is written on a slip of paper, and the slips are placed in a hat. If three slips of paper are selected at random, find the probability that the three numbers selected are greater than 4.

14. *Bike Riding* A bicycle club has 10 members. Six members ride Huffy bicycles, two members ride Roadmaster bicycles, and two members ride American Flier bicycles. If four of the members are selected at random, determine the probability that they all ride Huffy bicycles.

15. *Gift Certificates* The sales department at Atwell Studios consists of three people, the manufacturing department consists of six people, and the accounting department consists of two people. Three people will be selected at random from these people and will be given gift certificates to Sweet Tomatoes, a local restaurant. Find the probability that all those selected will be from the manufacturing department.

16. *Faculty-Student Committee* A committee of four is to be randomly selected from a group of seven teachers and eight students. Find the probability that the committee will consist of four students.

17. *Winning the Grand Prize* A lottery consists of 46 numbers. You select 6 numbers and if they match the 6 numbers selected by the lottery commission, you win the grand prize. Find the probability of winning the grand prize.

18. *Red Cards* You are dealt 5 cards from a standard deck of 52 cards. Find the probability that you are dealt 5 red cards.

TV Game Show In Exercises 19–22, a television game show has five doors, of which the contestant must pick two. Behind two of the doors are expensive cars, and behind the other three doors are consolation prizes. The contestant gets to keep the items behind the two doors she selects. Find the probability that the contestant wins

19. no cars.

20. both cars.

21. at least one car.

22. exactly one car.

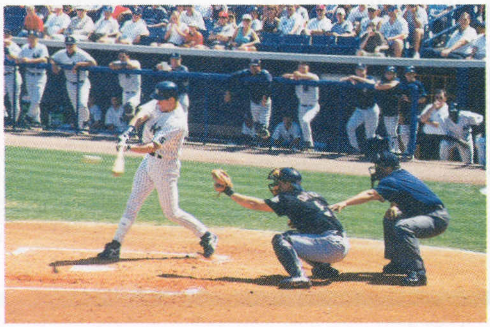
Baseball In Exercises 23–26, assume that a particular professional baseball team has 10 pitchers, 6 infielders, and 9 other players. If 3 players' names are selected at random, determine the probability that

23. all 3 are infielders.

24. none of the three is a pitcher.

25. 2 are pitchers and 1 is an infielder.

26. 1 is a pitcher and 2 are players other than pitchers and infielders.



Jury Selection In Exercises 27–30, a jury pool has 17 men and 22 women, from which 12 will be selected. Assuming that each person is equally likely to be selected and that the jury is selected at random, find the probability the jury consists of

27. all women.

28. 8 women and 4 men.

29. 6 men and 6 women.

30. at least 1 man.

Airline Routes In Exercises 31–34, an airline is given permission to fly 5 new routes of its choice. The airline is considering 15 new routes: 4 routes in Florida, 5 routes in Kentucky, and 6 routes in Virginia. If the airline selects the 5 new routes at random from the 15 possibilities, find the probability that

31. 3 are in Florida and 2 are in Virginia.

32. 4 are in Kentucky and 1 is in Florida.

33. 1 is in Florida, 2 are in Kentucky, and 2 are in Virginia.

34. at least one is in Virginia.

Theater In Exercises 35–38, five men and six women are going to be assigned to a specific row of seats in a theater. If the 11 tickets for the numbered seats are given out at random, find the probability that

35. five women are given the first five seats next to the center aisle.

36. at least one woman is in one of the first five seats.

37. exactly one woman is in one of the first five seats.

38. three women are seated in the first three seats and two men are seated in the next two seats.

39. *Work Shift* Among 24 employees who work at Wendy's, three are brothers. If 6 of the 24 are selected at random to work a late shift, find the probability that the three brothers are selected.

40. *Poker Probability* A full house in poker consists of three of one kind and two of another kind in a five-card hand. For example, if a hand contains three kings and two 5's, it is a full house. If 5 cards are dealt at random from a standard deck of 52 cards, without replacement, find the probability of getting three kings and two 5's.

41. *A Royal Flush* A royal flush consists of the ace, king, queen, jack, and 10 all in the same suit. If 7 cards are dealt at random from a standard deck of 52 cards, find the probability of getting a

a) royal flush in spades.

b) royal flush in any suit.

42. *Restaurant Staff* The staff of a restaurant consists of 25 people, including 8 waiters, 12 waitresses, and 5 cooks. For Mother's Day a total of 9 people will need to be selected to work. If the selections are made at random, determine the probability that 3 waiters, 4 waitresses, and 2 cooks will be selected.

43. *"Dead Man's Hand"* A pair of aces and a pair of 8's is often known as the "dead man's hand." (See the Did You Know on page 731.)

a) Determine the probability of being dealt the dead man's hand (any two aces, any two eights, and one other card that is not an ace or an eight) when 5 cards are dealt, without replacement, from a standard deck of 52 cards.

b) The actual cards "Wild Bill" Hickok was holding when he was shot were the aces of spades and clubs, the 8's of spades and clubs, and the 9 of diamonds. If you are dealt five cards without replacement, determine the probability of being dealt this exact hand.

Challenge Problems/Group Activities

44. *Alternate Seating* If three men and three women are to be assigned at random to six seats in a row at a theater, find the probability they will alternate by gender.

45. *Selecting Officers* A club consists of 15 people including Ali, Kendra, Ted, Alice, Marie, Dan, Linda, and Frank. From the 15 members a president, vice president, and treasurer will be selected at random, and an advisory committee of 5 other individuals will also be selected at random.

a) Find the probability that Ali is selected president, Kendra is selected vice president, Ted is selected treasurer, and the other 5 individuals named form the advisory committee.

b) Find the probability that 3 of the 8 individuals named are selected for the three officers' positions and the other 5 are selected for the advisory board.

46. ***A Marked Deck*** A number is written with a magic marker on each card of a deck of 52 cards. The number 1 is put on the first card, 2 on the second, and so on. The cards are then shuffled and cut. What is the probability that the top 4 cards will be in ascending order? (For example, the top card is 12, the second 22, the third 41, and the fourth 51.)

Recreational Exercise

47. ***Hair*** When the Isle of Flume took its most recent census, the population was 100,002 people. Nobody on the isle has more than 100,001 hairs on his or her head. Determine the probability that at least two people have exactly the same number of hairs on their head.

12.11 BINOMIAL PROBABILITY FORMULA

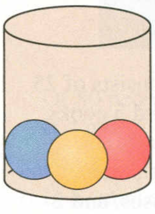


Figure 12.23

Suppose that a basket contains three identical balls, except for their color. One is red, one is blue, and one is yellow (Fig. 12.23). Suppose further that we are going to select three balls *with replacement* from the basket. We can determine specific probabilities by examining the tree diagram shown in Fig. 12.24. Note that 27 different selections are possible, as indicated in the sample space.

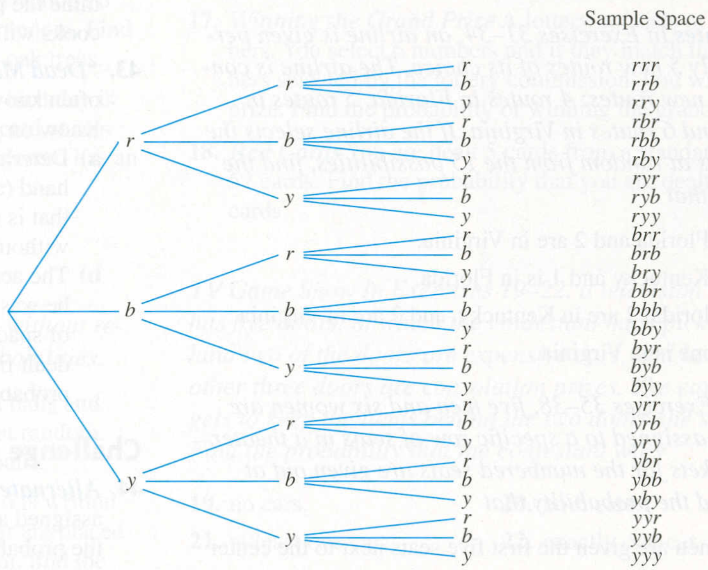


Figure 12.24

Our three selections may yield 0, 1, 2, or 3 red balls. We can determine the probability of selecting exactly 0, 1, 2, or 3 red balls by using the sample space. To determine the probability of selecting 0 red balls, we count those outcomes that do not contain a red ball. There are 8 of them (*bbb, bby, byb, byy, ybb, yby, yyb, yyy*). Thus, the probability of obtaining exactly 0 red balls is 8/27. We determine the probability of selecting exactly 1 red ball by counting the sample points that contain exactly 1 red ball. There are 12 of them. Thus, the probability is 12/27, or 4/9.

We can determine the probability of selecting exactly 2 red balls and exactly 3 red balls in a similar manner. The probabilities of selecting exactly 0, 1, 2, and 3 red balls are illustrated in Table 12.5.

TABLE 12.5 A Probability Distribution for Three Balls Selected with Replacement

Number of Red Balls Selected, (x)	Probability of Selecting the Number of Red Balls, $P(x)$
0	$\frac{8}{27}$
1	$\frac{12}{27}$
2	$\frac{6}{27}$
3	$\frac{1}{27}$
	Sum $= \frac{27}{27} = 1$

Note that the sum of the probabilities is 1. This table is an example of a ***probability distribution***, which shows the probabilities associated with each specific outcome of an experiment. *In a probability distribution, every possible outcome must be listed, and the sum of the probabilities must be 1.*

Let us specifically consider the probability of selecting 1 red ball in 3 selections. We see from Table 12.5 that this probability is $\frac{12}{27}$ or $\frac{4}{9}$. Can we determine this probability without developing a tree diagram? The answer is yes.

Suppose we consider selecting a red ball success, S, and a non–red ball failure, F. Furthermore, suppose we let p represent the probability of success and q the probability of failure on any trial. Then $p = \frac{1}{3}$ and $q = \frac{2}{3}$. We can obtain 1 success in three selections in the following ways:

SFF FSF FFS

We can compute the probabilities of each of these outcomes using the multiplication formula because each of the selections is independent.

$$P(\text{SFF}) = P(\text{S}) \cdot P(\text{F}) \cdot P(\text{F}) = p \cdot q \cdot q = pq^2 = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$$

$$P(\text{FSF}) = P(\text{F}) \cdot P(\text{S}) \cdot P(\text{F}) = q \cdot p \cdot q = pq^2 = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$$

$$P(\text{FFS}) = P(\text{F}) \cdot P(\text{F}) \cdot P(\text{S}) = q \cdot q \cdot p = pq^2 = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$$

$$\text{Sum} = \frac{12}{27} = \frac{4}{9}$$

We obtained an answer of 4/9, the same answer that was obtained using the tree diagram. Note that each of the 3 sets of outcomes above has one success and two failures. Rather than listing all the possibilities containing 1 success and 2 failures, we can use the combination formula to determine the number of possible combinations of 1 success in 3 trials. To do so, evaluate ${}_3C_1$.

$$ {}_3C_1 = \frac{3!}{(3-1)!1!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3 $$

Number of trials (3); Number of successes (1)

Thus, we see that there are 3 ways the 1 success could occur in 3 trials. To compute the probability of 1 success in 3 trials we can multiply the probability of success in any one trial, $p \cdot q^2$, by the number of ways the 1 success can be arranged among the 3 trials, ${}_3C_1$. Thus, the probability of selecting 1 red ball, $P(1)$, in 3 trials may be found as follows,

$$ P(1) = ({}_3C_1)p^1q^2 = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{12}{27} = \frac{4}{9} $$

The binomial probability formula, which we introduce shortly, explains how to obtain expressions like $P(1) = ({}_3C_1)p^1q^2$ and is very useful in finding certain types of probabilities.

To use the binomial probability formula the following three conditions must hold.

To Use the Binomial Probability Formula

1. There are n repeated independent trials.
2. Each trial has two possible outcomes, *success* and *failure*.
3. For each trial, the probability of success (and failure) remains the same.

Before going further, let's discuss why we can use the binomial probability formula to find the probability of selecting a specific number of red balls when three balls are selected with replacement. First, since each trial is performed *with replacement,* the three trials are independent of each other. Second, we may consider selecting a red ball as success and selecting any ball of another color as failure. Third, for each selection, the probability of success (selecting a red ball) is $\frac{1}{3}$, and the probability of failure (selecting a ball of another color) is $\frac{2}{3}$. Now let's discuss the binomial probability formula.

Binomial Probability Formula

The probability of obtaining exactly x successes, $P(x)$, in n independent trials is given by

$$ P(x) = ({}_nC_x)p^xq^{n-x} $$

where p is the probability of success on a single trial and q is the probability of failure on a single trial.

In the formula, p will be a number between 0 and 1, inclusive, and $q = 1 - p$. Therefore, if $p = 0.2$, then $q = 1 - 0.2 = 0.8$. If $p = 3/5$, then $q = 1 - 3/5 = 2/5$. Note that $p + q = 1$ and the values of p and q remain the same for each independent trial. The combination ${}_nC_x$ is called the *binomial coefficient*.

In Example 1, we use the binomial probability formula to solve the same problem we recently solved by using a tree diagram.

EXAMPLE 1 ***Selecting Colored Balls with Replacement***

A basket contains 3 balls: 1 red, 1 blue, and 1 yellow. Three balls are going to be selected with replacement from the basket. Find the probability that

a) no red balls are selected.
b) exactly 1 red ball is selected.
c) exactly 2 red balls are selected.
d) exactly 3 red balls are selected.

SOLUTION:

a) We will consider selecting a red ball a success and selecting a ball of any other color a failure. Since only 1 of the 3 balls is red, the probability of success on any single trial, p, is 1/3. The probability of failure on any single trial, q, is $1 - 1/3 = 2/3$. We are finding the probability of selecting 0 red balls, or 0 successes. Since x represents the number of successes, we let $x = 0$. There are 3 independent selections (or trials), so $n = 3$. In our calculations we will need to evaluate $(\frac{1}{3})^0$. Note that any nonzero number raised to a power of 0 is 1. Thus, $(\frac{1}{3})^0 = 1$. We determine the probability of 0 successes, or $P(0)$, as follows.

$$P(x) = ({}_nC_x)p^x q^{n-x}$$

$$P(0) = ({}_3C_0)\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{3-0}$$

$$= (1)(1)\left(\frac{2}{3}\right)^3$$

$$= \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

b) We are finding the probability of obtaining exactly 1 red ball or exactly 1 success in 3 independent selections. Thus, $x = 1$ and $n = 3$. We find the probability of exactly 1 success, or $P(1)$, as follows.

$$P(x) = ({}_nC_x)p^x q^{n-x}$$

$$P(1) = ({}_3C_1)\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{3-1}$$

$$= 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$$

$$= 3\left(\frac{1}{3}\right)\left(\frac{4}{9}\right) = \frac{4}{9}$$

c) We are finding the probability of selecting exactly 2 red balls in 3 independent trials. Thus, $x = 2$ and $n = 3$. We find $P(2)$ as follows.

$$P(x) = ({}_nC_x)p^x q^{n-x}$$

$$P(2) = {}_3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{3-2}$$

$$= 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$$

$$= 3\left(\frac{1}{9}\right)\left(\frac{2}{3}\right) = \frac{2}{9}$$

d) We are finding the probability of selecting exactly 3 red balls in 3 independent trials. Thus, $x = 3$ and $n = 3$. We find $P(3)$ as follows.

$$P(x) = ({}_nC_x)p^x q^{n-x}$$

$$\begin{aligned} P(3) &= ({}_3C_3)\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{3-3} \\ &= 1\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \\ &= 1\left(\frac{1}{27}\right)(1) = \frac{1}{27} \end{aligned}$$

All the probabilities obtained in Example 1 agree with the answers obtained by using the tree diagram. Whenever you obtain a value for $P(x)$, you should obtain a value between 0 and 1, inclusive. If you obtain a value greater than 1, you have made a mistake.

EXAMPLE 2 *Quality Control for Flashlights*

A manufacturer of flashlights knows that 0.5% of the flashlights produced by the company are defective.

a) Write the binomial probability formula that would be used to determine the probability that exactly x out of n flashlights produced are defective.

b) Write the binomial probability formula that would be used to find the probability that exactly 3 flashlights of 50 produced will be defective.

SOLUTION:

a) We want to find the probability that exactly x flashlights are defective where selecting a defective flashlight is considered success. The probability, P, that an individual flashlight is defective is 0.5%, or 0.005 in decimal form. The probability that a flashlight is not defective, q, is $- 0.005$, or 0.995. The general formula for finding the probability that exactly x out of n flashlights produced are defective is

$$P(x) = ({}_nC_x)p^x q^{n-x}$$

Substituting 0.005 for p and 0.995 for q, we obtain the formula

$$P(x) = ({}_nC_x)(0.005)^x(0.995)^{n-x}$$

b) We want to determine the probability that exactly 3 flashlights out of 50 produced are defective. Thus, $x = 3$ and $n = 50$. Substituting these values into the formula in part (a) gives

$$\begin{aligned} P(3) &= ({}_{50}C_3)(0.005)^3(0.995)^{50-3} \\ &= ({}_{50}C_3)(0.005)^3(0.995)^{47} \end{aligned}$$

The answer may be obtained using a scientific calculator.

EXAMPLE 3 *Weather Forecast Accuracy*

The local weatherperson has been accurate in her temperature forecast 80% of the time. Find the probability that she is accurate

a) exactly 3 of the next 5 days.

b) exactly 4 of the next 4 days.

SOLUTION:

a) We want to find the probability that the forecaster is successful (or accurate) exactly 3 of the next 5 days. Thus, $x = 3$ and $n = 5$. The probability of success on any one day, p, is 80%, or 0.8. The probability of failure, q, is $1 - 0.8 = 0.2$. Substituting these values into the binomial probability formula yields

$$
\begin{aligned}
P(x) &= ({}_nC_x)p^x q^{n-x} \\
P(3) &= ({}_5C_3)(0.8)^3(0.2)^{5-3} \\
&= 10(0.8)^3(0.2)^2 \\
&= 10(0.512)(0.04) \\
&= 0.2048
\end{aligned}
$$

Thus, the probability she is accurate in exactly 3 of the next 5 days is 0.2048.

b) We want to find the probability that she is accurate in each of the next 4 days. Thus, $x = 4$ and $n = 4$. We wish to find $P(4)$.

$$
\begin{aligned}
P(x) &= ({}_nC_x)p^x q^{n-x} \\
P(4) &= ({}_4C_4)(0.8)^4(0.2)^{4-4} \\
&= 1(0.8)^4(0.2)^0 \\
&= 1(0.4096)(1) \\
&= 0.4096
\end{aligned}
$$

TIMELY TIP Now would be a good time to spend a few minutes learning how to evaluate expressions such as $(0.8)^3(0.2)^2$ using your calculator. Read the manual that comes with your calculator to learn the procedure to follow to evaluate exponential expressions. Many scientific calculators have keys for evaluating factorials, and some can be used to evaluate permutations and combinations. Check to see if your calculator can be used to evaluate factorials, permutations, and combinations. Also, check with your instructor to see if these features on your calculator can be used on exams.

EXAMPLE 4 *Blue Eyes*

The probability that an individual selected at random has blue eyes is 0.4. Find the probability that

a) none of four people selected at random has blue eyes.

b) at least one of four people selected at random has blue eyes.

SOLUTION:

a) Success is selecting a person with blue eyes. Thus, $p = 0.4$ and $q = 1 - 0.4 = 0.6$. We want to find the probability of 0 successes in 4 trials. Thus, $x = 0$ and $n = 4$. We find the probability of 0 successes, or $P(0)$, as follows.

$$P(x) = ({}_nC_x)p^x q^{n-x}$$
$$P(0) = ({}_4C_0)(0.4)^0(0.6)^{4-0}$$
$$= 1(1)(0.6)^4$$
$$= 1(1)(0.1296)$$
$$= 0.1296$$

b) The probability that at least one person of the four has blue eyes can be found by subtracting from 1 the probability none of the people has blue eyes. We worked problems of this type in earlier sections of the chapter.

In part (a), we determined the probability that none of the people has blue eyes is 0.1296. Thus,

$$P(\text{at least 1 has blue eyes}) = 1 - P(\text{none has blue eyes})$$
$$= 1 - 0.1296$$
$$= 0.8704$$

SECTION 12.11 EXERCISES

Concept/Writing Exercises

1. What is a probability distribution?
2. What are the three requirements that must be met to use the binomial probability formula?
3. Write the binomial probability formula.
4. In the binomial probability formula what do p and q represent?

Practice the Skills

In Exercises 5–10, assume that each of the n trials is independent and that p is the probability of success on a given trial. Use the binomial probability formula to find P(x).

5. $n = 4, x = 2, p = 0.3$

6. $n = 3, x = 2, p = 0.6$

7. $n = 5, x = 2, p = 0.4$

8. $n = 3, x = 3, p = 0.9$

9. $n = 6, x = 0, p = 0.5$

10. $n = 5, x = 3, p = 0.4$

Problem Solving

11. ***A Dozen Eggs*** An egg distributor determines that the probability that any individual egg has a crack is 0.14.

a) Write the binomial probability formula to determine the probability that exactly x of n eggs are cracked.

b) Write the binomial probability formula to determine the probability that exactly 2 in a one-dozen egg carton are cracked.

12. ***Getting Audited*** The probability that a family selected at random will be audited by the Internal Revenue Service (IRS) is 0.0237.

a) Write the binomial probability formula to determine the probability that exactly x out of n families selected at random will be audited by the IRS.

b) Write the binomial probability formula to determine the probability that exactly 5 of 20 families selected at random will be audited by the IRS.

In Exercises 13–21, use the binomial probability formula to answer the question. Round answers to five decimal places.

13. *Cell Phones* Thirty percent of the cell phone users in Texas use a Motorola phone. Determine the probability that if six cell phone users in Texas are selected at random, exactly four of them use a Motorola cell phone.

14. *Traffic Tickets* In Georgia, the probability that a driver is actually given a ticket when he or she is pulled over for a traffic infraction by a member of the Georgia State Police is 0.6. If eight people who were pulled over are selected at random, determine the probability that exactly five of them were given tickets.

15. *Vision Surgery* Ninety-six percent of Dr. William's LASIK surgery patients end up with 20–30, or better, vision. Find the probability that exactly 2 of her next 3 patients end up with 20–30, or better, vision.

16. *Basketball* Jean Woody makes 80% of her free throws in a basketball game. Find the probability that she makes exactly four of the next six free throws.

17. *Dolphin Drug Care* When treated with the antibiotic resonocyllin, 92% of all dolphins are cured of a particular bacterial infection. If six dolphins with the particular bacterial infection are treated with resonocyllin, find the probability that exactly four are cured.

18. *Manufacturing Light bulbs* A quality control engineer at a GE light bulb plant finds that 1% of its bulbs are defective. Find the probability that exactly two of the next four bulbs made are defective.

19. *Water Heaters* The probability that a specific brand of water heater produces the water temperature it is set to produce is $\frac{4}{5}$. Find the probability that if five of these water heaters are selected at random, exactly four of them will produce the water temperature it is set to produce.

20. *TV Purchases* At a Circuit City store, $\frac{1}{4}$ of those purchasing color televisions purchase a large-screen TV. Find the probability that

a) none of the next four people who purchase a color television at Circuit City purchases a large-screen TV.

b) at least one of the next four people who purchase a color television at Circuit City purchases a large-screen TV.

21. *Supporting a Candidate* Sixty percent of the eligible voting residents of a certain community support Ms. Stein, the incumbent candidate. If five of the residents are selected at random, find the probability that

a) none supports Ms. Stein.

b) at least one supports Ms. Stein.

Challenge Problems/Group Activities

22. *Transportation to Work* In a random sample of 80 working mothers in Duluth, Minnesota, the following data indicating how they get to work were obtained.

Mode of Transportation	Number of Mothers
Car	40
Bus	20
Bike	16
Other	4

If this sample is representative of all working mothers in Duluth, find the probability that exactly three of five working mothers selected at random

a) take a car to work.

b) take a bus to work.

23. *Selecting 6 Cards* Six cards are selected from a deck of playing cards with replacement. Find the probability that

a) exactly three picture cards are obtained.

b) exactly two spades are obtained.

24. *Office Visit* The probability that a person visiting Dr. Guillermo Suarez's office is over 60 years old is 0.7. Find the probability that

a) exactly three of the next five people visiting the office are over 60 years old.

b) at least three of the next five people visiting the office are over 60 years old.

Recreational Exercise

25. *Aruba* The island of Aruba is well known for its beaches and predictable warm sunny weather. In fact, Aruba's weather is so predictable that the daily newspapers don't even bother to print a forecast. Strangely enough, however, on New Year's Eve, as the islanders were counting down the last 10 sec of 2002, it began to rain. What is the probability, from 0 to 1, that 72 hr later the sun will be shining?

CHAPTER 12 SUMMARY

IMPORTANT FACTS

Empirical probability

$$P(E) = \frac{\text{number of times event } E \text{ has occurred}}{\left(\begin{array}{c}\text{total number of times the}\\ \text{experiment has been performed}\end{array}\right)}$$

The law of large numbers

Probability statements apply in practice to a large number of trials, not to a single trial. It is the relative frequency over the long run that is accurately predictable, not individual events or precise totals.

Theoretical probability

$$P(E) = \frac{\text{number of outcomes favorable to } E}{\text{total number of possible outcomes}}$$

The probability of an event that cannot occur is 0. The probability of an event that must occur is 1. Every probability must be a number between 0 and 1 inclusively; that is

$$0 \le P(E) \le 1$$

The sum of the probabilities of all possible outcomes of an event is 1.

$$P(A) + P(\text{not } A) = 1$$

Odds against an event

$$\text{Odds against} = \frac{P(\text{event fails to occur})}{P(\text{event occurs})} = \frac{P(\text{failure})}{P(\text{success})}$$

Odds in favor of an event

$$\text{Odds in favor} = \frac{P(\text{event occurs})}{P(\text{event fails to occur})} = \frac{P(\text{success})}{P(\text{failure})}$$

Expected value

$$E = P_1A_1 + P_2A_2 + P_3A_3 + \cdots + P_nA_n$$

Fair price = expected value + cost to play

Counting principle

If a first experiment can be performed in M distinct ways and a second experiment can be performed in N distinct ways, then the two experiments in that specific order can be performed in $M \cdot N$ distinct ways.

Or and And problems

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Conditional probability

$$P(E_2 | E_1) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)}$$

The **number of permutations** of n items is $n!$.

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

Permutation formula

$$_nP_r = \frac{n!}{(n-r)!}$$

The number of different permutations of n objects where $n_1, n_2, \ldots, n_r$ of the objects are identical is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Combination formula

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Binomial probability formula

$$P(x) = (_nC_x)p^x q^{n-x}$$

CHAPTER 12 REVIEW EXERCISES

12.1–12.11

1. In your own words, explain the law of large numbers.
2. Explain how empirical probability can be used to determine whether a die is "loaded" (not a fair die).
3. *Bicycles* Of 40 people who purchase a bike at a bike shop, 8 purchased a mountain bike. Find the empirical probability that the next person who purchases a bike from that bike shop purchases a mountain bike.
4. *Cards* Select a card from a deck of cards 40 times with replacement and compute the empirical probability of selecting a heart.
5. *Television News* In a small town, 200 people were asked whether they watched ABC, CBS, NBC, or Fox news. The results are indicated below.

Network	Number of People
ABC	80
CBS	30
NBC	55
Fox	35

Find the empirical probability that the next person selected at random from the town watches ABC news.

Digits In Exercises 6–9, each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is written on a piece of paper, and all the pieces of paper are placed in a hat. One number is selected at random. Find the probability that the number selected is

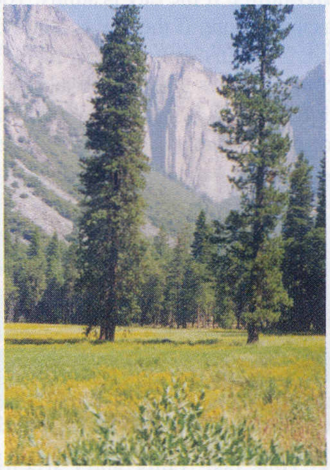
6. even.
7. odd or greater than 5.
8. greater than 2 or less than 5.
9. even and greater than 4.

National Parks In Exercises 10–13, assume that 240 people are selected at random and are asked to name their favorite national park. The results are summarized in the following chart.

National Park	Number of People
Grand Canyon	50
Yosemite	40
Rocky Mountain	35
Great Smoky Mountains	45
Other park	70

If one person from those surveyed is selected at random, find the probability that the person chose

10. Grand Canyon National Park.
11. Yosemite National Park.



Yosemite National Park, CA

12. either Rocky Mountain National Park or Great Smoky Mountains National Park.
13. a park other than Grand Canyon National Park.
14. *Gold Star* Planters Peanuts is having a contest. They indicate that 1 in 10 jars will have a gold star under the cover. Jason Dwyer purchases one jar. Find the odds
    a) against his jar having a star.
    b) in favor of his jar having a star.
15. *Vegetable Mix-up* Nicholas Delaney, a mischievous little boy, has removed labels on the eight cans of vegetables in the cabinet. Nicholas's father knows that there are three cans of corn, three cans of beans, and two cans of carrots. If the father selects and opens one can at random, find the odds against his selecting a can of corn.
16. *Horseracing* The odds against Buttermilk winning the Triple Crown in horse racing are 82:3. Find the probability that Buttermilk wins the Triple Crown.
17. *Jitterbug Contest* The probability that Bob and Sue Nenno will win the jitterbug contest is 0.7. Find the odds in favor of them winning.
18. *Raffle Tickets* A thousand raffle tickets are sold at $2 each. Three prizes of $200 and two prizes of $100 will be awarded. Assume that the tickets are replaced after each selection.
    a) Find the expectation of a person who purchases a ticket.
    b) Find the expectation of a person who purchases three tickets.
    c) Find the fair price to pay for a ticket.
19. *Expectation of a Card* If Cameron selects a picture card from a deck of cards, Lindsey will give him $9. If Cameron does not select a picture card, he must give Lindsey $3.

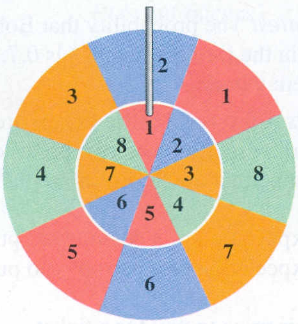
a) Find Cameron's expectation.
b) Find Lindsey's expectation.
c) If Cameron plays this game 100 times, how much can he expect to lose or gain?

20. ***Expected Attendance*** If the day is sunny, 1000 people will attend the baseball game. If the day is cloudy, only 500 people will attend. If it rains, only 100 people will attend. The local meteorologist states that the probability of a sunny day is 0.4, of a cloudy day is 0.5, and of a rainy day is 0.1. Find the number of people that are expected to attend.

21. ***Club Officers*** Tina, Jake, Gina, and Carla form a club. They plan to select a president and a vice president.
a) Construct a tree diagram showing all the possible outcomes.
b) List the sample space.
c) Find the probability that Gina is selected president and Jake is selected vice president.

22. ***A Coin and a Number*** A coin is flipped and then a number from 1 through 4 is selected at random from a bag.
a) Construct a tree diagram showing all the possible outcomes.
b) List the sample space.
c) Find the probability that a head is flipped and an odd number is selected.
d) Find the probability that a head is flipped or an odd number is selected.

Spinning Two Wheels *In Exercises 23–28, the outer and inner wheels are spun.*



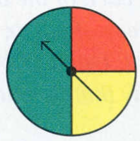
Assuming that the wheels are independent and the outcomes are equally likely, find the probability of obtaining

23. even numbers on both wheels.
24. numbers greater than 5 on both wheels.
25. an odd number on the outer wheel and a number less than 6 on the inner wheel.
26. an even number or a number less than 6 on the outer wheel.
27. an even number or a color other than green on the inner wheel.
28. gold on the outer wheel and a color other than gold on the inner wheel.

Candy Selections *In Exercises 29–32, assume that the Pollingers are going to purchase three bags of candy. The store has only 12 different bags of candies, including 5 different varieties made by Hershey, 4 different varieties made by Nestlé, and 3 different varieties made by H.B. Reese Candy Company. If Mrs. Pollinger selects 3 of these 12 varieties at random, find the probability she selects*

29. 3 varieties of Hershey candies.
30. no varieties of Nestlé candies.
31. at least 1 variety of Nestlé candy.
32. varieties of Hershey, Hershey, Reese in this order.

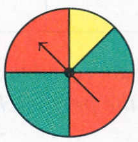
Spinner Probabilities *In Exercises 33–36, assume that the spinner cannot land on a line.*



If spun once, find

33. the probability that the spinner lands on yellow.
34. the odds against and the odds in favor of the spinner landing on yellow.
35. You are awarded \$5 if the spinner lands on red, \$10 if it lands on yellow, and \$20 if it lands on green. Find your expected value.
36. If the spinner is spun twice, find the probability that it lands on red and then green (assume independence).

Spinner Probabilities *In Exercises 37–40, assume that the spinner cannot land on a line.*



If spun once, find

37. the probability that the spinner does not land on green.

38. the odds in favor of and the odds against the spinner landing on green.

39. A person wins $10 if the spinner lands on green, wins $5 if the spinner lands on red, and loses $20 if the spinner lands on yellow. Find the expectation of a person who plays this game.

40. If the spinner is spun three times, find the probability that at least one spin lands on red.

Automobile Quality Control *In Exercises 41–44, a sample of 180 new cars was checked for defects. The following table shows the results of the survey.*

Car	Fewer than Six Defects	Six or More Defects	Total
American built	89	17	106
Foreign built	55	19	74
Total	144	36	180

Find the probability that if one car is selected from this sample, the car has

41. fewer than six defects, given that it is American built.

42. fewer than six defects, given that it is foreign built.

43. six or more defects, given that it is foreign built.

44. six or more defects, given that it is American built.

Neuroscience *In Exercises 45–48, assume that in a neuroscience course the students perform an experiment. Tests are given to determine if people are right brained, left brained, or have no predominance. It is also recorded whether they are right handed or left handed. The following chart shows the results obtained.*

	Right Brained	Left Brained	No Predominance	Total
Right handed	40	130	60	230
Left handed	120	30	20	170
Total	160	160	80	400

If one person who completed the survey is selected at random, find the probability the person selected is

45. right handed.

46. left brained, given that the person is left handed.

47. right handed, given that the person has no predominance.

48. right brained, given that the person is left handed.

49. ***Television Show*** Four contestants are on a television show. There are four different-colored rubber balls in a box, and each contestant gets to pick one from the box. Inside each ball is a slip of paper indicating the amount the contestant has won. The amounts are $10,000, $5000, $2000, and $1000.

a) In how many different ways can the contestants select the balls?

b) What is the expectation of a contestant?

50. ***Spelling Bee*** Five finalists remain in a high school spelling bee. Two will receive $50 each, two will receive $100 each, and one will receive $500. How many different arrangements of prizes are possible?

51. ***Candy Selection*** Mrs. Williams takes her 3 children shopping. Each of her children gets to select a different type of candy that only that child will eat. At the store there are only 10 boxes of candy left, and each is a different type. In how many ways can the 3 children select the candy?

52. ***Astronaut Selection*** Three of nine astronauts must be selected for a mission. One will be the captain, one will be the navigator, and one will perform scientific experiments. In how many ways can a three-person crew be selected so that each person has a different assignment?

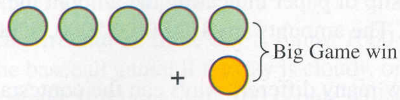
53. ***Medicine*** Dr. Goldberg has three doses of serum for influenza type A. Six patients in the office require the serum. In how many different ways could Dr. Goldberg dispense the serum?

54. ***Dogsled***

a) Ten of 15 huskies are to be selected to pull a dogsled. In how many ways can this selection be made?

b) How many different arrangements of the 10 huskies on a dogsled are possible?

55. ***Mega Millions*** The Big Game Mega Millions is a multistate lottery game offered in Georgia, Illinois, Maryland, Massachusetts, Michigan, New Jersey, New York, Ohio, Texas, Virginia, and Washington. To play, you select 5 numbers from 1 through 52 and 1 Big Money Ball number from 1 through 52. If you win the Big Game by matching all 6 numbers, your guaranteed minimum payoff is \$10 million. If you match the 5 numbers but do not match the Big Money number, your guaranteed payoff is \$175,000.



a) What is the probability you match the 5 numbers?
b) What is the probability you have a Big Game win?

56. **Parent-Teacher Committee** A committee of 6 is to be formed from 8 parents and 10 teachers. If the committee is to consist of 2 parents and 4 teachers, how many different combinations are possible?

57. **Selecting Test Subjects** In a psychology research laboratory, one room contains eight men and another room contains five women. Three men and two women are to be selected at random to be given a psychological test. How many different combinations of these people are possible?

58. **Choosing Two Aces** Two cards are selected at random, without replacement, from a deck of 52 cards. Find the probability that two aces are selected (use combinations).

Color Chips *In Exercises 59–62, a bag contains five red chips, three white chips, and two blue chips. Three chips are to be selected at random, without replacement. Find the probability that*

59. all are red.

60. the first two are red and the third is blue.

61. the first is red, the second is white, and the third is blue.

62. at least one is red.

Magazines *In Exercises 63–66, on a table in a doctor's office are six* Newsweek *magazines, five* U.S. News and World Report *magazines, and three* Time *magazines. If Ramona Cleary randomly selects three magazines, find the probability that*

63. three *U.S. News and World Report* magazines were selected.

64. two *Newsweek* magazines and one *Time* magazine were selected.

65. no *Newsweek* magazines were selected.

66. at least one *Newsweek* magazine was selected.

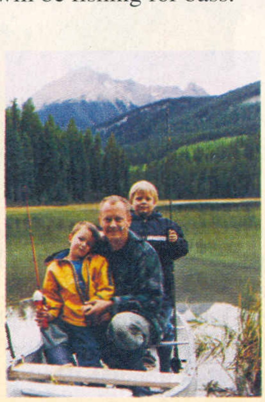
67. **New Homes** In the community of Spring Hill, 60% of the homes purchased cost more than \$125,000.
a) Write the binomial probability formula to determine the probability that exactly x of the next n homes purchased in Spring Hill cost more than \$125,000.
b) Write the binomial probability formula to determine the probability that exactly 75 of the next 100 home purchases cost more than \$125,000.

68. **Long-Stemmed Roses** At the Floyd's Flower Shop, $\frac{1}{5}$ of those ordering flowers select long-stemmed roses. Find the probability that exactly 3 of the next 5 customers ordering flowers select long-stemmed roses.

69. **Taking a Math Course** During any semester at City College, 60% of the students are taking a mathematics course. Find the probability that of four students selected at random
a) none is taking a mathematics course this semester.
b) at least one is taking a mathematics course this semester.

CHAPTER 12 TEST

1. **Fishing** Of 30 people who went fishing in Lake Mead, 22 were fishing for bass. Find the empirical probability that the next person who goes fishing in Lake Mead will be fishing for bass.



One Sheet of Paper *In Exercises 2–5, each of the numbers 1–9 is written on a sheet of paper, and the nine sheets of paper are placed in a hat. If one sheet of paper is selected at random from the hat, find the probability that the number selected is*

2. greater than 7.

3. odd.

4. even or greater than 4.

5. odd and greater than 4.

Two Sheets of Paper *In Exercises 6–9, if 2 of the same 9 sheets of paper mentioned above are selected, without replacement, from the hat, find the probability that*

6. both numbers are greater than 5.

7. both numbers are even.

8. the first number is odd and the second number is even.
9. neither of the numbers is greater than 6.
10. One card is selected at random from a deck of cards. Find the probability that the card selected is a red card or a picture card.

One Chip and One Die In Exercises 11–15, one colored chip—red, blue, or green—is selected at random, and a die is rolled.

11. Use the counting principle to determine the number of sample points in the sample space.
12. Construct a tree diagram illustrating all the possible outcomes, and list the sample space.

In Exercises 13–15, by observing the sample space of the chips and die, determine the probability of obtaining

13. the color blue and the number 1.
14. the color blue or the number 1.
15. a color other than red or an odd number.
16. *Passwords* A personal password for an Internet brokerage account is to consist of a digit, followed by two letters, followed by two digits. Find the number of personal codes possible if the first digit cannot be zero and repetition is permitted.
17. *Puppies* A litter of collie puppies consists of four males and five females. If one of the puppies is selected at random, find the odds

a) against the puppy being male.
b) in favor of the puppy being female.

18. *Tennis Odds* The odds against Aimee Calhoun winning the Saddlebrook Tennis Tournament are 5:2. Find the probability that Aimee wins the tournament.
19. *Pick a Card* You get to select one card at random from a deck of cards. If you pick a club, you win \$8. If you pick a heart, you win \$4. If you pick any other suit, you lose \$6. Find your expectation for this game.
20. *Cars and SUVs* The number of cars and the number of sport utility vehicles (SUVs) going through the toll gates of two bridges is recorded. The results are shown below.

Bridge	Cars	SUVs	Total
George Washington	120	106	226
Golden Gate	94	136	230
Total	214	242	456

The toll booths at Golden Gate bridge

If one of these vehicles going over the bridges is selected at random, find the probability that
a) it is a car.
b) it is going over the Golden Gate Bridge.
c) it is an SUV, given that it is going over the Golden Gate Bridge.
d) it is going over the George Washington Bridge, given that it is a car.

21. *Awarding Prizes* Three of six people are to be selected and given small prizes. One will be given a book, one will be given a calculator, and one will be given a \$10 bill. In how many different ways can these prizes be awarded?

Quality Control In Exercises 22 and 23, a bin contains a total of 20 batteries, of which 8 are defective. If you select 2 at random, without replacement, find the probability that

22. none of the batteries is good.
23. at least one battery is good.
24. *Apples from a Bucket* Five green apples and seven red apples are in a bucket. Five apples are to be selected at random, without replacement. Find the probability that two green apples and three red apples are selected.
25. *University Admission* The probability that a person is accepted for admission to a specific university is 0.1. Find the probability that exactly three of the next five people who apply to the university get accepted.

GROUP PROJECTS

The Probability of an Exact Measured Value

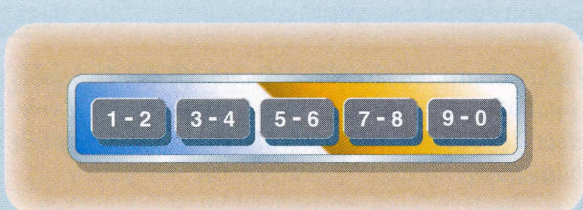
1. Your car's speedometer indicates that you are traveling at 65 mph. What is the probability that you are traveling at *exactly* 65 mph? Explain your answer.

Taking an Exam

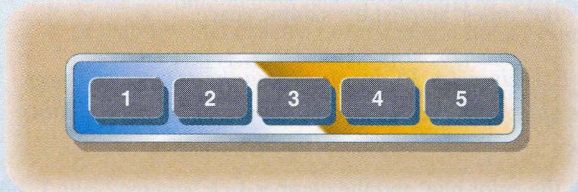
2. A 10-question multiple-choice exam is given, and each question has five possible answers. Pascal Gonyo takes this exam and guesses at every question. Use the binomial probability formula to find the probability (to 5 decimal places) that
   - a) he gets exactly 2 questions correct.
   - b) he gets no questions correct.
   - c) he gets at least 1 question correct (use the information from part (b) to answer this part).
   - d) he gets at least 9 questions correct.
   - e) Without using the binomial probability formula, determine the probability that he gets exactly 2 questions correct.
   - f) Compare your answers to parts (a) and (e). If they are not the same explain why.

Keyless Entry

3. Many cars have keyless entry. To open the lock you may press a 5-digit code on a set of buttons like that illustrated. The code may include repeated digits like 11433 or 55512.



   - a) How many different 5-digit codes can be made using the 10 digits if repetition is permitted?
   - b) How many different ways are there of pressing 5 buttons if repetition is allowed?
   - c) A burglar is going to press 5 buttons at random, with repetition allowed. Find the probability that the burglar hits the sequence to open the door.
   - d) Suppose that each button had only one number associated with it as illustrated below. How many different 5-digit codes can be made with the 5 digits if repetition is permitted?



   - e) Using the buttons labeled 1–5, how many different ways are there to press 5 buttons if repetition is allowed?
   - f) A burglar is going to press 5 buttons of those labeled 1–5 at random with repetition allowed. Find the probability that the burglar hits the sequence to open the door.
   - g) Is a burglar more likely, is he or she less likely, or does he or she have the same likelihood of pressing 5 buttons and opening the car door if the buttons are labeled as in the first illustration or as in the second illustration? Explain your answer.
   - h) Can you see any advantages in labeling the buttons as in the first illustration? Explain.*

*In actuality, in most cars that have key pads like that shown on the bottom left, each key acts as if it contains a single digit. For example, if your code is 7, 9, 5, 1, 3 the code 8, 0, 6, 2, 4 will unlock the door. The extra numbers, in effect, give the owner a false sense of security.