CHAPTER 2

Set building is a fundamental learning tool for even the smallest children. As babies, they learn to distinguish "me" from "mom" and "dad." As toddlers, they learn to distinguish and categorize objects as members of a set according to size, color, or shape. The TV show *Sesame Street* teaches children set building in the game "One of these things is not like the other."

SETS

One of the most basic human impulses is to sort and classify things. Consider yourself, for example. How many different sets are you a member of? You might start with some simple categories, such as whether you are male or female, your age group, and the state you live in. Then you might think about your family's ethnic group, socioeconomic group, and nationality. These are but some of the many ways you could describe yourself to other people.

Of what use is this activity of categorization? As you will see in this chapter, putting elements into sets helps you order and arrange your world. It allows you to deal with large quantities of information. Set building is a learning tool that helps answer the question, "What are the characteristics of this group?"

Sets underlie other mathematical topics, such as logic and abstract algebra. In fact, the book *Eléments de Mathématique,* written by a group of French mathematicians under the pseudonym Nicolas Bourbaki, states, "Nowadays it is possible, logically speaking, to derive the whole of known mathematics from a single source, the theory of sets."

2.1 SET CONCEPTS

We encounter sets in many different ways every day of our lives. A ***set*** is a collection of objects, which are called ***elements*** or ***members*** of the set. For example, the United States is a collection or set of 50 states. The 50 individual states are the members or elements of the set that is called the United States.

A set is ***well defined*** if its contents can be clearly determined. The set of U.S. presidents is a well-defined set because its contents, the presidents, can be named. The set of the three best movies is not a well-defined set because the word *best* is interpreted differently by different people. In this text, we use only well-defined sets.

Three methods are commonly used to indicate a set: (1) description, (2) roster form, and (3) set-builder notation.

The method of indicating a set by ***description*** is illustrated in Example 1.

EXAMPLE 1 *Description of Sets*

Write a description of the set containing the elements Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday.

SOLUTION: The set is the days of the week.

Listing the elements of a set inside a pair of ***braces***, { }, is called ***roster form***. The braces are an essential part of the notation because they identify the contents as a set. For example, $\{1, 2, 3\}$ is notation for the set whose elements are 1, 2, and 3, but (1, 2, 3) and [1, 2, 3] are not sets because parentheses and brackets do not indicate a set. For a set written in roster form, commas separate the elements of the set. The order in which the elements are listed is not important.

Sets are generally named with capital letters. For example, the name commonly selected for the set of ***natural numbers*** or ***counting numbers*** is N.

Natural Numbers

$N = \{1, 2, 3, 4, 5, \ldots\}$

The three dots after the 5, called an *ellipsis,* indicate that the elements in the set continue in the same manner. An ellipsis followed by a last element indicates that the elements continue in the same manner up to and including the last element. This notation is illustrated in Example 2(b).

EXAMPLE 2 *Roster Form of Sets*

Express the following in roster form.

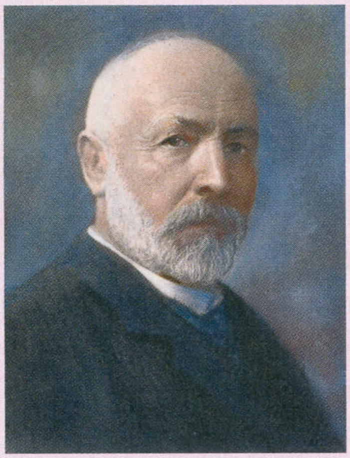
a) Set A is the set of natural numbers less than 6.
b) Set B is the set of natural numbers less than or equal to 50.
c) Set P is the set of planets in Earth's solar system.

SOLUTION:

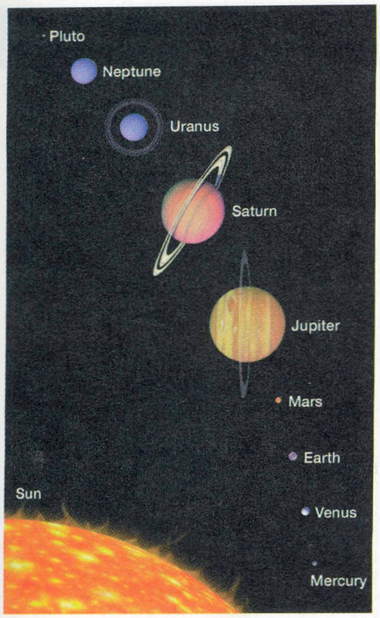
a) The natural numbers less than 6 are 1, 2, 3, 4, and 5. Thus, set A in roster form is $A = \{1, 2, 3, 4, 5\}$.

PROFILE IN MATHEMATICS

GEORG CANTOR



Georg Cantor (1845–1918), born in St. Petersburg, Russia, is recognized as the founder of set theory. Cantor's creative work in mathematics was nearly lost when his father insisted that he become an engineer rather than a mathematician. His two major books on set theory, *Foundations of General Theory of Aggregates* and *Contributions to the Founding of the Theory of Transfinite Numbers,* were published in 1883 and 1895, respectively.



The planets of Earth's solar system.

b) $B = \{1, 2, 3, 4, \ldots, 50\}$. The 50 after the ellipsis indicates that the elements continue in the same manner up to and including the number 50.

c) $P = \{$Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto$\}$

EXAMPLE 3 *The Word* Inclusive

Express the following in roster form.

a) The set of natural numbers between 5 and 8.

b) The set of natural numbers between 5 and 8, inclusive.

SOLUTION:

a) $A = \{6, 7\}$

b) $B = \{5, 6, 7, 8\}$. Note that the word *inclusive* indicates that the values of 5 and 8 are included in the set.

The symbol $\in$, read, is an element of, is used to indicate membership in a set. In Example 3, since 6 is an element of set A, we write $6 \in A$. This may also be written $6 \in \{6, 7\}$. We may also write $8 \notin A$, meaning that 8 is not an element of set A.

Set-builder notation (sometimes called *set-generator notation*) may be used to symbolize a set. Set-builder notation is frequently used in algebra. The following example illustrates its form.

D	$=$	$\{$	x	$\vert$	Condition(s) $\}$
↑	↑	↑	↑	↑	↑
Set D	is	the set of	all elements x	such that	the condition(s) x must meet in order to be a member of the set.

Consider $E = \{x \mid x \in N \text{ and } x > 10\}$. The statement is read: "Set E is the set of all the elements x such that x is a natural number and x is greater than 10." The conditions that x must meet to be a member of the set are $x \in N$, which means that x must be a natural number, and $x > 10$, which means that x must be greater than 10. The numbers that meet both conditions are the set of natural numbers greater than 10. Set E in roster form is

$$E = \{11, 12, 13, 14, \ldots\}$$

EXAMPLE 4 *Using Set-Builder Notation*

a) Write set $B = \{1, 2, 3, 4, 5\}$ in set-builder notation.

b) Write, in words, how you would read set B in set-builder notation.

SOLUTION:

a) Since set B consists of the natural numbers less than 6, we write

$$B = \{x \mid x \in N \text{ and } x < 6\}$$

Another acceptable answer is $B = \{x \mid x \in N \text{ and } x \leq 5\}$.

b) Set B is the set of all elements x such that x is a natural number and x is less than 6. ▲

EXAMPLE 5 *Roster Form to Set-Builder Notation*

a) Write set $C = \{$North America, South America, Europe, Asia, Australia, Africa, Antarctica$\}$ in set-builder notation.

b) Write in words how you would read set C in set-builder notation.

SOLUTION:

a) $C = \{x \mid x \text{ is a continent}\}$.

b) Set C is the set of all elements x such that x is a continent. ▲

EXAMPLE 6 *Set-Builder Notation to Roster Form*

Write set $A = \{x \mid x \in N \text{ and } 2 \leq x < 8\}$ in roster form.

SOLUTION: $A = \{2, 3, 4, 5, 6, 7\}$ ▲

EXAMPLE 7 *Busiest Ports*

The chart shows the 10 busiest U.S. ports in 2000, ranked by tonnage handled. Also given is a map of Texas and its ports. Let set T be the set of ports in Texas that are among the 10 busiest ports in the United States. Write set T in roster form.

Ten Busiest Ports in the U.S., 2000	Total Tonnage
South Louisiana, LA, Port of	217,756,734
Houston, TX	191,419,265
New York, NY and NJ, Port of	138,669,879
New Orleans, LA	90,768,449
Corpus Christi, TX	83,124,950
Beaumont, TX	82,652,554
Huntington-Tristate, WV	78,867,987
Long Beach, CA	70,149,684
Baton Rouge, LA	65,631,084
Texas City, TX	61,585,891

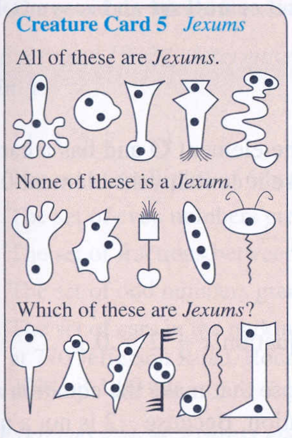
Source: Corps of Engineers, Department of the U.S. Army, U.S. Department of Defense

SOLUTION: By examining the map and the chart we find that four cites appear on both the map and the chart. They are Beaumont, Corpus Christi, Houston, and Texas City. Thus, set $T = \{$Beaumont, Corpus Christi, Houston, Texas City$\}$. ▲

A set is said to be ***finite*** if it either contains no elements or the number of elements in the set is a natural number. The set $B = \{2, 4, 6, 8, 10\}$ is a finite set because the number of elements in the set is 5, and 5 is a natural number. A set that is not finite is said to be ***infinite***. The set of counting numbers is one example of an infinite set. Infinite sets are discussed in more detail in Section 2.6.

DID YOU KNOW

Creature Cards



We learn to group objects according to what we see as the relevant distinguishing characteristics. One way used by educators to measure this ability is through visual cues. An example can be seen in this test, called "Creature Cards," offered by the Education Development Center. How would you describe membership in the set of Jexums?

Another important concept is equality of sets.

> Set A is **equal** to set B, symbolized by $A = B$, if and only if set A and set B contain exactly the same elements.

For example, if set $A = \{1, 2, 3\}$ and set $B = \{3, 1, 2\}$, then $A = B$ because they contain exactly the same elements. The order of the elements in the set is not important. If two sets are equal, both must contain the same number of elements. The number of elements in a set is called its *cardinal number*.

> The **cardinal number** of set A, symbolized by $n(A)$, is the number of elements in set A.

Both set $A = \{1, 2, 3\}$ and set $B = \{\text{England, Brazil, Japan}\}$ have a cardinal number of 3; that is, $n(A) = 3$, and $n(B) = 3$. We can say that set A and set B both have a cardinality of 3.

Two sets are said to be *equivalent* if they contain the same number of elements.

> Set A is **equivalent** to set B if and only if $n(A) = n(B)$.

Any sets that are equal must also be equivalent. Not all sets that are equivalent are equal, however. The sets $D = \{\text{a, b, c}\}$ and $E = \{\text{apple, orange, pear}\}$ are equivalent, since both have the same cardinal number, 3. Because the elements differ, however, the sets are not equal.

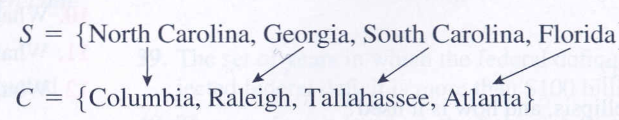
Two sets that are equivalent or have the same cardinality can be placed in ***one-to-one correspondence***. Set A and set B can be placed in one-to-one correspondence if every element of set A can be matched with exactly one element of set B and every element of set B can be matched with exactly one element of set A. For example, there is a one-to-one correspondence between the student names on a class list and the student identification numbers because we can match each name with a student identification number.

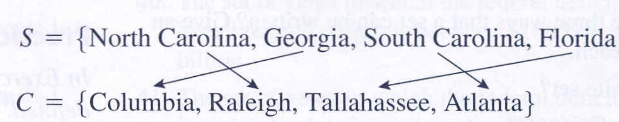
Consider set S, states, and set C, state capitals.

$$S = \{\text{North Carolina, Georgia, South Carolina, Florida}\}$$

$$C = \{\text{Columbia, Raleigh, Tallahassee, Atlanta}\}$$

Two different one-to-one correspondences for sets S and C follow.





Other one-to-one correspondences between sets S and C are possible. Do you know which capital goes with which state?

Null or Empty Set

Some sets do not contain any elements, such as the set of zebras that are in this room.

The set that contains no elements is called the **empty set** or **null set** and is symbolized by { } or Ø.

Note that {Ø} is not the empty set. This set contains the element Ø and has a cardinality of 1. The set {0} is also not the empty set because it contains the element 0. It also has a cardinality of 1.

EXAMPLE 8 *Natural Number Solutions*

Indicate the set of natural numbers that satisfies the equation $x + 2 = 0$.

SOLUTION: The values that satisfy the equation are those that make the equation a true statement. Only the number −2 satisfies this equation. Because −2 is not a natural number, the solution set of this equation is { } or Ø. ▲

Universal Set

Another important set is a ***universal set***.

A **universal set,** symbolized by U, is a set that contains all the elements for any specific discussion.

When a universal set is given, only the elements in the universal set may be considered when working the problem. If, for example, the universal set for a particular problem is defined as $U = \{1, 2, 3, 4, \ldots, 10\}$, then only the natural numbers 1 through 10 may be used in that problem.

SECTION 2.1 EXERCISES

Concept/Writing Exercises

In Exercises 1–12, answer each question with a complete sentence.

1. What is a set?
2. What is an ellipsis, and how is it used?
3. What are the three ways that a set can be written? Give an example of each.
4. What is a finite set?
5. What is an infinite set?
6. What are equal sets?
7. What are equivalent sets?
8. What is the cardinal number of a set?
9. What is the empty set?
10. What are the two ways to indicate the empty set?
11. What does a one-to-one correspondence of two sets mean?
12. What is a universal set?

Practice the Skills

In Exercises 13–18, determine whether each set is well defined.

13. The set of people who own large dogs
14. The set of the best Internet web sites

15. The set of states that have a common border with Colorado
16. The set of the four states in the United States having the largest areas
17. The set of astronauts who walked on the moon
18. The set of the nicest entertainers

In Exercises 19–24, determine whether each set is finite or infinite.

19. $\{1, 3, 5, 7, \ldots\}$
20. The set of multiples of 6 between 0 and 90
21. The set of even numbers greater than 19
22. The set of fractions between 1 and 2
23. The set of odd numbers greater than 15
24. The set of cars in the parking lot at the Home Depot store at 770 Jefferson Road, Rochester, New York, on August 1, 2004, at 1:00 P.M.

In Exercises 25–34, express each set in roster form. You may need to use a world almanac or some other reference source.

25. The set of oceans in the world
26. The set of states in the United States whose names begin with the letter I
27. The set of natural numbers between 10 and 178
28. $C = \{x \mid x + 6 = 10\}$
29. $B = \{x \mid x \in N \text{ and } x \text{ is even}\}$
30. The set of states west of the Mississippi River that have a common border with the state of Florida
31. The set of football players over the age of 70 who are still playing in the National Football League
32. The set of states in the United States that have no common border with any other state
33. $E = \{x \mid x \in N \text{ and } 6 \leq x < 72\}$
34. The set of professional baseball players in the major leagues who have hit at least 70 home runs in a season prior to 2002

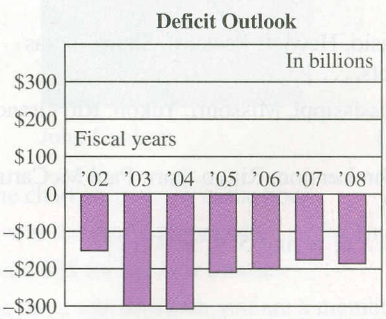
The list above and to the right shows the estimated price of the seven best-selling digital cameras, ranked by market share, in October 2000. In Exercises 35–38, use the list to represent each of the sets in roster form. Let the seven cameras in the list represent the universal set.

35. The set of best-selling digital cameras with an estimated price greater than $500
36. The set of best-selling digital cameras with an estimated price less than $300
37. The set of best-selling digital cameras with an estimated price between $250 and $650
38. The set of best-selling digital cameras with an estimated price between $500 and $800

Camera	Estimated Price
1. Sony Mavica FD-73	$400
2. Olympus D-360L	$290
3. Sony DSC-S50	$550
4. Sony DSC-S70	$750
5. Kodak DC215	$310
6. H-P Photo Smart C315	$300
7. Sony Mavica FD-90	$700

Source: PC Data (Rochester Democrat and Chronicle)

The following graph shows the federal deficit, in billions of dollars, for 2002 and 2003 and the projected federal deficit for the years 2004–2008. In exercises 39–42, use the graph to represent each of the sets in roster form.



Source: White House Office of Management and Budget

39. The set of years in which the federal deficit or the projected federal deficit is more than $100 billion
40. The set of years in which the federal deficit or the projected federal deficit is between $100 billion and $250 billion
41. The set of years in which the federal deficit or the projected federal deficit is less than $250 billion
42. The set of years in which the federal deficit or the projected federal deficit is more than $250 billion

In Exercises 43–50, express each set in set-builder notation.

43. $B = \{4, 5, 6, 7, 8, 9, 10\}$
44. $A = \{1, 2, 3, 4, 5, 6, 7\}$
45. $C = \{3, 6, 9, 12, \dots\}$
46. $D = \{5, 10, 15, 20, \dots\}$
47. E is the set of odd natural numbers
48. A is the set of national holidays in the United States in September
49. C is the set of months that contain less than 30 days
50. $F = \{15, 16, 17, \dots, 100\}$

In Exercises 51–58, write a description of each set.

51. $A = \{1, 2, 3, 4, 5, 6, 7\}$
52. $D = \{4, 8, 12, 16, 20, \dots\}$
53. $V = \{a, e, i, o, u\}$
54. $S = \{$Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, Sneezy$\}$

55. $C = \{$Casio, Hewlett-Packard, Sharp, Texas Instruments, ...$\}$
56. $B = \{$Mississippi, Missouri, Yukon, Rio Grande, Arkansas$\}$
57. $B = \{$John Lennon, Ringo Starr, Paul McCartney, George Harrison$\}$
58. $E = \{x \mid x \in N \text{ and } 5 < x \leq 12\}$

The following list shows the top 10 media markets, in order, for advertisements for the 2000 elections of the president and members of Congress, through October 10, 2000. In Exercises 59–62, use the list to represent each of the sets in roster form. Let the 10 markets represent the universal set.

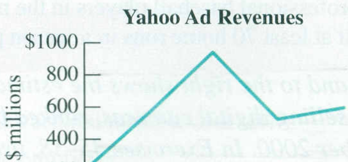
59. $\{x \mid x$ is a city in which the number of advertisements was greater than 18,000$\}$
60. $\{x \mid x$ is a city in which the number of advertisements was less than 10,000$\}$
61. $\{x \mid x$ is a city in which the number of advertisements was between 12,500 and 13,000$\}$
62. $\{x \mid x$ is a city in which the number of advertisements was between 13,000 and 14,000$\}$

Market	Number of Ads
1. St. Louis	18,755
2. Kansas City	14,872
3. Seattle	14,234
4. Detroit	13,490
5. Spokane	13,191
6. Grand Rapids	12,436
7. Flint-Saginaw	11,797
8. Philadelphia	11,006
9. Louisville	10,345
10. Scranton	9016

Source: The Hotline and the Brennan Center for Justice

St. Louis, MO

The following graph shows the advertising revenues for Yahoo, in millions of dollars, for the years 1998–2002. In Exercises 63–66, use the graph to represent each of the sets in roster form.



Source: *Newsweek*

63. $\{x \mid x$ is a year in which advertising revenues exceeded \$400 million$\}$
64. $\{x \mid x$ is a year in which advertising revenues were less than \$300 million$\}$
65. $\{x \mid x$ is a year in which advertising revenues exceeded \$500 million but were less than \$800 million$\}$

66. $\{x \mid x$ is a year in which advertising revenues exceeded \$300 million but were less than \$500 million$\}$

In Exercises 67–74, state whether each statement is true or false. If false, give the reason.

67. $\{b\} \in \{a, b, c, d, e, f\}$
68. $b \in \{a, b, c, d, e, f\}$
69. $h \in \{a, b, c, d, e, f\}$
70. Cat in the Hat $\in$ {characters created by Dr. Seuss}
71. $3 \notin \{x \mid x \in N$ and x is odd$\}$
72. Maui $\in$ {capital cities in the United States}
73. *Titanic* $\in$ {top 10 motion pictures with the greatest revenues}
74. $2 \in \{x \mid x$ is an odd natural number$\}$

In Exercises 75–78, for the sets $A = \{2, 4, 6, 8\}$, $B = \{1, 3, 7, 9, 13, 21\}$, $C = \{\ \}$, and $D = \{\#, \&, \%, \square, *\}$, determine

75. $n(A)$.
76. $n(B)$.
77. $n(C)$.
78. $n(D)$.

In Exercises 79–84, determine whether the pairs of sets are equal, equivalent, both, or neither.

79. A = {circle, triangle, square}, B = {triangle, circle, square}
80. $A = \{7, 9, 10\}$, $B = \{a, b, c\}$
81. A = {grapes, apples, oranges}, B = {grapes, peaches, apples, oranges}
82. A is the set of collies. B is the set of dogs.
83. A is the set of letters in the word *tap*. B is the set of letters in the word *ant*.
84. A is the set of states. B is the set of state capitals.

Problem Solving

85. Set-builder notation is often more versatile and efficient than listing a set in roster form. This versatility is illustrated with the two sets.

$$A = \{x \mid x \in N \text{ and } x > 2\}$$
$$B = \{x \mid x > 2\}$$

a) Write a description of set A and set B.
b) Explain the difference between set A and set B. (*Hint:* Is $4\frac{1}{2} \in A$? Is $4\frac{1}{2} \in B$?)
c) Write set A in roster form.
d) Can set B be written in roster form? Explain your answer.

86. Start with sets

$$A = \{x \mid 2 < x \leq 5 \text{ and } x \in N\}$$

and

$$B = \{x \mid 2 < x \leq 5\}$$

a) Write a description of set A and set B.
b) Explain the difference between set A and set B.
c) Write set A in roster form.
d) Can set B be written in roster form? Explain your answer.

A cardinal number answers the question "How many?" An ***ordinal number*** *describes the relative position that an element occupies. For example, Molly's desk is the third desk from the aisle.*

In Exercises 87–90, determine whether the number used is a cardinal number or an ordinal number.

87. John Grisham has written 12 books.

John Grisham

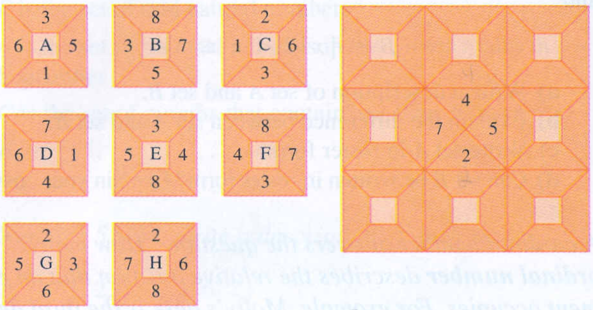
88. Study the chart on page 25 in the book.
89. Lincoln was the sixteenth president of the United States.
90. Emily paid \$35 for her new blouse.
91. Describe three sets of which you are a member.
92. Describe three sets that have no members.
93. Write a short paragraph explaining why the universal set and the empty set are necessary in the study of sets.

Challenge Problem/Group Activity

94. a) In a given exercise, a universal set is not specified, but we know that actor Brad Pitt is a member of the universal set. Describe five different possible universal sets of which Brad Pitt is a member.
b) Write a description of one set that includes all the universal sets in part (a).

Recreational Mathematics

95. ***Face to Face*** Place the eight squares on the left into the diagram on the right so that two squares with a common border will have the same number on both sides of the border. Do not turn the squares or rearrange the numbers within each square.



Internet/Research Activity

96. Georg Cantor is recognized as the founder and a leader in the development of set theory. Do research and write a paper on his life and his contributions to set theory and to the field of mathematics. References include history of mathematics books, encyclopedias, and the Internet.

2.2 SUBSETS

In our complex world, we often break larger sets into smaller more manageable sets, called *subsets*. For example, consider the set of people in your class. Suppose we categorize the set of people in your class according to the first letter of their last name (the A's, B's, C's, etc.). When we do this, each of these sets may be considered a subset of the original set. Each of these subsets can be separated further. For example, the set of people whose last name begins with the letter A can be categorized as either male or female or by their age. Each of these collections of people is also a subset. A given set may have many different subsets.

> Set A is a **subset** of set B, symbolized by $A \subseteq B$, if and only if all the elements of set A are also elements of set B.

The symbol $A \subseteq B$ indicates that "set A is a subset of set B." The symbol $\nsubseteq$ is used to indicate "is not a subset." Thus, $A \nsubseteq B$ indicates that set A is not a subset of set B. *To show that set A is not a subset of set B, we must find at least one element of set A that is not an element of set B.*

EXAMPLE 1 *A Subset?*

Determine whether set A is a subset of set B.

a) $A = \{\text{blue jay, robin, cardinal}\}$
$B = \{\text{blue jay, robin, chickadee, cardinal}\}$

b) $A = \{2, 3, 4, 5\}$ $B = \{2, 3\}$

c) $A = \{x \mid x \text{ is a yellow fruit}\}$

$B = \{x \mid x \text{ is a red fruit}\}$

d) $A = \{\text{cassette, compact disc, videotape}\}$

$B = \{\text{compact disc, videotape, cassette}\}$

SOLUTION:

a) All the elements of set A are contained in set B, so $A \subseteq B$.

b) The elements 4 and 5 are in set A but not in set B, so $A \not\subseteq B$ (A is not a subset of B). In this example, however, all the elements of set B are contained in set A; therefore, $B \subseteq A$.

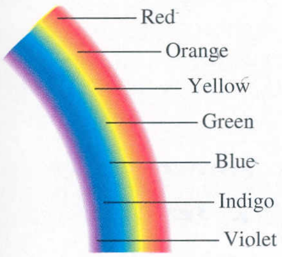
c) There are fruits, such as bananas, that are in set A that are not in set B, so $A \not\subseteq B$.

d) All the elements of set A are contained in set B, so $A \subseteq B$. Note that set A = set B.

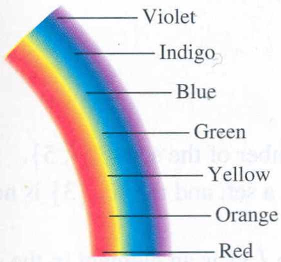
▲

DID YOU KNOW

Rainbows



Colors of primary rainbow



Colors of secondary rainbow

Most rainbows we see are primary rainbows, but there are rare moments when a second, fainter rainbow can be seen behind the first. In this secondary rainbow, the light pattern has been reversed. Both rainbows contain the same set of colors, so each set of colors is a subset of the other.

Proper Subsets

Set A is a **proper subset** of set B, symbolized by $A \subset B$, if and only if all the elements of set A are elements of set B and set $A \neq$ set B (that is, set B must contain at least one element not in set A).

Consider the sets $A = \{\text{red, blue, yellow}\}$ and $B = \{\text{red, orange, yellow, green, blue, violet}\}$. Set A is a *subset* of set B, $A \subseteq B$, because every element of set A is also an element of set B. Set A is also a *proper subset* of set B, $A \subset B$, because set A and set B are not equal. Now consider $C = \{\text{car, bus, train}\}$ and $D = \{\text{train, car, bus}\}$. Set C is a subset of set D, $C \subseteq D$, because every element of set C is also an element of set D. Set C, however, is not a proper subset of set D, $C \not\subset D$, because set C and set D are equal sets.

EXAMPLE 2 *A Proper Subset?*

Determine whether set A is a proper subset of set B.

a) $A = \{\text{refrigerator, microwave, dishwasher}\}$

$B = \{\text{stove, refrigerator, microwave, dishwasher, garbage disposal}\}$

b) $A = \{a, b, c, d\}$ $B = \{a, c, b, d\}$

SOLUTION:

a) All the elements of set A are contained in set B, and sets A and B are not equal; thus, $A \subset B$.

b) Set A = set B, so $A \not\subset B$. (However, $A \subseteq B$.)

▲

Every set is a subset of itself, but no set is a proper subset of itself. For all sets A, $A \subseteq A$, but $A \not\subset A$. For example, if $A = \{1, 2, 3\}$, then $A \subseteq A$ because every element of set A is contained in set A, but $A \not\subset A$ because set A = set A.

Let $A = \{\ \}$ and $B = \{1, 2, 3, 4\}$. Is $A \subseteq B$? To show $A \not\subseteq B$, you must find at least one element of set A that is not an element of set B. As this cannot be done, $A \subseteq B$ must be true. Using the same reasoning, we can show that *the empty set is a subset of every set, including itself.*

EXAMPLE 3 *Element or Subset?*

Determine whether the following are true or false.

a) $3 \in \{3, 4, 5\}$

b) $\{3\} \in \{3, 4, 5\}$

c) $\{3\} \in \{\{3\}, \{4\}, \{5\}\}$

d) $\{3\} \subseteq \{3, 4, 5\}$

e) $3 \subseteq \{3, 4, 5\}$

f) $\{\ \} \subseteq \{3, 4, 5\}$

SOLUTION:

a) $3 \in \{3, 4, 5\}$ is a true statement because 3 is a member of the set $\{3, 4, 5\}$.

b) $\{3\} \in \{3, 4, 5\}$ is a false statement because $\{3\}$ is a set, and the set $\{3\}$ is not an element of the set $\{3, 4, 5\}$.

c) $\{3\} \in \{\{3\}, \{4\}, \{5\}\}$ is a true statement because $\{3\}$ is an element in the set. The elements of the set $\{\{3\}, \{4\}, \{5\}\}$ are themselves sets.

d) $\{3\} \subseteq \{3, 4, 5\}$ is a true statement because every element of the first set is an element of the second set.

e) $3 \subseteq \{3, 4, 5\}$ is a false statement because the 3 is not in braces, so it is not a set and thus cannot be a subset. The 3 is an element of the set as indicated in part (a).

f) $\{\ \} \subseteq \{3, 4, 5\}$ is a true statement because the empty set is a subset of every set.

Number of Subsets

How many distinct subsets can be made from a given set? The empty set has no elements and has exactly one subset, the empty set. A set with one element has two subsets. A set with two elements has four subsets. A set with three elements has eight subsets. This information is illustrated in Table 2.1 on page 53. How many subsets will a set with four elements contain?

By continuing this table with larger and larger sets, we can develop a general formula for finding the number of distinct subsets that can be made from any given set.

The **number of distinct subsets** of a finite set A is 2^n, where n is the number of elements in set A.

TABLE 2.1 Number of Subsets

Set	Subsets	Number of Subsets
{ }	{ }	$1 = 2^0$
$\{a\}$	$\{a\}$ { }	$2 = 2^1$
$\{a, b\}$	$\{a, b\}$ $\{a\}, \{b\}$ { }	$4 = 2 \times 2 = 2^2$
$\{a, b, c\}$	$\{a, b, c\}$ $\{a, b\}, \{a, c\}, \{b, c\}$ $\{a\}, \{b\}, \{c\}$ { }	$8 = 2 \times 2 \times 2 = 2^3$

EXAMPLE 4 *Distinct Subsets*

a) Determine the number of distinct subsets for the set {S, L, E, D}.

b) List all the distinct subsets for the set {S, L, E, D}.

c) How many of the distinct subsets are proper subsets?

SOLUTION:

a) Since the number of elements in the set is 4, the number of distinct subsets is $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

b)

{S, L, E, D}	{S, L, E}	{S, L}	{S}	{ }
	{S, L, D}	{S, E}	{L}	
	{S, E, D}	{S, D}	{E}	
	{L, E, D}	{L, E}	{D}	
		{L, D}		
		{E, D}		

c) There are 15 proper subsets. Every subset except {S, L, E, D} is a proper subset. ▲

EXAMPLE 5 *Variations of Ice Cream*

Shanna Ruben is going to purchase ice cream at Friendly's Restaurant. To her ice cream she can add any of the following toppings: hot fudge, whipped cream, cherries, butterscotch topping, caramel topping, chopped nuts, Reese's Pieces, M & M's, Gummy Bears. How many different variations of the ice cream and toppings can be made?

SOLUTION: Shanna can order the ice cream with no extra toppings, any one topping, any two toppings, any three toppings, and so on, up to all nine toppings. One technique used in problem solving is to consider similar problems that you have solved previously. If you think about this problem, you will realize that this problem is the same as, "How many distinct subsets can be made from a set with nine elements?" The number of different variations of the ice cream is the same as the number of possible subsets of a set that has nine elements. There are 2^9 or 512 possible subsets of a set with nine elements, so there are 512 possible variations of the ice cream and toppings. ▲

DID YOU KNOW

The Ladder of Life

Scientists use sets to classify and categorize knowledge. In biology, the science of classifying all living things is called *taxonomy* and was probably practiced by the earliest cave-dwellers. Over 2000 years ago, Aristotle formalized animal classification with his "ladder of life": higher animals, lower animals, higher plants, lower plants.

A zebra, *Equus burchelli*, is a member of the genus *Equus*, as is the horse, *Equus caballus*. Both the zebra and the horse are members of the universal set called the kingdom of animals and the same family, Equidae; they are members of different species (*E. burchelli* and *E. caballus*), however.

Contemporary biologists use a system of classification called the Linnaean system, named after Swedish biologist Carolus Linnaeas (1707–1778). The Linnaean system starts with the smallest unit (member) and assigns it to a specific genus (set) and species (subset).

Even more general groupings of living things are made according to shared characteristics. The groupings, from most general to most specific are: kingdom, phylum, class, order, family, genus, and species. Each of the groupings is classified into sub groupings. For example, in the sixteenth century, living organisms were classified into two kingdoms, plants and animals. Today, living organisms are classified into six kingdoms called animalia, plantae, archaea, eubacteria, fungi, and protista.

SECTION 2.2 EXERCISES

Concept/Writing Exercises

In Exercises 1–6, answer each question with a complete sentence.

1. What is a subset?
2. What is a proper subset?
3. Explain the difference between a subset and a proper subset.
4. Write the formula for determining the number of distinct subsets for a set with n distinct elements.
5. Write the formula for determining the number of distinct proper subsets for a set with n distinct elements.

6. Can any set be a proper subset of itself? Explain.

Practice the Skills

In Exercises 7–24, answer true or false. If false, give the reason.

7. gold $\subseteq$ {gold, silver, sapphire, emerald}
8. { } $\in$ {knee, ankle, shoulder, hip}
9. { } $\subseteq$ {Tigger, Pooh, Christopher Robin}
10. red $\subset$ {red, green, blue}
11. $5 \notin \{2, 4, 6\}$
12. {Pete, Mike, Amy} $\subseteq$ {Amy, Kaitlyn, Brianna}
13. $\{ \} = \{\emptyset\}$
14. {engineer} $\subseteq$ {architect, physician, attorney, engineer}
15. $\emptyset = \{ \}$
16. $0 = \{ \}$
17. $\{0\} = \emptyset$
18. $\{3, 8, 11\} \subseteq \{3, 8, 11\}$
19. {swimming} $\in$ {sailing, waterskiing, swimming}
20. $\{3, 5, 9\} \not\subseteq \{3, 9, 5\}$
21. $\{ \} \subseteq \{ \}$
22. $\{1\} \in \{\{1\}, \{2\}, \{3\}\}$
23. {US Airways, Delta, American} $\subset$ {American, US Airways, Delta}
24. $\{b, a, t\} \subseteq \{t, a, b\}$

In Exercises 25–32, determine whether $A = B$, $A \subseteq B$, $B \subseteq A$, $A \subset B$, $B \subset A$, or none of these. (There may be more than one answer.)

25. A = {Pepsi, Mountain Dew, Coke, Sprite}
    B = {Pepsi, Coke}
26. $A = \{x \mid x \in N \text{ and } x < 6\}$
    $B = \{x \mid x \in N \text{ and } 1 \leq x \leq 5\}$
27. Set A is the set of states east of the Mississippi River. Set B is the set of states that border the Atlantic Ocean.
28. $A = \{1, 3, 5, 7, 9\}$
    $B = \{3, 9, 5, 7, 6\}$
29. $A = \{x \mid x$ is a brand of ice cream$\}$
    B = {Breyers, Ben & Jerry's, Häagen-Dazs}
30. $A = \{x \mid x$ is a sport that uses a ball$\}$
    B = {basketball, soccer, tennis}
31. Set A is the set of natural numbers between 2 and 7. Set B is the set of natural numbers greater than 2 and less than 7.
32. Set A is the set of toys requiring batteries. Set B is the set of toys requiring AA batteries.

In Exercises 33–38, list all the subsets of the sets given.

33. $D = \emptyset$
34. $A = \{\bigcirc\}$
35. B = {pen, pencil}
36. C = {apple, peach, banana}

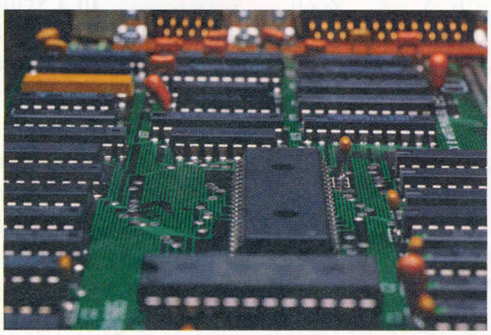
Problem Solving

37. For set $A = \{a, b, c, d\}$
    a) list all the subsets of set A.
    b) state which of the subsets in part (a) are not proper subsets of set A.
38. A set contains nine elements.
    a) How many subsets does it have?
    b) How many proper subsets does it have?

In Exercises 39–50, if the statement is true for all sets A and B, write "true." If it is not true for all sets A and B, write "false." Assume that $A \neq \emptyset$, $U \neq \emptyset$, and $A \subset U$.

39. If $A \subseteq B$, then $A \subset B$.
40. If $A \subset B$, then $A \subseteq B$.
41. $A \subseteq A$
42. $A \subset A$
43. $\emptyset \subset A$
44. $\emptyset \subseteq A$
45. $A \subseteq U$
46. $\emptyset \subset \emptyset$
47. $\emptyset \subset U$
48. $U \subseteq \emptyset$
49. $\emptyset \subseteq \emptyset$
50. $U \subset \emptyset$

51. ***Building a House*** The Jacobsens are planning to build a house in a new development. They can either build the base model offered by the builder or add any of the following options: deck, hot tub, security system, hardwood flooring. How many different variations of the house are possible?
52. ***Computer Upgrade*** Jason Jackson is considering having his computer upgraded. He can leave the computer as it is, or he can upgrade any of the following set of items: RAM, modem, video card, hard drive, processor, sound card. How many possible options for upgrading does Jason have?



53. ***Telephone Features*** A customer with Verizon can order telephone service with some, all, or none of the following features: call waiting, call forwarding, caller identification, three-way calling, voice mail, fax line. How many different variations of the set of features are possible?

54. ***Hamburger Variations*** Customers ordering hamburgers at Vic and Irv's Hamburger stand are always asked, "What do you want on it?" The choices are ketchup, mustard, relish, hot sauce, onions, lettuce, tomato. How many different variations are there for ordering a hamburger?

55. If $E \subseteq F$ and $F \subseteq E$, what other relationship exists between E and F? Explain.

56. How can you determine whether the set of boys is equivalent to the set of girls at a roller-skating rink?

57. For the set $D = \{a, b, c\}$
   a) is a an element of set D? Explain.
   b) is c a subset of set D? Explain.
   c) is $\{a, b\}$ a subset of set D? Explain.

Challenge Problems/Group Activity

58. ***Hospital Expansion*** A hospital has four members on the board of directors: Arnold, Benitez, Cathy, and Dominique.
   a) When the members vote on whether to add a wing to the hospital, how many different ways can they vote (abstentions are not allowed)? For example, Arnold—yes, Benitez—no, Cathy—no, and Dominique—yes is one of the many possibilities.
   b) Make a listing of all the possible outcomes of the vote. For example, the vote described in part (a) could be represented as (YNNY).
   c) How many of the outcomes given in part (b) would result in a majority supporting the addition of a wing to the hospital? That is, how many of the outcomes have three or more Y's?

Recreational Mathematics

59. How many elements must a set have if the number of proper subsets of the set is $\frac{1}{2}$ of the total number of subsets of the set?

60. If $A \subset B$ and $B \subset C$, must $A \subset C$?

61. If $A \subset B$ and $B \subseteq C$, must $A \subset C$?

62. If $A \subseteq B$ and $B \subseteq C$, must $A \subset C$?

Internet/Research Activity

63. On page 54, we discussed the ladder of life. Do research and indicate all the different classifications in the Linnaean system, from most general to the most specific, in which a koala belongs.

2.3 VENN DIAGRAMS AND SET OPERATIONS

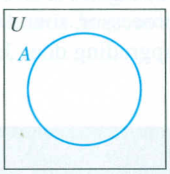


Figure 2.1

A useful technique for picturing set relationships is the Venn diagram, named for the English mathematician John Venn (1834–1923). Venn invented the diagrams and used them to illustrate ideas in his text on symbolic logic, published in 1881.

In a Venn diagram, a rectangle usually represents the universal set, U. The items inside the rectangle may be divided into subsets of the universal set. The subsets are usually represented by circles. In Fig. 2.1, the circle labeled A represents set A, which is a subset of the universal set.

Two sets may be represented in a Venn diagram in any of four different ways (see Fig. 2.2 on page 57). Two sets A and B are ***disjoint*** when they have no elements in common. Two disjoint sets A and B are illustrated in Fig. 2.2(a). If set A is a proper subset of set B, $A \subset B$, the two sets may be illustrated as in Fig. 2.2(b). If set A contains exactly the same elements as set B, that is, $A = B$, the two sets may be illustrated as in Fig. 2.2(c). Two sets A and B with some elements in common are shown in Fig. 2.2(d), which is regarded as the most general form of a Venn diagram.

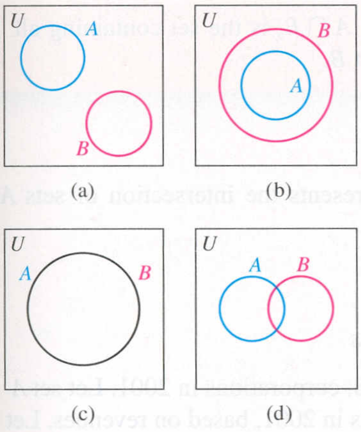


Figure 2.2

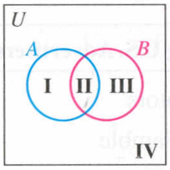


Figure 2.3

If we label the regions of the diagram in Fig. 2.2(d) using I, II, III, and IV, we can illustrate the four possible cases with this one diagram, Fig. 2.3.

CASE 1: DISJOINT SETS When sets A and B are disjoint, they have no elements in common. Therefore, region II of Fig. 2.3 is empty.

CASE 2: SUBSETS When $A \subseteq B$, every element of set A is also an element of set B. Thus, there can be no elements in region I of Fig. 2.3. If $B \subseteq A$, however, then region III of Fig. 2.3 is empty.

CASE 3: EQUAL SETS When set A = set B, all the elements of set A are elements of set B and all the elements of set B are elements of set A. Thus, regions I and III of Fig. 2.3 are empty.

CASE 4: OVERLAPPING SETS When sets A and B have elements in common, those elements are in region II of Fig. 2.3. The elements that belong to set A but not to set B are in region I. The elements that belong to set B but not to set A are in region III.

In each of the four cases, any element not belonging to set A or set B is placed in region IV.

Venn diagrams will be helpful in understanding set operations. The basic operations of arithmetic are $+$, $-$, $\times$, and $\div$. When we see these symbols, we know what procedure to follow to determine the answer. Some of the operations in set theory are $'$, $\cup$, and $\cap$. They represent complement, union, and intersection, respectively.

Complement

The **complement** of set A, symbolized by A', is the set of all the elements in the universal set that are not in set A.

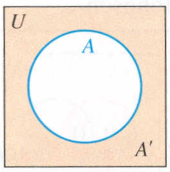


Figure 2.4

In Fig. 2.4, the shaded region outside of set A within the universal set represents the complement of set A, or A'.

EXAMPLE 1 *A Set and Its Complement*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\} \text{ and } A = \{1, 3, 4\}$$

find A' and illustrate the relationship among sets U, A, and A' in a Venn diagram.

SOLUTION: The elements in U that are not in set A are 2, 5, 6, 7, 8. Thus, $A' = \{2, 5, 6, 7, 8\}$. The Venn diagram is illustrated in Fig. 2.5.

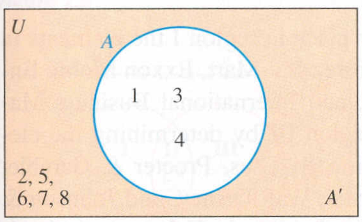
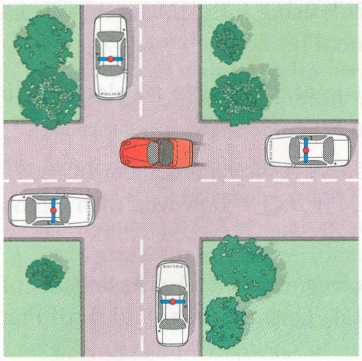


Figure 2.5

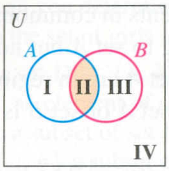
Intersection

The word *intersection* brings to mind the area common to two crossing streets. The red car in the figure on the next page is in the intersection of the two streets. The set operation is defined as follows.



The **intersection** of sets A and B, symbolized by $A \cap B$, is the set containing all the elements that are common to both set A and set B.

The shaded region, region II, in Fig. 2.6 represents the intersection of sets A and B.



$A \cap B$

Figure 2.6

EXAMPLE 2 *Sets with Overlapping Regions*

Let the universal set, U, represent the set of all U.S. corporations in 2001. Let set A represent the set of the 10 largest U.S. corporations in 2001, based on revenues. Let set B represent the set of the 10 leading U.S. advertisers in 2001 (see the table). Draw a Venn diagram illustrating the relationship between set A and set B.

Ten Largest U.S. Corporations	Ten Leading U.S. Advertisers
Wal-Mart	General Motors
Exxon Mobil	Procter & Gamble
General Motors	Ford Motor Company
Ford Motor Company	PepsiCo
Enron	Pfizer
General Electric	DaimlerChrysler
Citigroup	AOL Time Warner
ChevronTexaco	Phillip Morris
International Business Machines	Walt Disney
Phillip Morris	Johnson & Johnson

Source: *Fortune*, www.adage.com

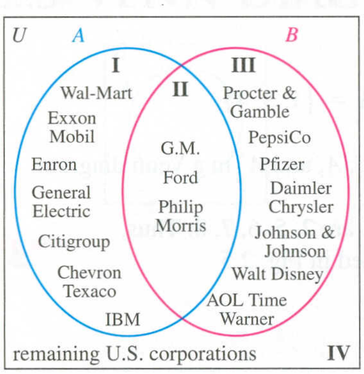


Figure 2.7

SOLUTION: First determine the intersection of sets A and B. General Motors, Ford Motor Company, and Phillip Morris are common to both sets. Therefore,

$$A \cap B = \{\text{General Motors, Ford Motor Company, Phillip Morris}\}$$

Place these elements in region II of Fig. 2.7. Now place in region I the elements in set A that have not been placed in region II. Therefore, Wal-Mart, Exxon Mobil, Enron, General Electric, Citigroup, ChevronTexaco, and International Business Machines (IBM) are placed in region I. Complete region III by determining the elements in set B that have not been placed in region II. Thus, Procter & Gamble, PepsiCo, Pfizer, DaimlerChrysler, AOL Time Warner, Walt Disney, and Johnson & Johnson are placed in region III. Finally, place those elements in U that are not in either set outside both circles. This group includes the remaining U.S. corporations, which are placed in region IV. ▲

EXAMPLE 3 *The Intersection of Sets*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 4, 6\}$$
$$B = \{1, 3, 6, 7, 9\}$$
$$C = \{\ \}$$

find

a) $A \cap B$. b) $A \cap C$. c) $A' \cap B$. d) $(A \cap B)'$.

SOLUTION:

a) $A \cap B = \{1, 2, 4, 6\} \cap \{1, 3, 6, 7, 9\} = \{1, 6\}$. The elements common to both set A and set B are 1 and 6.

b) $A \cap C = \{1, 2, 4, 6\} \cap \{\ \} = \{\ \}$. There are no elements common to both set A and set C.

c) $A' = \{3, 5, 7, 8, 9, 10\}$
$A' \cap B = \{3, 5, 7, 8, 9, 10\} \cap \{1, 3, 6, 7, 9\}$
$= \{3, 7, 9\}$

d) To find $(A \cap B)'$, first determine $A \cap B$.
$A \cap B = \{1, 6\}$ from part (a)
$(A \cap B)' = \{1, 6\}' = \{2, 3, 4, 5, 7, 8, 9, 10\}$

Union

The word *union* means to unite or join together, as in marriage, and that is exactly what is done when we perform the operation of union.

The **union** of set A and set B, symbolized by $A \cup B$, is the set containing all the elements that are members of set A or of set B (or of both sets).

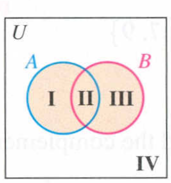


Figure 2.8

The three shaded regions of Fig. 2.8, regions I, II, and III, together represent the union of sets A and B. If an element is common to both sets, it is listed only once in the union of the sets.

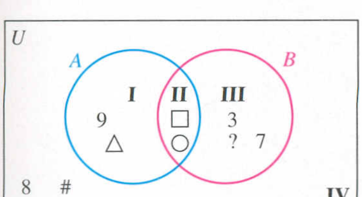


Figure 2.9

EXAMPLE 4 *Determining Sets from a Venn Diagram*

Use the Venn diagram in Fig. 2.9 to determine the following sets.

a) U b) A c) B' d) $A \cap B$
e) $A \cup B$ f) $(A \cup B)'$ g) $n(A \cup B)$

SOLUTION:

a) The universal set consists of all the elements within the rectangle, that is, the elements in regions I, II, III, and IV. Thus, $U = \{9, \triangle, \square, \bigcirc, 3, 7, ?, \#, 8\}$.

b) Set A consists of the elements in regions I and II. Thus, $A = \{9, \triangle, \square, \bigcirc\}$.

c) B' consists of the elements outside set B, or the elements in regions I and IV. Thus, $B' = \{9, \triangle, \#, 8\}$.

d) $A \cap B$ consists of the elements that belong to both set A and set B (region II). Thus, $A \cap B = \{\square, \bigcirc\}$.

e) $A \cup B$ consists of the elements that belong to set A or set B (regions I, II, or III). Thus, $A \cup B = \{9, \triangle, \square, \bigcirc, 3, 7, ?\}$.

f) $(A \cup B)'$ consists of the elements in U that are not in $A \cup B$. Thus, $(A \cup B)' = \{\#, 8\}$.

g) $n(A \cup B)$ represents the *number of elements* in the union of sets A and B. Thus, $n(A \cup B) = 7$, as there are seven elements in the union of sets A and B. ▲

EXAMPLE 5 *The Union of Sets*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 4, 6\}$$
$$B = \{1, 3, 6, 7, 9\}$$
$$C = \{\ \}$$

find

a) $A \cup B$. b) $A \cup C$. c) $A' \cup B$. d) $(A \cup B)'$.

SOLUTION:

a) $A \cup B = \{1, 2, 4, 6\} \cup \{1, 3, 6, 7, 9\} = \{1, 2, 3, 4, 6, 7, 9\}$

b) $A \cup C = \{1, 2, 4, 6\} \cup \{\ \} = \{1, 2, 4, 6\}$. Note that $A \cup C = A$.

c) To determine $A' \cup B$, we must determine A'.

$$A' = \{3, 5, 7, 8, 9, 10\}$$
$$A' \cup B = \{3, 5, 7, 8, 9, 10\} \cup \{1, 3, 6, 7, 9\} = \{1, 3, 5, 6, 7, 8, 9, 10\}$$

d) Find $(A \cup B)'$ by first determining $A \cup B$, and then find the complement of $A \cup B$.

$$A \cup B = \{1, 2, 3, 4, 6, 7, 9\} \text{ from part (a)}$$
$$(A \cup B)' = \{1, 2, 3, 4, 6, 7, 9\}' = \{5, 8, 10\}$$ ▲

EXAMPLE 6 *Union and Intersection*

Given

$$U = \{a, b, c, d, e, f, g\}$$
$$A = \{a, b, e, g\}$$
$$B = \{a, c, d, e\}$$
$$C = \{b, e, f\}$$

find

a) $(A \cup B) \cap (A \cup C)$. b) $(A \cup B) \cap C'$. c) $A' \cap B'$.

SOLUTION:

a) $(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, g\} \cap \{a, b, e, f, g\}$
$= \{a, b, e, g\}$

b) $(A \cup B) \cap C' = \{a, b, c, d, e, g\} \cap \{a, c, d, g\}$
$= \{a, c, d, g\}$

c) $A' \cap B' = \{c, d, f\} \cap \{b, f, g\}$
$= \{f\}$

The Meaning of *and* and *or*

The words *and* and *or* are very important in many areas of mathematics. We use these words in several chapters in this book, including the probability chapter. The word ***or*** is generally interpreted to mean ***union***, whereas ***and*** is generally interpreted to mean ***intersection***. Suppose $A = \{1, 2, 3, 5, 6, 8\}$ and $B = \{1, 3, 4, 7, 9, 10\}$. Then the elements that belong to set A *or* set B are 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. These are the elements in the union of the sets. The elements that belong to set A *and* set B are 1 and 3. These are the elements in the intersection of the sets.

The Relationship Between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$

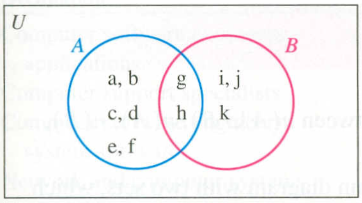


Figure 2.10

Having looked at unions and intersections, we can now determine a relationship between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$. Suppose set A has eight elements, set B has five elements, and $A \cap B$ has two elements. How many elements are in $A \cup B$? Let's make up some arbitrary sets that meet the criteria specified and draw a Venn diagram. If we let $A = \{a, b, c, d, e, f, g, h\}$, then set B must contain five elements, two of which are also in set A. Let $B = \{g, h, i, j, k\}$. We construct a Venn diagram by filling in the intersection first, as shown in Fig. 2.10. The number of elements in $A \cup B$ is 11. The elements g and h are in both sets, and if we add $n(A) + n(B)$, we are counting these elements twice.

To find the number of elements in the union of sets A and B, we can add the number of elements in sets A and B and then subtract the number of elements common to both sets.

For any finite sets *A* and *B*,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

EXAMPLE 7 *How Many Visitors Speak Spanish or French?*

The results of a survey of visitors at the Grand Canyon showed that 25 speak Spanish, 14 speak French, and 4 speak both Spanish and French. How many speak Spanish or French?

SOLUTION: If we let set A be the set of visitors who speak Spanish and let set B be the set of visitors who speak French, then we need to determine $n(A \cup B)$. We can use the above formula to find $n(A \cup B)$.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$n(A \cup B) = 25 + 14 - 4$$
$$= 35$$

Thus, 35 of the visitors surveyed speak either Spanish or French. ▲

EXAMPLE 8 ***The Number of Elements in Set A or Set B***

Set A contains 6 letters and 5 numbers. Set B contains 4 letters and 9 numbers. Two letters and 1 number are common to both sets A and B. Find the number of elements in set A or set B.

SOLUTION: You are asked to find the number of elements in set A or set B, which is $n(A \cup B)$. Because $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if you can determine $n(A)$, $n(B)$, and $n(A \cap B)$, you can solve the problem. Set A contains 6 letters and 5 numbers, so $n(A) = 11$. Set B contains 4 letters and 9 numbers, so $n(B) = 13$. Because 2 letters and 1 number are common to both sets, $n(A \cap B) = 3$.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 11 + 13 - 3 = 21$$

Thus, the number of elements in set A or set B is 21. ▲

SECTION 2.3 EXERCISES

Concept/Writing Exercises

In Exercises 1–5, use Fig. 2.2 as a guide to draw a Venn diagram that illustrates the situation described.

1. Set A and set B are disjoint sets.
2. $A \subset B$
3. $B \subset A$
4. $A = B$
5. Set A and set B are overlapping sets.
6. If we are given set A, how do we obtain A complement, A'?
7. How do we obtain the union of two sets A and B, $A \cup B$?
8. In a Venn diagram with two overlapping sets, which region(s) represents $A \cup B$?
9. How do we obtain the intersection of two sets A and B, $A \cap B$?
10. In a Venn diagram with two overlapping sets, which region(s) represents $A \cap B$?
11. a) Which set operation is the word *or* generally interpreted to mean?
    b) Which set operation is the word *and* generally interpreted to mean?
12. Give the relationship between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$.
13. When constructing a Venn diagram with two sets, which region of the diagram do we generally complete first?
14. When constructing a Venn diagram with two sets, which region of the diagram do we generally complete last?

Practice the Skills/Problem Solving

15. *Restaurants* For the sets U, A, and B, construct a Venn diagram and place the elements in the proper regions.

U = {Pizza Hut, Papa John's, McDonald's, Burger King, Wendy's, Roy Rogers, Taco Bell, Subway, Del Taco, Denny's}

$A = \{$Pizza Hut, Papa John's, Wendy's, Roy Rogers, Taco Bell, Denny's$\}$

$B = \{$McDonald's, Burger King, Wendy's, Taco Bell, Subway, Denny's$\}$

16. *Appliances and Electronics* For the sets U, A, and B, construct a Venn diagram and place the elements in the proper regions.

$U = \{$microwave oven, washing machine, dryer, refrigerator, dishwasher, compact disc player, videocassette recorder, computer, camcorder, television$\}$

$A = \{$microwave oven, washing machine, dishwasher, computer, television$\}$

$B = \{$washing machine, dryer, refrigerator, compact disc player, computer, television$\}$

17. *Occupations* The following table shows the fastest-growing occupations based on employment in 2000 and the estimated employment for that profession for 2010. Let the occupations in the table represent the universal set.

Fastest-Growing Occupations, 2000–2010

Occupation	Employment (in thousands of jobs) 2000	2010
Computer software engineers, applications	380	760
Computer support specialists	506	996
Computer software engineers, system software	317	601
Network and computer systems administrators	229	416
Network systems/data communications analysts	119	211
Desktop publishers	38	63
Database administrators	106	176
Personal/home care aides	414	672
Computer systems analysts	431	689
Medical assistants	329	516

Source: U.S. Department of Labor, Bureau of Labor Statistics

Let A = the set of fastest-growing occupations whose 2000 employment was at least 250,000.

Let B = the set of fastest-growing occupations whose estimated employment in 2010 is at least 650,000.

Construct a Venn diagram illustrating the sets.

18. *Basketball Statistics* The table above and to the right shows the number of times certain basketball players were selected as the National Basketball Association's most valuable player and the number of times the player was the league scoring champion. Let these players represent the universal set.

Player	Most Valuable Player	League Scoring Champion
Michael Jordan	4	10
Dominique Wilkins	0	1
Wilt Chamberlain	2	7
Kareem Abdul-Jabbar	4	2
Shaquille O'Neal	1	2
Earvin Johnson	3	0
Kobe Bryant	0	0
Jerry West	0	1
George Gervin	0	4
Alan Iverson	1	1

Source: National Basketball Association

Michael Jordan

Let A = the set of basketball players that were selected most valuable player at least one time.

Let B = the set of basketball players that were league scoring champion at least one time.

Construct a Venn diagram illustrating the sets.

19. Let U represent the set of U.S. colleges and universities. Let A represent the set of U.S. colleges and universities in the state of North Dakota. Describe A'.

20. Let U represent the set of marbles in a box. Let set B represent the set of marbles that contain some blue coloring. Describe B'.

In Exercises 21–26,

U is the set of insurance companies in the U.S.

A is the set of insurance companies that offer life insurance.

B is the set of insurance companies that offer car insurance.

Describe each of the following sets in words.

21. A'
22. B'
23. $A \cup B$
24. $A \cap B$
25. $A \cap B'$
26. $A \cup B'$

In Exercises 27–32,

U is the set of U.S. corporations.

A is the set of U.S. corporations whose headquarters are in the state of New York.

B is the set of U.S. corporations whose chief executive officer is a woman.

C is the set of U.S. corporations that employ at least 100 people.

Describe the following sets.

27. $A \cap B$
28. $A \cup C$
29. $B' \cap C$
30. $A \cap B \cap C$
31. $A \cup B \cup C$
32. $A' \cup C'$

In Exercises 33–40, use the Venn diagram in Fig. 2.11 to list the set of elements in roster form.

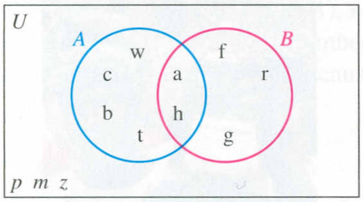


Figure 2.11

33. A
34. B
35. $A \cap B$
36. U
37. $A \cup B$
38. $(A \cup B)'$
39. $A' \cap B'$
40. $(A \cap B)'$

In Exercises 41–48, use the Venn diagram in Fig. 2.12 to list the set of elements in roster form.

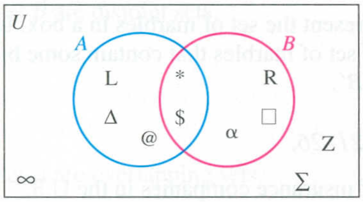


Figure 2.12

41. A
42. B
43. U
44. $A \cup B$
45. $A \cap B$
46. $A \cup B'$
47. $A' \cap B$
48. $(A \cup B)'$

In Exercises 49–58, let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$A = \{1, 2, 4, 5, 8\}$$
$$B = \{2, 3, 4, 6\}$$

Determine the following.

49. $A \cup B$
50. $A \cap B$
51. B'
52. $A \cup B'$
53. $(A \cup B)'$
54. $A' \cap B'$
55. $(A \cup B)' \cap B$
56. $(A \cup B) \cap (A \cup B)'$
57. $(B \cup A)' \cap (B' \cup A')$
58. $A' \cup (A \cap B)$

In Exercises 59–68, let

$$U = \{a, b, c, d, e, f, g, h, i, j, k\}$$
$$A = \{a, c, d, f, g, i\}$$
$$B = \{b, c, d, f, g\}$$
$$C = \{a, b, f, i, j\}$$

Determine the following.

59. B'
60. $B \cup C$
61. $A \cap C$
62. $A \cup B'$
63. $(A \cap C)'$
64. $(A \cap B) \cup C$
65. $A \cup (C \cap B)'$
66. $A \cup (C' \cup B')$
67. $(A' \cup C) \cup (A \cap B)$
68. $(C \cap B) \cap (A' \cap B)$

Problem Solving

In Exercises 69–82, let

$$U = \{x \mid x \in N \text{ and } x < 10\}$$
$$A = \{x \mid x \in N \text{ and } x \text{ is odd and } x < 10\}$$
$$B = \{x \mid x \in N \text{ and } x \text{ is even and } x < 10\}$$
$$C = \{x \mid x \in N \text{ and } x < 6\}$$

Determine the following.

69. $A \cap B$
70. $A \cup B$
71. $A' \cup B$
72. $(B \cup C)'$
73. $A \cap C'$
74. $A \cap B'$
75. $(B \cap C)'$
76. $(A \cup C) \cap B$
77. $(C \cap B) \cup A$
78. $(C' \cup A) \cap B$
79. $(A' \cup C) \cap B$
80. $(A \cap B)' \cup C$
81. $(A' \cup B') \cap C$
82. $(A' \cap C) \cup (A \cap B)$
83. When will a set and its complement be disjoint? Explain and give an example.
84. When will $n(A \cap B) = 0$? Explain and give an example.
85. *Visiting California* The results of a survey of visitors in Hollywood, California, showed that 27 visited the Hollywood Bowl, 38 visited Disneyland, and 16 visited both the

Hollywood Bowl and Disneyland. How many people visited either the Hollywood Bowl or Disneyland?

Hollywood Bowl

86. ***Chorus and Band*** At Henniger High School, 46 students sang in the chorus or played in the stage band, 30 students played in the stage band, and 4 students sang in the chorus and played in the stage band. How many students sang in the chorus?

87. Consider the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

a) Show that this relation holds for $A = \{a, b, c, d\}$ and $B = \{b, d, e, f, g, h\}$.
b) Make up your own sets A and B, each consisting of at least six elements. Using these sets, show that the relation holds.
c) Use a Venn diagram and explain why the relation holds for any two sets A and B.

88. The Venn diagram in Fig. 2.13 shows a technique of labeling the regions to indicate membership of elements in a particular region. Define each of the four regions with a set statement. (*Hint:* $A \cap B'$ defines region I.)

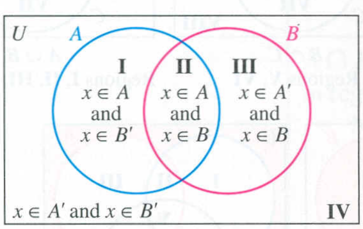


Figure 2.13

In Exercises 89–98, let $U = \{0, 1, 2, 3, 4, 5, \ldots\}$, $A = \{1, 2, 3, 4, \ldots\}$, $B = \{4, 8, 12, 16, \ldots\}$, *and* $C = \{2, 4, 6, 8, \ldots\}$. *Determine the following.*

89. $A \cup B$
90. $A \cap B$
91. $B \cap C$
92. $B \cup C$
93. $A \cap C$
94. $A' \cap C$
95. $B' \cap C$
96. $(B \cup C)' \cup C$
97. $(A \cap C) \cap B'$
98. $U' \cap (A \cup B)$

Challenge Problems/Group Activities

In Exercises 99–106, determine whether the answer is $\emptyset$, A, *or* U. *(Assume* $A \neq \emptyset, A \neq U$.)

99. $A \cup A'$
100. $A \cap A'$
101. $A \cup \emptyset$
102. $A \cap \emptyset$
103. $A' \cup U$
104. $A \cap U$
105. $A \cup U$
106. $A \cup U'$

In Exercises 107–112, determine the relationship between set A and set B if

107. $A \cap B = B$.
108. $A \cup B = B$.
109. $A \cap B = \emptyset$.
110. $A \cup B = A$.
111. $A \cap B = A$.
112. $A \cup B = \emptyset$.

Difference of Two Sets Another set operation is the **difference of two sets**. The difference of two sets A and B, symbolized $A - B$, is defined as

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Thus, $A - B$ is the set of elements that belong to set A but not to set B. For example, if $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 5, 9, 10\}$, and $B = \{1, 3, 4, 5, 6, 7\}$, then $A - B = \{2, 9, 10\}$ and $B - A = \{1, 3, 6, 7\}$.

In Exercises 113–116, let $U = \{a, b, c, d, e, f, g, h, i, j, k\}$, $A = \{b, c, e, f, g, h\}$, *and* $B = \{a, b, c, g, i\}$. *Determine the following.*

113. $A - B$
114. $B - A$
115. $A' - B$
116. $A - B'$

In Exercises 117–122, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, $A = \{2, 4, 5, 7, 9, 11, 13\}$, *and* $B = \{1, 2, 4, 5, 6, 7, 8, 9, 11\}$. *Determine the following.*

117. $A - B$
118. $B - A$
119. $(A - B)'$
120. $A - B'$
121. $(B - A)'$
122. $A \cap (A - B)$

Recreational Mathematics

123. ***What Am I?*** The poem below gives clues for a 10-letter word discussed in this section. Use the clues to determine the word.

My first is in card and also cat.
My fourth is in top and also in pat.

My third and my seventh are one and the same—
you'll find them in math and also in game.
My sixth and my eighth, find one you will find two—
you will find them in open, you will find them in shoe.
My second is in work and also in told.
My fifth is in label and also in bold.
My ninth is in number and also in change.
My tenth is in table but never in range.
I have more than one meaning, some say I'm "not."
Although you may want to receive me a lot.

124. *Wordgram* Hidden in the box are the following words discussed in this chapter: DIAGRAM, UNION, INTERSECTION, SUBSET. You will find them by going letter to letter either vertically or horizontally. A letter can only be used once when spelling out a word. Find the words and make sure you understand the meaning of each word.

N	O	I	E	L	A
T	C	T	S	E	T
S	E	U	B	P	U
R	D	S	R	I	N
E	I	A	G	O	N
T	N	I	R	A	M

2.4 VENN DIAGRAMS WITH THREE SETS AND VERIFICATION OF EQUALITY OF SETS

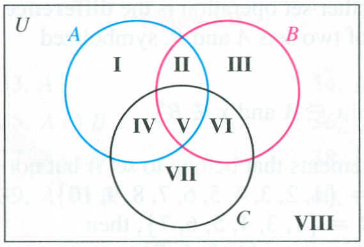


Figure 2.14

Venn diagrams can be used to illustrate three or more sets. For three sets, *A*, *B*, and *C*, the diagram is drawn so the three sets overlap (Fig. 2.14), creating eight regions. The diagrams in Fig. 2.15 emphasize selected regions of three intersecting sets. *When constructing Venn diagrams with three sets, we generally start with region V and work outward*, as explained in the following procedure.

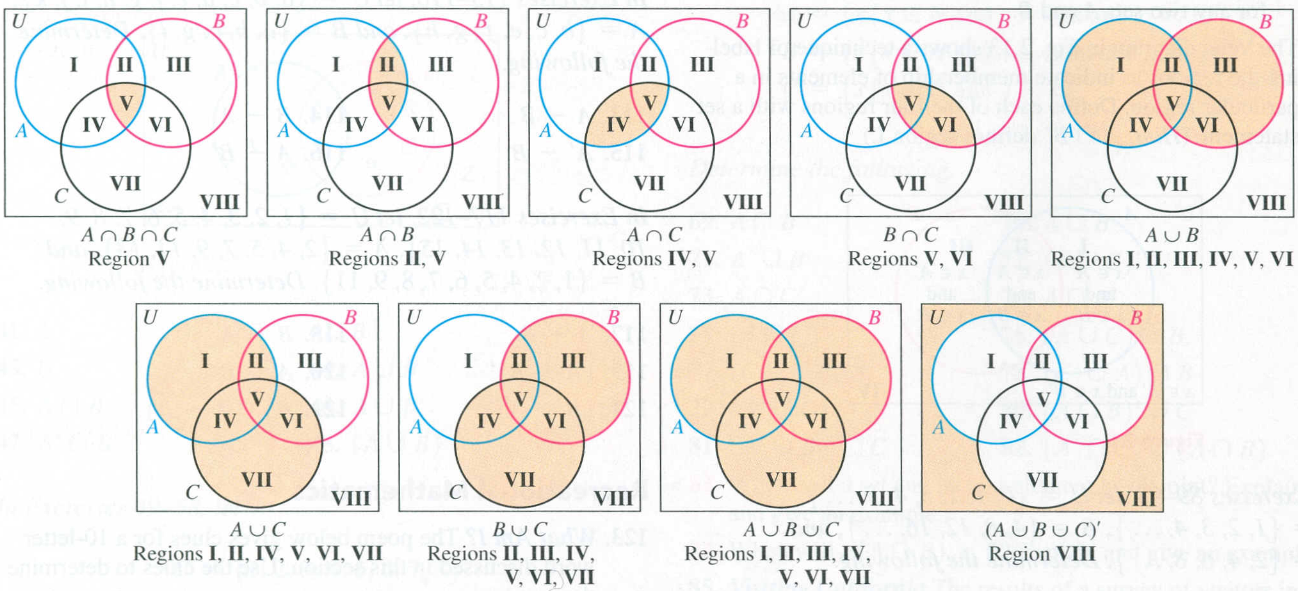


Figure 2.15

General Procedure for Constructing Venn Diagrams with Three Sets, *A*, *B*, and *C*

1. Determine the elements to be placed in region V by finding the elements that are common to all three sets, $A \cap B \cap C$.
2. Determine the elements to be placed in region II. Find the elements in $A \cap B$. The elements in this set belong in regions II and V. Place the elements in the set $A \cap B$ that are not listed in region V in region II. The elements in regions IV and VI are found in a similar manner.
3. Determine the elements to be placed in region I by determining the elements in set *A* that are not in regions II, IV, and V. The elements in regions III and VII are found in a similar manner.
4. Determine the elements to be placed in region VIII by finding the elements in the universal set that are not in regions I through VII.

Example 1 illustrates the general procedure.

EXAMPLE 1 *Constructing a Venn Diagram for Three Sets*

Construct a Venn diagram illustrating the following sets.

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$
$$A = \{1, 5, 8, 9, 10, 12\}$$
$$B = \{2, 4, 5, 9, 10\}$$
$$C = \{1, 3, 5, 8, 9, 11\}$$

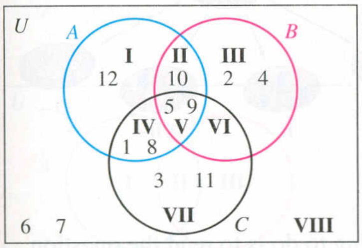


Figure 2.16

SOLUTION: First find the intersection of all three sets. Because the elements 5 and 9 are in all three sets, $A \cap B \cap C = \{5, 9\}$. The elements 5 and 9 are placed in region V in Fig. 2.16. Next complete region II by determining the intersection of sets *A* and *B*.

$$A \cap B = \{5, 9, 10\}$$

$A \cap B$ consists of regions II and V. The elements 5 and 9 have already been placed in region V, so 10 must be placed in region II.

Now determine what numbers go in region IV.

$$A \cap C = \{1, 5, 8, 9\}$$

Since 5 and 9 have already been placed in region V, place the 1 and 8 in region IV. Now determine the numbers to go in region VI.

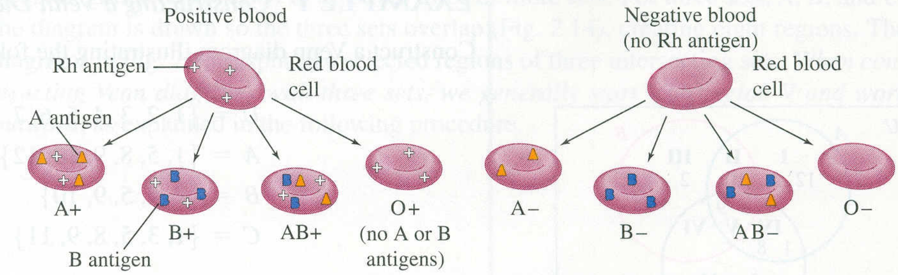
$$B \cap C = \{5, 9\}$$

Since both the 5 and 9 have been placed in region V, there are no numbers to be placed in region VI. Now complete set *A*. The only element of set *A* that has not previously been placed in regions II, IV, or V is 12. Therefore, place the element 12 in region I. The element 12 that is placed in region I is only in set A and not in set B or set C. Using set B, complete region III using the same general procedure used to determine the numbers in region I. Using set C, complete region VII by using the same procedure used to complete regions I and III. To determine the elements in region VIII, find the elements in U that have not been placed in regions I–VII. The elements 6 and 7 have not been placed in regions I–VII, so place them in region VIII. ▲

Venn diagrams can be used to illustrate and analyze many everyday problems. One example follows.

EXAMPLE 2 *Blood Types*

Human blood is classified (typed) according to the presence or absence of the specific antigens A, B, and Rh in the red blood cells. Antigens are highly specified proteins and carbohydrates that will trigger the production of antibodies in the blood to fight infection. Blood containing the Rh antigen is labeled positive, +, while blood lacking the Rh antigen is labeled negative, −. Blood lacking both A and B antigens is called type O. Sketch a Venn diagram with three sets A, B, and Rh and place each type of blood listed in the proper region. A person has only one type of blood.



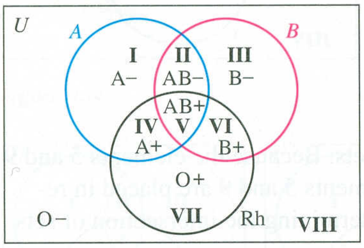


Figure 2.17

SOLUTION: As illustrated in Chapter 1, the first thing to do is to read the question carefully and make sure you understand what is given and what you are asked to find. There are three antigens A, B, and Rh. Therefore, begin by naming the three circles in a Venn diagram with the three antigens; see Fig. 2.17.

Any blood containing the Rh antigen is positive, and any blood not containing the Rh antigen is negative. Therefore, all blood in the Rh circle is positive, and all blood outside the Rh circle is negative. The intersection of all three sets, region V, is AB+. Region II contains only antigens A and B and is therefore AB−. Region I is A− because it contains only antigen A. Region III is B−, region IV is A+, and region VI is B+. Region VII is O+, containing only the Rh antigen. Region VIII, which lacks all three antigens, is O−. ▲

Verification of Equality of Sets

In this chapter, for clarity we may refer to operations on sets, such as $A \cup B'$ or $A \cap B \cap C$, as *statements involving sets* or simply as *statements*. Now we discuss how to determine if two statements involving sets are equal.

Consider the question: Is $A' \cup B = A' \cap B$ *for all sets A and B*? For the specific sets $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 3\}$, and $B = \{2, 4, 5\}$, is $A' \cup B = A' \cap B$? To answer the question, we do the following.

Find $A' \cup B$	Find $A' \cap B$
$A' = \{2, 4, 5\}$	$A' = \{2, 4, 5\}$
$A' \cup B = \{2, 4, 5\}$	$A' \cap B = \{2, 4, 5\}$

For these sets, $A' \cup B = A' \cap B$, because both set statements are equal to $\{2, 4, 5\}$. At this point you may believe that $A' \cup B = A' \cap B$ for all sets A and B.

If we select the sets $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$, and $B = \{2, 3\}$, we see that $A' \cup B = \{2, 3, 4\}$ and $A' \cap B = \{2\}$. For this case, $A' \cup B \neq A' \cap B$. Thus, we have proved that $A' \cup B \neq A' \cap B$ for all sets A and B by using a *counterexample*. A counterexample, as explained in Chapter 1, is an example that shows a statement is not true.

In Chapter 1, we explained that proofs involve the use of deductive reasoning. Recall that deductive reasoning begins with a general statement and works to a specific conclusion. To verify, or determine whether set statements are equal for any two sets selected, we use deductive reasoning with Venn diagrams. Venn diagrams are used because they can illustrate general cases. To determine if statements that contain sets, such as $(A \cup B)'$ and $A' \cap B'$, are equal for all sets A and B, we use the regions of Venn diagrams. If both statements represent the same regions of the Venn diagram, then the statements are equal for all sets A and B. See Example 3.

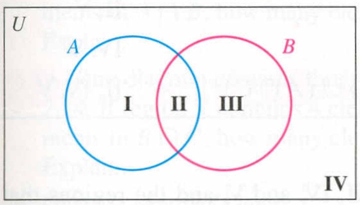


Figure 2.18

EXAMPLE 3 *Equality of Sets*

Determine whether $(A \cap B)' = A' \cup B'$ for all sets A and B.

SOLUTION: Draw a Venn diagram with two sets A and B, as in Fig. 2.18. Label the regions as indicated.

Find $(A \cap B)'$

Set	Corresponding Regions
A	I, II
B	II, III
$A \cap B$	II
$(A \cap B)'$	I, III, IV

Find $A' \cup B'$

Set	Corresponding Regions
A'	III, IV
B'	I, IV
$A' \cup B'$	I, III, IV

Both statements are represented by the same regions, I, III, and IV, of the Venn diagram. Thus, $(A \cap B)' = A' \cup B'$ for all sets A and B. ▲

In Example 3, when we proved that $(A \cap B)' = A' \cup B'$, we started with two general sets and worked to the specific conclusion that both statements represented

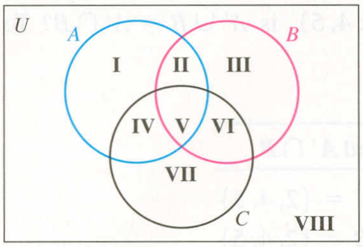


Figure 2.19

the same regions of the Venn diagram. We showed that $(A \cap B)' = A' \cup B'$ *for all sets A and B*. No matter what sets we choose for A and B, this statement will be true. For example, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{3, 4, 6, 10\}$, and $B = \{1, 2, 4, 5, 6, 8\}$.

$$(A \cap B)' = A' \cup B'$$
$$\{4, 6\}' = \{3, 4, 6, 10\}' \cup \{1, 2, 4, 5, 6, 8\}'$$
$$\{1, 2, 3, 5, 7, 8, 9, 10\} = \{1, 2, 5, 7, 8, 9\} \cup \{3, 7, 9, 10\}$$
$$\{1, 2, 3, 5, 7, 8, 9, 10\} = \{1, 2, 3, 5, 7, 8, 9, 10\}$$

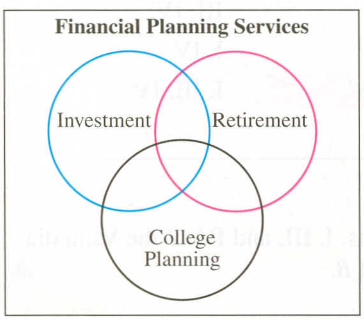
We can also use Venn diagrams to prove statements involving three sets.

MATHEMATICS *Everywhere*

Using Venn Diagrams

A financial planning company uses the following Venn diagram to categorize the financial planning services the company offers. From the diagram, we can see that the company offers financial planning in an "intersection" of the areas investment, retirement, and college planning.

We categorize items on a daily basis, from filing items to planning meals to planning social activities. Children are taught how to categorize items at an early age when they learn how to classify items according to color, shape, and size. Biologists categorize items when they classify organisms according to shared characteristics. A Venn diagram is a very useful tool to help order and arrange items and to picture the relationship between sets.



EXAMPLE 4 *Equality of Sets*

Determine whether $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets, A, B, and C.

SOLUTION: Because the statements include three sets, A, B, and C, three circles must be used. The Venn diagram illustrating the eight regions is shown in Fig. 2.19.

First we will find the regions that correspond to $A \cap (B \cup C)$, and then we will find the regions that correspond to $(A \cap B) \cup (A \cap C)$. If both answers are the same, the statements are equal.

Find $A \cap (B \cup C)$

Set	Corresponding Regions
A	I, II, IV, V
$B \cup C$	II, III, IV, V, VI, VII
$A \cap (B \cup C)$	II, IV, V

Find $(A \cap B) \cup (A \cap C)$

Set	Corresponding Regions
$A \cap B$	II, V
$A \cap C$	IV, V
$(A \cap B) \cup (A \cap C)$	II, IV, V

The regions that correspond to $A \cap (B \cup C)$ are II, IV, and V, and the regions that correspond to $(A \cap B) \cup (A \cap C)$ are also II, IV, and V. The results show that both statements are represented by the same regions, namely, II, IV, and V, and therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

In Example 4, we proved that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C. Show that this statement is true for the specific sets $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 7\}$, $B = \{2, 3, 4, 5, 7, 9\}$, and $C = \{1, 4, 7, 8, 10\}$.

De Morgan's Laws

In set theory, logic, and other branches of mathematics, a pair of related theorems known as De Morgan's laws make it possible to transform statements and formulas into alternative and often more convenient forms. In set theory, ***De Morgan's laws*** are symbolized as follows.

De Morgan's Laws

1. $(A \cup B)' = A' \cap B'$
2. $(A \cap B)' = A' \cup B'$

Law 2 was verified in Example 3. We suggest that you verify law 1 at this time. The laws were expressed verbally by William of Ockham in the fourteenth century. In the nineteenth century, Augustus De Morgan expressed them mathematically. De Morgan's laws will be discussed more thoroughly in Chapter 3, Logic.

SECTION 2.4 EXERCISES

Concept/Writing Exercises

1. How many regions are created when constructing a Venn diagram with three overlapping sets?
2. When constructing a Venn diagram with three overlapping sets, which region do you generally complete first?
3. When constructing a Venn diagram with three overlapping sets, after completing region V, which regions do you generally complete next?
4. A Venn diagram contains three sets, A, B, and C, as in Fig. 2.14. If region V contains 6 elements and there are 10 elements in $A \cap B$, how many elements belong in region II? Explain.
5. A Venn diagram contains three sets, A, B, and C, as in Fig. 2.14. If region V contains 4 elements and there are 12 elements in $B \cap C$, how many elements belong in region VI? Explain.
6. Give De Morgan's laws.
7. a) For $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 4, 5\}$, and $B = \{1, 4, 5\}$, does $A \cup B = A \cap B$?
   b) By observing the answer to part (a), can we conclude that $A \cup B = A \cap B$ for all sets A and B? Explain.
   c) Using a Venn diagram, determine if $A \cup B = A \cap B$ for all sets A and B.
8. What type of reasoning do we use when using Venn diagrams to verify or determine whether set statements are equal?

Practice the Skills

9. Construct a Venn diagram illustrating the following sets.

$U = \{a, b, c, d, e, f, g, h, i, j\}$
$A = \{c, d, e, g, h, i\}$
$B = \{a, c, d, g\}$
$C = \{c, f, i, j\}$

10. Construct a Venn diagram illustrating the following sets.

U = {Delaware, Pennsylvania, New Jersey, Georgia, Connecticut, Massachusetts, Maryland, South Carolina, New Hampshire, Virginia, New York, North Carolina, Rhode Island}

A = {New York, New Jersey, Pennsylvania, Massachusetts, New Hampshire}

B = {Delaware, Connecticut, Georgia, Maryland, New York, Rhode Island}

C = {New York, South Carolina, Rhode Island, Massachusetts}

11. Construct a Venn diagram illustrating the following sets.

U = {football, basketball, baseball, gymnastics, lacrosse, soccer, tennis, volleyball, swimming, wrestling, cross-country, track, golf, fencing}

A = {football, basketball, soccer, lacrosse, volleyball}

B = {baseball, lacrosse, tennis, golf, volleyball}

C = {swimming, gymnastics, fencing, basketball, volleyball}

12. Construct a Venn diagram illustrating the following sets.

U = {*The Lion King, Aladdin, Cinderella, Beauty and the Beast, Snow White and the Seven Dwarfs, Toy Story, 101 Dalmatians, The Little Mermaid, Jurassic Park*}

A = {*Aladdin, Toy Story, The Lion King, Snow White and the Seven Dwarfs*}

B = {*Snow White and the Seven Dwarfs, Toy Story, The Lion King, Beauty and the Beast*}

C = {*Snow White and the Seven Dwarfs, Toy Story, Beauty and the Beast, Cinderella, 101 Dalmatians*}

The Lion King

13. Construct a Venn diagram illustrating the following sets.

U = {peach, pear, banana, apple, grape, melon, carrot, corn, orange, spinach}

A = {pear, grape, melon, carrot}

B = {peach, pear, banana, spinach, corn}

C = {pear, banana, apple, grape, melon, spinach}

14. Construct a Venn diagram illustrating the following sets.

U = {Louis Armstrong, Glenn Miller, Stan Kenton, Charlie Parker, Duke Ellington, Benny Goodman, Count Basie, John Coltrane, Dizzy Gillespie, Miles Davis, Thelonius Monk}

A = {Stan Kenton, Count Basie, Dizzy Gillespie, Duke Ellington, Thelonius Monk}

B = {Louis Armstrong, Glenn Miller, Count Basie, Duke Ellington, Miles Davis}

C = {Count Basie, Miles Davis, Stan Kenton, Charlie Parker, Duke Ellington}

15. ***Olympic Medals*** Consider the chart, which shows countries that won at least 25 medals in the 2000 Summer Olympics. Let the teams shown in the chart represent the universal set.

	Gold	Silver	Bronze	Total
United States	39	25	33	97
Russia	32	28	28	88
China	28	16	15	59
Australia	16	25	17	58
Germany	14	17	26	57
France	13	14	11	38
Italy	13	8	13	34
Cuba	11	11	7	29
Great Britain	11	10	7	28
Korea	8	9	11	28
Romania	11	6	9	26
Netherlands	12	9	4	25

Source: *2001 Time Almanac*

Let A = set of teams that won at least 58 medals.

Let B = set of teams that won at least 20 gold medals.

Let C = set of teams that won at least 10 bronze medals.

Construct a Venn diagram that illustrates the sets A, B, and C.

16. ***Popular TV Shows*** Let U = {*Friends, CSI, NFL Monday Night Football, Survivor II, E.R., Who Wants to Be a Millionaire—Tues., Who Wants to Be a Millionaire—Wed., Who Wants to Be a Millionaire—Thurs., Who Wants to Be a Millionaire—Sun., Everybody Loves Raymond, 60 Minutes, Law& Order, West Wing*} Set A represents the five most popular prime time shows on television in 2001–2002, set B represents the five most popular prime-time shows on television in 2000–2001, and set C represents the five most popular prime-time shows on television in 1999–2000 (according to Nielsen Media Research). Then

A = {*Friends, CSI, E.R., Everybody Loves Raymond, Law & Order*}

B = {*Survivor II, E.R., Who Wants to Be a Millionaire—Wed., Who Wants to Be a Millionaire—Tues., Friends*}

C = {*Who Wants to Be a Millionaire—Tues., Who Wants to Be a Millionaire—Thurs., Who Wants to Be a Millionaire—Sun., E.R., Friends*}

Construct a Venn diagram illustrating the sets.

Graduate Schools The accompanying chart on page 73 shows the *U.S News and World Report* top 10 rankings of graduate schools in 2002 in the fields of medicine, education, and business. The universal set is the set of all U.S. graduate schools. In Fig. 2.20, the set indicated as Medical represents the 10 highest-rated medical schools, the set indicated as Education represents the 10 highest-rated graduate education schools, and the set listed as Business represents the 10 highest-rated graduate business schools.

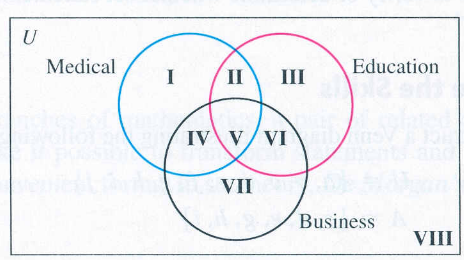


Figure 2.20

Top-Rated Graduate Schools

Medical	Education	Business
1. Harvard	1. Harvard	1. Stanford
2. Johns Hopkins	2. Stanford	2. Harvard
3. Duke	3. Columbia	3. Northwestern
4. University of Pennsylvania	4. UCLA	4. University of Pennsylvania
5. Washington University in St. Louis	5. Vanderbilt	5. M.I.T.
6. Columbia	6. University of California–Berkeley	6. Columbia
7. University of California–San Francisco	7. University of Michigan	7. University of California–Berkeley
8. Yale	8. University of Pennsylvania	8. Duke
9. Stanford	9. University of Wisconsin–Madison	9. University of Chicago
9. University of Michigan (tie)	10. Northwestern	10. University of Michigan

Source: *U.S. News*, www.usnews.com

Indicate in Fig. 2.20 in which region, I–VIII, each of the following belongs.

17. Harvard **18.** Yale

19. Boston College **20.** University of California–Berkeley

21. Northwestern **22.** Duke

Rankings of Metropolitan Areas The table above and to the right, taken from *Places Rated Almanac, Millennium Edition,* shows that almanac's rankings of the overall top 10 metropolitan areas across the United States and Canada. The table also shows the top 10 ranked areas for the categories of transportation and education. The universal set is the set of all metropolitan areas in the United States and Canada. In Fig. 2.21, the set indicated as Overall represents the set of metropolitan areas listed in the table under overall rankings.

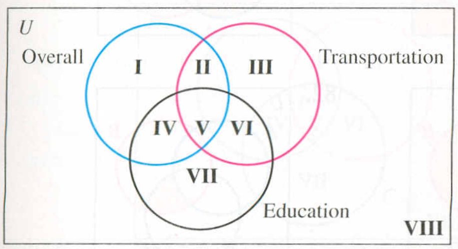


Figure 2.21

The set indicated as Transportation represents the set of metropolitan areas listed in the table under transportation, and the set indicated as Education represents the set of metropolitan areas listed in the table under education.

Rankings of Metropolitan Areas

Overall	Transportation	Education
1. Salt Lake City–Odgen	1. Chicago	1. Raleigh–Durham–Chapel Hill
2. Washington, D.C.	2. Pittsburgh	2. Boston
3. Seattle-Bellevue-Everett	3. New York	3. Albany-Schenectady-Troy
4. Tampa–St. Petersburg–Clearwater	4. Cincinnati	4. St. Louis
5. Denver	5. Detroit	5. Chicago
6. Raleigh–Durham–Chapel Hill	6. Denver	6. Rochester, NY
7. Toronto	7. Atlanta	7. Austin–San Marcos
8. Houston	8. Toronto	8. San Francisco
9. Minneapolis–St. Paul	9. St. Louis	9. Washington, D.C.
10. Phoenix-Mesa	10. Minneapolis–St. Paul	10. Saskatoon

Indicate in Fig. 2.21 in which region, I–VIII, each of the following metropolitan areas belongs.

23. Washington, D.C. **24.** Pittsburgh

25. Denver **26.** Houston

27. Rochester, NY **28.** Chicago

Figures *In Exercises 29–40, indicate in Fig. 2.22 the region in which each of the figures would be placed.*

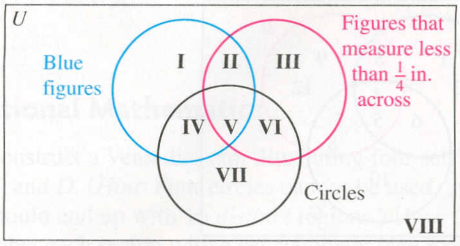


Figure 2.22

29.  **30.**  **31.** 

32.  **33.**  **34.** 

35. 
36. 
37. 
38.
39. 
40. 

Senate Bills *During a session of the U.S. Senate, three bills were voted on. The votes of six senators are shown below the figure. Determine in which region of Fig. 2.23 each senator would be placed. The set labeled Bill 1 represents the set of senators who voted yes on Bill 1, and so on.*

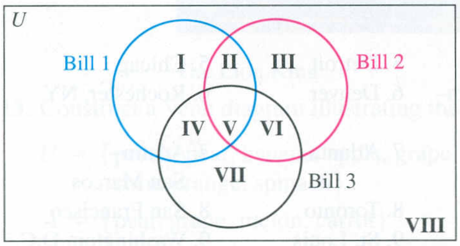


Figure 2.23

Senator	Bill 1	Bill 2	Bill 3
41. Grump	yes	no	no
42. Happi	no	no	yes
43. Turwilliger	no	no	no
44. Dillinger	yes	yes	yes
45. Isaitere	no	yes	yes
46. Smith	no	yes	no

In Exercises 47–60, use the Venn diagram in Fig. 2.24 to list the sets in roster form.

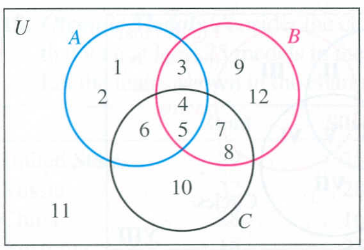


Figure 2.24

47. A
48. U
49. B
50. C
51. $A \cap B$
52. $A \cap C$
53. $(B \cap C)'$
54. $A \cap B \cap C$
55. $A \cup B$
56. $B \cup C$
57. $(A \cup C)'$
58. $A \cup B \cup C$
59. A'
60. $(A \cup B \cup C)'$

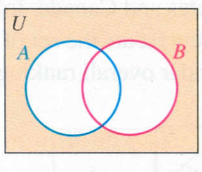
In Exercises 61–68, use Venn diagrams to determine whether the following statements are equal for all sets A and B.

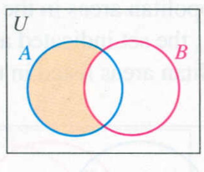
61. $(A \cup B)'$, $A' \cap B'$
62. $(A \cap B)'$, $A' \cup B$
63. $A' \cup B'$, $A \cap B$
64. $(A \cup B)'$, $(A \cap B)'$
65. $A' \cup B'$, $(A \cup B)'$
66. $A' \cap B'$, $A \cup B'$
67. $(A' \cap B)'$, $A \cup B'$
68. $A' \cap B'$, $(A' \cap B')'$

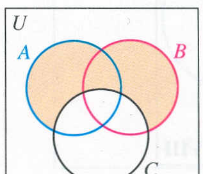
In Exercises 69–78, use Venn diagrams to determine whether the following statements are equal for all sets A, B, and C.

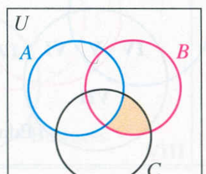
69. $A \cap (B \cup C)$, $(A \cap B) \cup C$
70. $A \cup (B \cap C)$, $(B \cap C) \cup A$
71. $A \cap (B \cup C)$, $(B \cup C) \cap A$
72. $A \cup (B \cap C)$, $A' \cap (B \cup C)$
73. $A \cap (B \cup C)$, $(A \cap B) \cup (A \cap C)$
74. $A \cup (B \cap C)$, $(A \cup B) \cap (A \cup C)$
75. $A \cap (B \cup C)'$, $A \cap (B' \cap C')$
76. $(A \cup B) \cap (B \cup C)$, $B \cup (A \cap C)$
77. $(A \cup B)' \cap C$, $(A' \cup C) \cap (B' \cup C)$
78. $(C \cap B)' \cup (A \cap B)'$, $A \cap (B \cap C)$

In Exercises 79–82, use set statements to write a description of the shaded area. Use union, intersection and complement as necessary. More than one answer may be possible.

79. 

80. 

81. 

82. 

Problem Solving

83. Let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 6, 7\}$$
$$C = \{6, 7, 9\}$$

a) Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for these sets.

b) Make up your own sets A, B, and C. Verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for your sets A, B, and C.

c) Use Venn diagrams to verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for all sets A, B, and C.

84. Let

$$U = \{a, b, c, d, e, f, g, h, i\}$$
$$A = \{a, c, d, e, f\}$$
$$B = \{c, d\}$$
$$C = \{a, b, c, d, e\}$$

a) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for these sets.

b) Make up your own sets, A, B, and C. Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for your sets.

c) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for all sets A, B, and C.

85. *Blood Types* A hematology text gives the following information on percentages of the different types of blood worldwide.

Type	Positive Blood, %	Negative Blood, %
A	37	6
O	32	6.5
B	11	2
AB	5	0.5

Construct a Venn diagram similar to the one in Example 2 and place the correct percent in each of the eight regions.

86. Define each of the eight regions in Fig. 2.25 using sets A, B, and C and a set operation. (*Hint:* $A \cap B' \cap C'$ defines region I.)

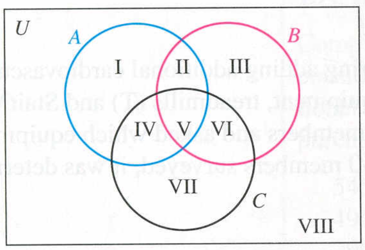


Figure 2.25

87. *Categorizing Contracts* J & C Mechanical Contractors wants to classify its projects. The contractors categorize set A as the set of office building construction projects, set B as the set of plumbing projects, and set C as the set of projects with a budget greater than $300,000.

a) Draw a Venn diagram that can be used to categorize the company projects according to the listed criteria.

b) Determine the region of the diagram that contains office building construction and plumbing projects with a budget greater than $300,000. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

c) Determine the region of the diagram that contains plumbing projects with a budget greater than $300,000 that are not office building construction projects. Describe the region using sets A, B and C with set operations. Use union, intersection, and complement as necessary.

d) Determine the region of the diagram that contains office building construction and nonplumbing projects whose budget is less than or equal to $300,000. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

Challenge Problem/Group Activity

88. We were able to determine the number of elements in the union of two sets with the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Can you determine a formula for finding the number of elements in the union of three sets? In other words, write a formula to determine $n(A \cup B \cup C)$. [*Hint*: The formula will contain each of the following: $n(A)$, $n(B)$, $n(C)$, $n(A \cap B \cap C')$, $n(A \cap B' \cap C)$, $n(A' \cap B \cap C)$, and $2n(A \cap B \cap C)$.]

Recreational Mathematics

89. a) Construct a Venn diagram illustrating four sets, A, B, C, and D. (*Hint*: Four circles cannot be used, and you should end up with 16 *distinct* regions.) Have fun!

b) Label each region with a set statement (see Exercise 86). Check all 16 regions to make sure that *each is distinct*.

90. *Triangle Seek* In the word seek on the next page, you are looking for six-letter words that form triangles. You need to trace out a triangle as you move from the first letter to the last letter and back to the first letter. The first letter of the word can be in any position in the triangle. Triangles may overlap other triangles, and the triangles can point up

or down. The word list is below. The word PROPER is shown as an example.

S U B T R H M D F R T S

T S J O N B U I E L E M

W E S N E R N I T V O

P R S A K C P A R N E P

O R P O N R R O Y I M T

I R J I E S O C L U S

F M U N G R T R N E P P

Word List

PROPER

SUBSET

FINITE

NUMBER

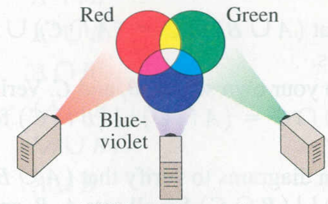
ROSTER

PENCIL

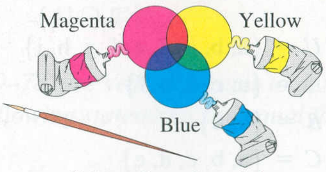
REGION

Internet/Research Activity

91. The two Venn diagrams illustrate what happens when colors are added or subtracted. Do research in an art text, an encyclopedia, the Internet, or another source and write a report explaining the creation of the colors in the Venn diagrams, using such terms as union of colors and subtraction (or difference) of colors.

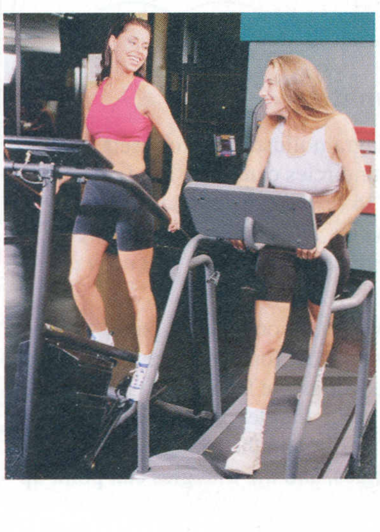


Additive color mixing



Subtractive color mixing

2.5 APPLICATIONS OF SETS



We can solve practical problems involving sets by using the problem-solving process discussed in Chapter 1: Understand the problem, devise a plan, carry out the plan, and then examine and check the results. First determine: What is the problem? or What am I looking for? To devise the plan, list all the facts that are given and how they are related. *Look for key words or phrases* such as "only set *A*," "set *A* and set *B*," "set *A* or set *B*," "set *A* and set *B* and not set *C*." Remember that *and* means intersection, *or* means union, and *not* means complement. The problems we solve in this section contain two or three sets of elements, which can be represented in a Venn diagram. Our plan will generally include drawing a Venn diagram, labeling the diagram, and filling in the regions of the diagram.

Whenever possible, follow the procedure in Section 2.4 for completing the Venn diagram and then answer the questions. Remember, when drawing Venn diagrams, we generally start with the intersection of the sets and work outward.

EXAMPLE 1 *Fitness Equipment*

Fitness for Life health club is considering adding additional cardiovascular equipment. It is considering two types of equipment, treadmills (T) and StairMasters (S). The health club surveyed a sample of members and asked which equipment they had used in the previous month. Of 150 members surveyed, it was determined that

102 used the treadmills.

71 used the StairMasters.

40 used both types.

Of those surveyed,

a) how many did not use either the treadmill or the StairMaster?
b) how many used the treadmill but not the StairMaster?
c) how many used the StairMaster but not the treadmill?
d) how many used either the treadmill or StairMaster?

SOLUTION: The problem provides the following information.

The number of members surveyed is 150: $n(U)$ is 150.
The number of members surveyed who used the treadmill is 102: $n(T) = 102$.
The number of members surveyed who used the StairMaster is 71: $n(S) = 71$.
The number of members surveyed who used both the treadmill and the StairMaster is 40: $n(T \cap S) = 40$.

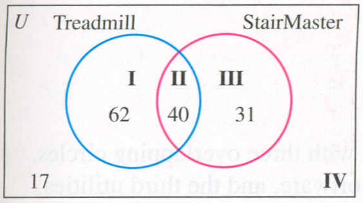


Figure 2.26

We illustrate this information on the Venn diagram shown in Fig. 2.26. We already know that $T \cap S$ corresponds to region II. As $n(T \cap S) = 40$, we write 40 in region II. Set T consists of regions I and II. We know that set T, the members who used the treadmill, contains 102 members. Therefore, region I contains $102 - 40$, or 62 members. We write the number 62 in region I. Set S consists of regions II and III. As $n(S) = 71$, the total in these two regions must be 71. Region II contains 40, leaving $71 - 40$ or 31 for region III. We write 31 in region III.

The total number of members surveyed who used the treadmill or the StairMaster is found by adding the numbers in regions I, II, and III. Therefore, $n(T \cup S) = 62 + 40 + 31 = 133$. The number in region IV is the difference between $n(U)$ and $n(T \cup S)$. There are $150 - 133$, or 17 members in region IV.

a) The members surveyed who did not use either the treadmill or the StairMaster are those members of the universal set who are not contained in set T or set S. The 17 members in region IV did not use the treadmill or StairMaster.

b) The 62 members in region I are those members surveyed who used the treadmill but not the StairMaster.

c) The 31 members in region III are those members surveyed who used the StairMaster but not the treadmill.

d) The members in regions I, II, or III are those members surveyed who used either the treadmill or the StairMaster. Thus, $62 + 40 + 31$ or 133 members surveyed used either the treadmill or the StairMaster. Notice that the 40 members in region II who use both types of equipment are included in those members surveyed who used either the treadmill or the StairMaster. ▲

Similar problems involving three sets can be solved, as illustrated in Example 2.

EXAMPLE 2 *Software Purchases*

CompUSA is considering expanding their computer software department. They are considering additional space for three types of computer software: games, educational software, and utility programs. The following information regarding software purchases was obtained from a survey of 893 customers.

545 purchased games.
497 purchased educational software.
290 purchased utility programs.
297 purchased games and educational software.

196 purchased educational software and utility programs.
205 purchased games and utility programs.
157 purchased all three types of software.

Use a Venn diagram to answer the following questions. How many customers purchased

a) none of these types of software?
b) only games?
c) at least one of these types of software?
d) exactly two of these types of software?

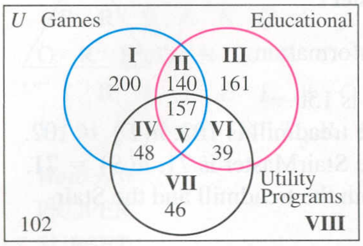


Figure 2.27

SOLUTION: Begin by constructing a Venn diagram with three overlapping circles. One circle represents games, another educational software, and the third utilities. See Fig 2.27. Label the eight regions.

Whenever possible, work from the center of the diagram outwards. First fill in region V. Since 157 customers purchased all three types of software, we place 157 in region V. Next determine the number to be placed in region II. Regions II and V together represent the customers who purchased both games and educational software. Since 297 customers purchased both of these types of software, the sum of the numbers in these regions must be 297. Since 157 have already been placed in region V, $297 - 157 = 140$ must be placed in region II. Now we determine the number to be placed in region IV. Since 205 customers purchased both games and utility programs, the sum of the numbers in regions IV and V must be 205. Since 157 have already been placed in region V, $205 - 157 = 48$ must be placed in region IV. Now determine the number to be placed in region VI. A total of 196 customers purchased educational software and utility programs. The numbers in regions V and VI must total 196. Since 157 have already been placed in region V, the number to be placed in region VI is $196 - 157 = 39$.

Now that we have determined the numbers for regions V, II, IV, and VI, we can determine the numbers to be placed in regions I, III, and VII. We are given that 545 customers purchased games. The sum of the numbers in regions I, II, IV, and V must be 545. To determine the number to be placed in region I, subtract the amounts in regions II, IV, and V from 545. There must be $545 - 140 - 48 - 157 = 200$ in region I. Determine the numbers to be placed in regions III and VII in a similar manner.

$$\text{Region III} = 497 - 140 - 157 - 39 = 161$$
$$\text{Region VII} = 290 - 48 - 157 - 39 = 46$$

Now that we have determined the numbers in regions I through VII, we can determine the number to be placed in region VIII. Adding the numbers in regions I through VII yields a sum of 791. The difference between the total number of customers surveyed, 893, and the sum of the numbers in regions I through VII must be placed in region VIII.

$$\text{Region VIII} = 893 - 791 = 102$$

Now that we have completed the Venn diagram, we can answer the questions.

a) One hundred two customers did not purchase any of these types of software. These customers are indicated in region VIII.

b) Region I represents those customers who purchased only games. Thus, 200 customers purchased only games.

c) The words *at least one* mean "one or more." All those in regions I through VII purchased at least one of the types of software. The sum of the numbers in regions I through VII is 791, so 791 customers purchased at least one of the types of software.

d) The customers in regions II, IV, and VI purchased exactly two of the types of software. Summing the numbers in these regions 140 + 48 + 39 we find that 227 customers purchased exactly two of these types of software. Notice that we did not include the customers in region V. Those customers purchased all three types of software.

The procedure to work problems like those given in Example 2 is generally the same. Start by completing region V. Next complete regions II, IV, and VI. Then complete regions I, III, and VII. Finally, complete region VIII. When you are constructing Venn diagrams, be sure to check your work carefully.

TIMELY TIP When constructing a Venn diagram, the most common mistake made by students is forgetting to subtract the number in region V from the respective values in determining the numbers to be placed in regions II, IV, and VI.

EXAMPLE 3 *Birds at the Feeders*

In a bird sanctuary, 41 different species of birds are being studied. Three large bird feeders are constructed, each providing a different type of bird feed. One feeder has sunflower seeds. A second feeder has a mixture of seeds, and the third feeder has small pieces of fruit. The following information was obtained.

20 species ate sunflower seeds.
22 species ate the mixture.
11 species ate the fruit.
10 species ate the sunflower seeds and the mixture.
4 species ate the sunflower seeds and the fruit.
3 species ate the mixture and the fruit.
1 species ate all three.

Use a Venn diagram to answer the following questions. How many species ate

a) none of the foods?
b) the sunflower seeds, but neither of the other two foods?
c) the mixture *and* the fruit, but not the sunflower seeds?
d) the mixture *or* the fruit, but not the sunflower seeds?
e) exactly one of the foods?

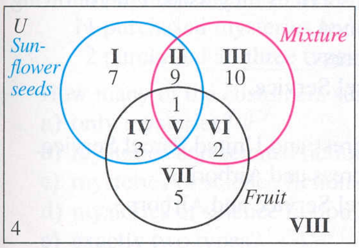


Figure 2.28

SOLUTION: The Venn diagram is constructed using the procedure we outlined in Example 2. The diagram is illustrated in Fig. 2.28. We suggest you construct the diagram by yourself now and check your diagram with Fig. 2.28.

a) Four species did not eat any of the food (see region VIII).

b) Seven species (see region I) ate the sunflower seeds but neither of the other two foods.

c) Those in region VI ate both the mixture and fruit but not the sunflower seeds. Therefore, two species satisfy the criteria.

d) The word *or* in this type of problem means one or the other or both. All the species in regions II, III, IV, V, VI, and VII ate the mixture or the fruit or both. Those in regions II, IV, and V also ate the sunflower seeds. The species that ate the mixture or fruit, but not the sunflower seeds, are found by adding the numbers in regions III, VI, and VII. There are $10 + 2 + 5 = 17$ species that satisfy the criteria.

e) Those species indicated in regions I, III, and VII ate exactly one of the foods. Therefore, $7 + 10 + 5 = 22$ species ate exactly one of the three types of bird food. ▲

SECTION 2.5 EXERCISES

Practice the Skills/Problem Solving

In Exercises 1–15, draw a Venn diagram to obtain the answers.

1. *Landscape Purchases* Agway Lawn and Garden collected the following information regarding purchases from 130 of its customers.
   74 purchased shrubs.
   70 purchased trees.
   41 purchased both shrubs and trees.
   Of those surveyed,
   a) how many purchased only shrubs?
   b) how many purchased only trees?
   c) how many did not purchase either of these items?

2. *Study Locations* At a local college, a survey was taken to determine where students studied on campus. Of 160 students surveyed, it was determined that
   79 studied in the library.
   65 studied in the student lounge.
   43 studied in both the library and the student lounge.
   Of those interviewed,
   a) how many studied in only the library?
   b) how many studied in only the student lounge?
   c) how many did not study in either location?

3. *Real Estate* The Maiello's are moving to Wilmington, Delaware. Their real estate agent located 83 houses listed for sale, in the Wilmington area, in their price range. Of these houses listed for sale,
   47 had a family room.
   42 had a deck.
   30 had a family room and a deck.
   How many had
   a) a family room but not a deck?
   b) a deck but not a family room?
   c) either a family room or a deck?

4. *Toothpaste Taste Test* A drug company is considering manufacturing a new toothpaste. They are considering two flavors, regular and mint. In a sample of 120 people, it was found that
   74 liked the regular.
   62 liked the mint.
   35 liked both types.
   a) How many liked only the regular?
   b) How many liked only the mint?
   c) How many liked either one or the other or both?

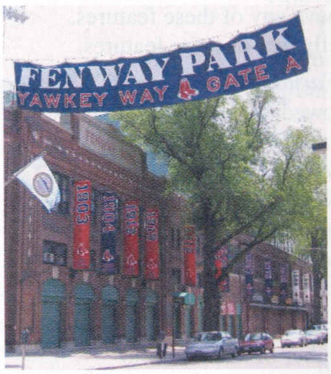
5. *Overnight Delivery Services* In San Diego, California, a sample of 444 businesses was surveyed to determine which overnight mailing services they used. The following information was determined.
   189 used Federal Express.
   205 used United Parcel Service.
   122 used Airborne.
   57 used Federal Express and United Parcel Service.
   34 used Federal Express and Airborne.
   30 used United Parcel Service and Airborne.
   22 used all three.
   How many used
   a) none of these services?
   b) only Airborne?
   c) exactly one of the services?
   d) exactly two of the services?
   e) Federal Express or United Parcel Service?

6. ***Professional Teams*** Thirty-three U.S. cities with large populations were surveyed to determine whether they had a professional baseball team, a professional football team, or a professional basketball team. The following information was determined.

 16 had baseball.
 17 had football.
 15 had basketball.
 11 had baseball and football.
 7 had baseball and basketball.
 9 had football and basketball.
 5 had all three teams.

How many had

a) only a football team?
b) baseball and football, but not basketball?
c) baseball or football?
d) baseball or football, but not basketball?
e) exactly two teams?



7. ***Book Purchases*** A survey of 85 customers was taken at Barnes & Noble regarding the types of books purchased. The survey found that

 44 purchased mysteries.
 33 purchased science fiction.
 29 purchased romance novels.
 13 purchased mysteries and science fiction.
 5 purchased science fiction and romance novels.
 11 purchased mysteries and romance novels.
 2 purchased all three types of books.

How many of the customers surveyed purchased

a) only mysteries?
b) mysteries and science fiction, but not romance novels?
c) mysteries or science fiction?
d) mysteries or science fiction, but not romance novels?
e) exactly two types?

8. ***Resorts*** In a survey of 65 resorts, it was found that

 34 provided refrigerators in the guest rooms.
 30 provided laundry services.
 37 provided child care services.
 15 provided refrigerators in the guest rooms and laundry services.
 17 provided refrigerators in the guest rooms and child care services.
 19 provided laundry services and child care services.
 7 provided all three features.

How many of the resorts provided

a) only refrigerators in the guest rooms?
b) exactly one of the features?
c) at least one of the features?
d) exactly two of the features?
e) none of the features?

9. ***Transportation*** Seventy businesspeople in Sacramento, California, were asked how they traveled to work during the previous month. The following information was determined.

 28 used public transportation.
 18 rode in a car pool.
 37 drove alone.
 5 used public transportation and rode in a car pool.
 9 used public transportation and drove alone.
 4 rode in a car pool and drove alone.
 3 used all three forms of transportation.

How many of those surveyed

a) only used public transportation?
b) only drove alone?
c) used public transportation and rode in a car pool, but did not drive alone?
d) used public transportation or rode in a car pool, but did not drive alone?
e) used none of these forms of transportation?

10. ***Colleges and Universities*** In a survey of four-year colleges and universities, it was found that

 356 offered a liberal arts degree.
 293 offered a computer engineering degree.
 285 offered a nursing degree.
 193 offered a liberal arts degree and a computer engineering degree.
 200 offered a liberal arts degree and a nursing degree.
 139 offered a computer engineering degree and a nursing degree.
 68 offered a liberal arts degree, a computer engineering degree, and a nursing degree.
 26 offered none of these degrees.

a) How many four-year colleges and universities were surveyed?

Of the four-year colleges and universities surveyed, how many offered

b) a liberal arts degree and a nursing degree, but not a computer engineering degree?
c) a computer engineering degree, but neither a liberal arts degree nor a nursing degree?
d) exactly two of these degrees?
e) at least one of these degrees?

11. ***Fastest-Growing Cities*** The following census information was collected regarding the 25 largest U.S. cities based on the 2000 population.

5 cities were in Texas.
13 cities had a population greater than 750,000.
11 cities had a population increase greater than 10% from 1990 to 2000.
3 cities had a population greater than 750,000 and were in Texas.
4 cities had a population increase greater than 10% from 1990 to 2000 and were in Texas.
6 cities had a population greater than 750,000 and a population increase greater than 10% from 1990 to 2000.
3 cities had a population greater than 750,000, a population increase greater than 10% from 1990 to 2000, and were in Texas.

How many of the 25 largest cities

a) with a population greater than 750,000 were not in Texas?
b) were in Texas or had a population greater than 750,000?
c) were in Texas and had a population greater than 750,000, but did not have a population increase greater than 10% from 1990 to 2000?
d) were not in Texas, did not have a population greater than 750,000 and did not have a population increase greater than 10% from 1990 to 2000?

12. *Appetizers Survey* Da Tulio's Restaurant hired Dennis Goldstein to determine what kind of appetizers customers liked. He surveyed 100 people, with the following results: 78 liked shrimp cocktail, 56 liked mozzarella sticks, and 35 liked both shrimp cocktail and mozzarella sticks. Every person interviewed liked one or the other or both kinds of appetizers. Does this result seem correct? Explain your answer.

13. *Discovering an Error* An immigration agent sampled cars going from the United States into Canada. In his report, he indicated that of the 85 cars sampled,

35 cars were driven by women.
53 cars were driven by U.S. citizens.
43 cars had two or more passengers.
27 cars were driven by women who are U.S. citizens.
25 cars were driven by women and had two or more passengers.
20 cars were driven by U.S. citizens and had two or more passengers.
15 cars were driven by women who are U.S. citizens and had two or more passengers.

After his supervisor reads the report, she explains to the agent that he made a mistake. Explain how his supervisor knew that the agent's report contained an error.

Challenge Problems/Group Activities

14. *Parks* A survey of 300 parks showed the following.

15 had only camping.
20 had only hiking trails.
35 had only picnicking.
185 had camping.
140 had camping and hiking trails.
125 had camping and picnicking.
210 had hiking trails.

Find the number of parks that

a) had at least one of these features.
b) had all three features.
c) did not have any of these features.
d) had exactly two of these features.

15. *Surveying Farmers* A survey of 500 farmers in a midwestern state showed the following.

125 grew only wheat.
110 grew only corn.
90 grew only oats.
200 grew wheat.
60 grew wheat and corn.
50 grew wheat and oats.
180 grew corn.

Find the number of farmers who

a) grew at least one of the three.
b) grew all three.
c) did not grow any of the three.
d) grew exactly two of the three.

Recreational Mathematics

16. *Number of Elements* A universal set U consists of 12 elements. If sets A, B, and C are proper subsets of U and $n(U) = 12$, $n(A \cap B) = n(A \cap C) = n(B \cap C) = 6$, $n(A \cap B \cap C) = 4$, and $n(A \cup B \cup C) = 10$ determine

a) $n(A \cup B)$ b) $n(A' \cup C)$ c) $n(A \cap B)'$

2.6 INFINITE SETS

On page 44, we state that a finite set is a set in which the number of elements is zero or the number of elements can be expressed as a natural number. On page 45, we define a one-to-one correspondence. To determine the number of elements in a finite set, we can place the set in a one-to-one correspondence with a subset of the set of counting numbers. For example, the set $A = \{\#, ?, \$\}$ can be placed in one-to-one correspondence with set $B = \{1, 2, 3\}$, a subset of the set of counting numbers.

$$\begin{aligned} A &= \{\#, ?, \$\} \\ &\;\;\;\downarrow\;\downarrow\;\downarrow \\ B &= \{1, 2, 3\} \end{aligned}$$

Because the cardinal number of set B is 3, the cardinal number of set A is also 3. Any two sets such as set A and set B that can be placed in a one-to-one correspondence must have the same number of elements (therefore the same cardinality) and must be equivalent sets. Note that $n(A)$ and $n(B)$ both equal 3.

The German mathematician Georg Cantor (1845–1918), known as the father of set theory, thought about sets that were not bounded. He called an unbounded set an *infinite set* and provided the following definition.

> An **infinite set** is a set that can be placed in a one-to-one correspondence with a proper subset of itself.

In Example 1, we use Cantor's definition of an infinite set to show that the set of counting numbers is infinite.

EXAMPLE 1 *The Set of Natural Numbers*

Show that $N = \{1, 2, 3, 4, 5, \ldots, n, \ldots\}$ is an infinite set.

SOLUTION: To show that the set N is infinite, we establish a one-to-one correspondence between the counting numbers and a proper subset of itself. By removing the first element from the set of counting numbers, we get the set $\{2, 3, 4, 5, \ldots\}$, which is a proper subset of the set of counting numbers. Now we establish the one-to-one correspondence.

$$\begin{aligned} \text{Counting numbers} &= \{1, 2, 3, 4, 5, \ldots, \quad n \quad, \ldots\} \\ &\quad\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow \qquad\;\; \downarrow \\ \text{Proper subset} &= \{2, 3, 4, 5, 6, \ldots, n+1, \ldots\} \end{aligned}$$

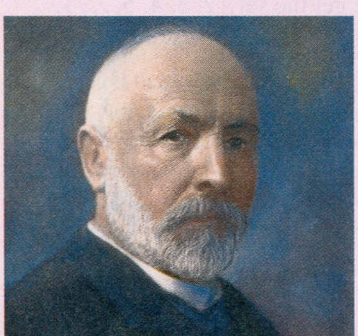
Note that for any number, n, in the set of counting numbers, its corresponding number in the proper subset is one greater, or $n + 1$. We have now shown the desired one-to-one correspondence, and thus the set of counting numbers is infinite. ▲

PROFILE IN MATHEMATICS

LEOPOLD KRONECKER

Mathematician Leopold Kronecker (1823–1891), Cantor's former mentor, ridiculed Cantor's theories and prevented Cantor from gaining a position at the University of Berlin. Although Cantor's work on infinite sets is now considered a masterpiece, it generated heated controversy when originally published. Cantor's claim that the infinite set was unbounded offended the religious views of the time that God had created a complete universe, which could not be wholly comprehended by humans. Eventually Cantor was given the recognition due to him, but by then the criticism had taken its toll on his health. He had several nervous breakdowns and spent his last days in a mental hospital.

GEORG CANTOR



Note in Example 1 that we showed the pairing of the general terms $n \rightarrow (n + 1)$. Showing a one-to-one correspondence of infinite sets requires showing the pairing of the general terms in the two infinite sets.

In the set of counting numbers, n represents the general term. For any other set of numbers, the general term will be different. The general term in any set should be written in terms of n such that when 1 is substituted for n in the general term, we get the first number in the set; when 2 is substituted for n in the general term, we get the second number in the set; when 6 is substituted for n in the general term, we get the sixth number in the set; and so on.

Consider the set $\{4, 9, 14, 19, \ldots\}$. Suppose we want to write the general term for this set (or sequence) or numbers. What would the general term be? The numbers differ by 5, so the general term will be of the form $5n$ plus or minus some number. Substituting 1 for n yields 5(1), or 5. Because the first number in the set is 4, we need to subtract 1 from the 5. Thus, the general term is $5n - 1$. Note that when $n = 1$, the value is $5(1) - 1$ or 4; when $n = 2$, the value is $5(2) - 1$ or 9; when $n = 3$, the value is $5(3) - 1$ or 14; and so on. Therefore, we write the set of numbers with the general term as

$$\{4, 9, 14, 19, \ldots, 5n - 1, \ldots\}$$

Now that you are aware of how to determine the general term of a set of numbers, we can do some more problems involving sets.

EXAMPLE 2 *The Set of Even Numbers*

Show that the set of even counting numbers $\{2, 4, 6, \ldots, 2n, \ldots\}$ is an infinite set.

SOLUTION: First create a proper subset of the set of even counting numbers by removing the first number from the set. Then establish a one-to-one correspondence.

$$\begin{array}{ll} \text{Even counting numbers:} & \{2, 4, 6, 8, \ldots, 2n, \ldots\} \\ & \;\;\downarrow\;\downarrow\;\downarrow\;\downarrow \qquad\quad \downarrow \\ \text{Proper subset:} & \{4, 6, 8, 10, \ldots, 2n + 2, \ldots\} \end{array}$$

A one-to-one correspondence exists between the two sets, so the set of even counting numbers is infinite. ▲

EXAMPLE 3 *The Set of Multiples of Three*

Show that the set $\{3, 6, 9, 12, \ldots, 3n, \ldots\}$ is an infinite set.

SOLUTION:

$$\begin{array}{ll} \text{Given set:} & \{3, 6, 9, 12, 15, \ldots, 3n, \ldots\} \\ & \;\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow \qquad \downarrow \\ \text{Proper subset:} & \{6, 9, 12, 15, 18, \ldots, 3n + 3, \ldots\} \end{array}$$

Therefore, the given set is an infinite set. ▲

Countable Sets

In his work with infinite sets, Cantor developed ideas on how to determine the cardinal number of an infinite set. He called the cardinal number of infinite sets "transfinite cardinal numbers" or "transfinite powers." He defined a set as ***countable*** if it is finite or if it can be placed in a one-to-one correspondence with the set of counting numbers. All infinite sets that can be placed in a one-to-one correspondence with the set of counting numbers have cardinal number, ***aleph-null***, symbolized $\aleph_0$ (the first Hebrew letter, aleph, with a zero subscript, read "null").

EXAMPLE 4 *The Cardinal Number of the Set of Even Numbers*

Show that the set of even counting numbers has cardinal number $\aleph_0$.

SOLUTION: In Example 2, we showed that a set of even counting numbers is infinite by setting up a one-to-one correspondence between the set and a proper subset of itself.

Now we will show that it is countable and has cardinality $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the set of even counting numbers.

$$\begin{array}{ll} \text{Counting numbers:} & N = \{1, 2, 3, 4, \ldots, \;\; n, \ldots\} \\ & \qquad\;\; \downarrow\;\downarrow\;\downarrow\;\downarrow \qquad\;\; \downarrow \\ \text{Even counting numbers:} & E = \{2, 4, 6, 8, \ldots, 2n, \ldots\} \end{array}$$

For each number n in the set of counting numbers, its corresponding number is $2n$. Since we found a one-to-one correspondence between the set of counting numbers and the set of even counting numbers, the set of even counting numbers is countable. Thus, the cardinal number of the set of even counting numbers is $\aleph_0$; that is, $n(E) = \aleph_0$. As we mentioned earlier, the set of even counting numbers is an infinite set since it can be placed in a one-to-one correspondence with a proper subset of itself. Therefore, the set of even counting numbers is both infinite and countable. ▲

> Any set that can be placed in a one-to-one correspondence with the set of counting numbers has cardinality $\aleph_0$ and is countable.

EXAMPLE 5 *The Cardinal Number of the Set of Odd Numbers*

Show that the set of odd counting numbers has cardinality $\aleph_0$.

SOLUTION: To show that the set of odd counting numbers has cardinality $\aleph_0$, we need to show a one-to-one correspondence between the counting numbers and the odd counting numbers.

$$\begin{array}{ll} \text{Counting numbers:} & N = \{1, 2, 3, 4, 5, \ldots, \qquad n \qquad\}\\ & \qquad\;\; \downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow \qquad\quad \downarrow \\ \text{Odd counting numbers:} & O = \{1, 3, 5, 7, 9, \ldots, 2n - 1, \ldots\} \end{array}$$

Since there is a one-to-one correspondence, the odd counting numbers have cardinality $\aleph_0$; that is, $n(O) = \aleph_0$. ▲

We have shown that both the odd and even counting numbers have cardinality $\aleph_0$. Merging the odd counting numbers with the even counting numbers gives the set of counting numbers, and we may reason that

$$\aleph_0 + \aleph_0 = \aleph_0$$

This result may seem strange, but it is true. What could such a statement mean? Well, consider a hotel with infinitely many rooms. If all the rooms are occupied, then the hotel is, of course, full. If more guests appear wanting accommodations, will they be turned away? The answer is *no,* for if the room clerk were to reassign each guest to a new room with a room number twice that of the present room, then all the odd-numbered rooms would become unoccupied and there would be space for more guests!

. . . where there's always room for one more. . .

In Cantor's work, he showed that there are different orders of infinity. Sets that are countable and have cardinal number $\aleph_0$ are the lowest order of infinity. Cantor showed that the set of integers and the set of rational numbers (fractions of the form p/q, where $q \neq 0$) are infinite sets with cardinality $\aleph_0$. He also showed that the set of real numbers (discussed in Chapter 5) could not be placed in a one-to-one correspondence with the set of counting numbers and that they have a higher order of infinity, aleph-one, $\aleph_1$.

SECTION 2.6 EXERCISES

Concept/Writing Exercises

1. What is an infinite set as defined in this section?
2. a) What is a countable set?
   b) How can we determine if a given set has cardinality $\aleph_0$?

Practice the Skills

In Exercises 3–12, show that the set is infinite by placing it in a one-to-one correspondence with a proper subset of itself. Be sure to show the pairing of the general terms in the sets.

3. $\{7, 8, 9, 10, 11, \ldots\}$
4. $\{12, 13, 14, 15, 16, \ldots\}$
5. $\{3, 5, 7, 9, 11, \ldots\}$
6. $\{20, 22, 24, 26, 28, \ldots\}$
7. $\{4, 7, 10, 13, 16, \ldots\}$
8. $\{4, 8, 12, 16, 20, \ldots\}$
9. $\{6, 11, 16, 21, 26, \ldots\}$
10. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
11. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\}$
12. $\{\frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \frac{9}{11}, \frac{10}{11}, \ldots\}$

In Exercises 13–22, show that the set has cardinal number $\aleph_0$ by establishing a one-to-one correspondence between the set of counting numbers and the given set. Be sure to show the pairing of the general terms in the sets.

13. $\{6, 12, 18, 24, 30, \ldots\}$
14. $\{50, 51, 52, 53, 54, \ldots\}$
15. $\{4, 6, 8, 10, 12, \ldots\}$
16. $\{0, 2, 4, 6, 8, \ldots\}$
17. $\{2, 5, 8, 11, 14, \ldots\}$
18. $\{4, 9, 14, 19, 24, \ldots\}$
19. $\{5, 8, 11, 14, 17, \ldots\}$
20. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\}$
21. $\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\}$
22. $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$

Challenge Problems/Group Activities

In Exercises 23–26, show that the set has cardinality $\aleph_0$ by establishing a one-to-one correspondence between the set of counting numbers and the given set.

23. $\{1, 4, 9, 16, 25, 36, \ldots\}$
24. $\{2, 4, 8, 16, 32, \ldots\}$
25. $\{3, 9, 27, 81, 243, \ldots\}$
26. $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \ldots\}$

Recreational Mathematics

In Exercises 27–31, insert the symbol <, >, or = in the shaded area to make a true statement.

27. $\aleph_0$ ■ $\aleph_0 + \aleph_0$

28. $2\aleph_0$ ■ $\aleph_0 + \aleph_0$

29. $2\aleph_0$ ■ $\aleph_0$

30. $\aleph_0 + 5$ ■ $\aleph_0 - 3$

31. $n(N)$ ■ $\aleph_0$

32. There are a number of paradoxes (a statement that appears to be true and false at the same time) associated with infinite sets and the concept of infinity. One of these, called *Zeno's Paradox*, is named after the mathematician Zeno, born about 496 BC in Italy. According to Zeno's paradox, suppose Achelles starts out 1 meter behind a tortoise. Also, suppose that Achelles walks 10 times as fast as the tortoise crawls. When Achelles reaches the point where the tortoise started, the tortoise is 1/10 of a meter ahead of Achelles. When Achelles reaches the point where the tortoise was 1/10 of a meter ahead, the tortoise is now 1/100 of a meter ahead. And so on. According to Zeno's Paradox, Achelles gets closer and closer to the tortoise but never catches up to the tortoise.
   a) Do you believe the reasoning process is sound? If not, explain why not.
   b) In actuality, if this were a real situation, would Achelles ever pass the tortoise?

Internet/Research Activities

33. Do research to explain how Cantor proved that the set of rational numbers has cardinal number $\aleph_0$.

34. Do research to explain how it can be shown that the real numbers do not have cardinal number $\aleph_0$.

CHAPTER 2 SUMMARY

IMPORTANT FACTS

Or is generally interpreted to mean *union*.

And is generally interpreted to mean *intersection*.

DE MORGAN'S LAWS

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

For any sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Number of distinct subsets of a finite set with n elements is 2^n.

Symbol	Meaning
$\in$	is an element of
$\notin$	is not an element of
$n(A)$	number of elements in set A
$\emptyset$ or $\{\ \}$	the empty set
U	the universal set
$\subseteq$	is a subset of
$\not\subseteq$	is not a subset of
$\subset$	is a proper subset of
$\not\subset$	is not a proper subset of
$'$	complement
$\cup$	union
$\cap$	intersection
$\aleph_0$	aleph-null

CHAPTER 2 REVIEW EXERCISES

2.1, 2.2, 2.3, 2.4, 2.6

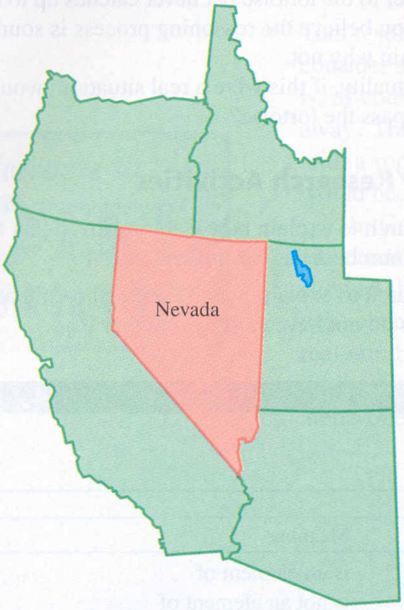
In Exercises 1–14, state whether each is true or false. If false, give a reason.

1. The set of counties located in the state of Alabama is a well-defined set.
2. The set of the three best beaches in the United States is a well-defined set.
3. maple $\in$ {oak, elm, maple, sycamore}
4. $\{\ \} \subset \emptyset$
5. $\{3, 6, 9, 12, \dots\}$ and $\{2, 4, 6, 8, \dots\}$ are disjoint sets.
6. $\{a, b, c, d, e\}$ is an example of a set in roster form.
7. {computer, calculator, pencil} = {calculator, computer, diskette}
8. {apple, orange, banana, pear} is equivalent to {tomato, corn, spinach, radish}.
9. If $A = \{a, e, i, o, u\}$, then $n(A) = 5$.
10. $A = \{1, 4, 9, 16, \dots\}$ is a countable set.
11. $A = \{1, 4, 7, 10, \dots, 31\}$ is a finite set.
12. $\{3, 6, 7\} \subseteq \{7, 6, 3, 5\}$.
13. $\{x \mid x \in N \text{ and } 3 < x \leq 9\}$ is a set in set-builder notation.
14. $\{x \mid x \in N \text{ and } 2 < x \leq 12\} \subseteq \{1, 2, 3, 4, 5, \dots, 20\}$

In Exercises 15–18, express each set in roster form.

15. Set A is the set of odd natural numbers between 5 and 16.

16. Set B is the set of states that border Nevada.



17. $C = \{x \mid x \in N \text{ and } x < 297\}$

18. $D = \{x \mid x \in N \text{ and } 8 < x \leq 96\}$

In Exercises 19–22, express each set in set-builder notation.

19. Set A is the set of natural numbers between 52 and 100.

20. Set B is the set of natural numbers greater than 63.

21. Set C is the set of natural numbers less than 3.

22. Set D is the set of natural numbers between 23 and 41, inclusive.

In Exercises 23–26, express each set with a written description.

23. $A = \{x \mid x$ is a letter of the English alphabet from E through M inclusive$\}$

24. $B = \{$penny, nickel, dime, quarter, half-dollar$\}$

25. $C = \{x, y, z\}$

26. $D = \{x \mid 3 \leq x < 9\}$

In Exercises 27–32, let

$$U = \{1, 2, 3, 4, \dots, 10\}$$
$$A = \{1, 3, 5, 6\}$$
$$B = \{5, 6, 9, 10\}$$
$$C = \{1, 6, 10\}$$

In Exercises 27–32, determine the following.

27. $A \cap B$ **28.** $A \cup B'$

29. $A' \cap B$ **30.** $(A \cup B)' \cup C$

31. The number of subsets of set B

32. The number of proper subsets of set A

33. For the following sets, construct a Venn diagram and place the elements in the proper region.

$U = \{$lion, tiger, leopard, cheetah, puma, lynx, panther, jaguar$\}$

$A = \{$tiger, puma, lynx$\}$

$B = \{$lion, tiger, jaguar, panther$\}$

$C = \{$tiger, lynx, cheetah, panther$\}$

In Exercises 34–39, use Fig. 2.29 to determine the sets.

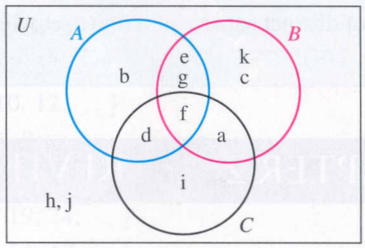


Figure 2.29

34. $A \cup B$ **35.** $A \cap B'$

36. $A \cup B \cup C$ **37.** $A \cap B \cap C$

38. $(A \cup B) \cap C$ **39.** $(A \cap B) \cup C$

Construct a Venn diagram to determine whether the following statements are true for all sets A, B, and C.

40. $(A' \cup B')' = A \cap B$

41. $(A \cup B') \cup (A \cup C') = A \cup (B \cap C)'$

Breakfast Cereals *In Exercises 42–47, use the following chart, which shows selected breakfast cereals and some of their nutritional contents for a 1-cup serving. Let the cereals shown represent the universal set.*

Cereal	Fat (grams)	Fiber (grams)	Sugar (grams)
Kellogg's Corn Flakes	0	1	2
Kellogg's Product 19	0	1	4
Kellogg's Fruit Loops	1	1	15
Kellogg's All-Bran	2	20	12
Kellogg's Raisin Bran	1.5	8	18
General Mills Wheaties	1	3	4
Kellogg's Special K	0	less than 1	4
General Mills Cinnamon Toast Crunch	3.5	1	10
General Mills Cheerios	2	3	1
General Mills Cookie Crisp	1	0	13

Let A be the set of cereals that contain at least 1 gram of fat.
Let B be the set of cereals that contain at least 3 grams of fiber.
Let C be the set of cereals that contain at least 4 grams of sugar.
Indicate in Fig. 2.30 in which region, I–VIII, each of the following cereals belongs.

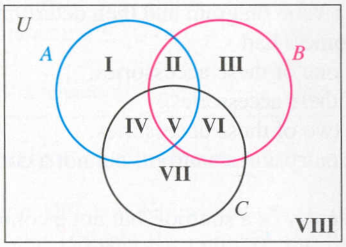


Figure 2.30

42. General Mills Cheerios
43. Kellogg's Raisin Bran
44. Kellogg's Corn Flakes
45. General Mills Cookie Crisp
46. Kellogg's Fruit Loops
47. Kellogg's Special *K*

2.5

48. ***Pizza Survey*** A pizza chain was willing to pay $1 to each person interviewed about his or her likes and dislikes of types of pizza crust. Of the people interviewed, 200 liked thin crust, 270 liked thick crust, 70 liked both, and 50 did not like pizza at all. What was the total cost of the survey?

49. ***Cookie Preferences*** The Cookie Shoppe conducted a survey to determine its customers' preferences.

200 people liked chocolate chip cookies.
190 people liked peanut butter cookies.
210 people liked sugar cookies.
100 people liked chocolate chip cookies and peanut butter cookies.
150 people liked peanut butter cookies and sugar cookies.
110 people liked chocolate chip cookies and sugar cookies.
70 people liked all three.
5 people liked none of these cookies.

Draw a Venn diagram, then determine how many people
a) completed the survey.
b) liked only peanut butter cookies.
c) liked peanut butter cookies and chocolate chip cookies, but not sugar cookies.
d) liked peanut butter cookies or sugar cookies, but not chocolate chip cookies.

50. ***TV Choices*** *TV Guide* sent a questionnaire to selected subscribers asking which of the following three reality-based shows they watched on a regular basis. The three shows asked about were *Survivor I, Survivor II*, and *Survivor III*. The results of the 510 questionnaires that were returned showed that

175 watched *Survivor I.*
227 watched *Survivor II.*
285 watched *Survivor III.*
100 watched *Survivor I* and *Survivor II.*
87 watched *Survivor II* and *Survivor III.*
96 watched *Survivor I* and *Survivor III.*
59 watched all three shows.

Construct a Venn diagram and determine how many people
a) watched only *Survivor I.*
b) watched exactly one of these shows.
c) watched *Survivor II* and *Survivor III,* but not *Survivor I.*
d) watched *Survivor I* or *Survivor III,* but not *Survivor II.*
e) watched exactly two of these shows.

2.6

In Exercises 51 and 52, show that the sets are infinite by placing each set in a one-to-one correspondence with a proper subset of itself.

51. $\{2, 4, 6, 8, 10, \ldots\}$ **52.** $\{3, 5, 7, 9, 11, \ldots\}$

In Exercises 53 and 54, show that each set has cardinal number $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the given set.

53. $\{5, 8, 11, 14, 17, \ldots\}$ **54.** $\{4, 9, 14, 19, 24, \ldots\}$

CHAPTER 2 TEST

In Exercises 1–9, state whether each is true or false. If the statement is false, explain why.

1. $\{1, y, 8, \$\}$ is equivalent to $\{p, \#, 5, 2\}$.
2. $\{3, 5, 9, h\} = \{9, 5, 3, j\}$
3. $\{\text{star, moon, sun}\} \subset \{\text{star, moon, sun, planet}\}$
4. $\{7\} \subseteq \{x \mid x \in N \text{ and } x < 7\}$
5. $\{\ \} \not\subset \{0\}$
6. $\{p, q, r\}$ has seven subsets.
7. If $A \cap B = \{\ \}$, then A and B are disjoint sets.
8. For any set A, $A \cup A' = \{\ \}$.
9. For any set A, $A \cap U = A$.

In Exercises 10 and 11, use set

$$A = \{x \mid x \in N \text{ and } x < 9\}.$$

10. Write set A in roster form.
11. Write a description of set A.

In Exercises 12–15, use the following information.

$$U = \{3, 5, 7, 9, 11, 13, 15\}$$
$$A = \{3, 5, 7, 9\}$$
$$B = \{7, 9, 11, 13\}$$
$$C = \{3, 11, 15\}$$

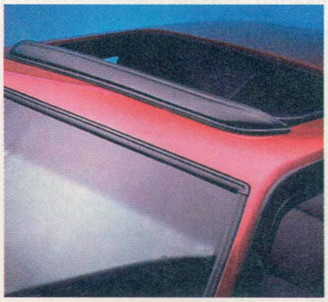
Determine the following.

12. $A \cap B$
13. $A \cup C'$
14. $A \cap (B \cap C)'$
15. $n(A \cap B')$
16. Using the sets provided for Exercises 12–15, draw a Venn diagram illustrating the relationship among the sets.
17. Use a Venn diagram to determine whether

$$A \cap (B \cup C') = (A \cap B) \cup (A \cap C')$$

for all sets A, B, and C. Show your work.

18. **Car Accessories** Auto Accessories Unlimited surveyed 155 customers to determine information regarding car accessories their car had. The results of the surveys showed
    76 had keyless entry.
    90 had a sunroof.
    107 had a compact disc player.
    54 had keyless entry and a compact disc player.
    57 had a sunroof and a compact disc player.
    52 had keyless entry and a sunroof.
    35 had all three accessories.



Construct a Venn diagram and then determine how many customers had
a) exactly one of these accessories.
b) none of these accessories.
c) at least two of these accessories.
d) keyless entry and a sunroof, but not a compact disc player.
e) keyless entry or a sunroof, but not a compact disc player.
f) only a compact disc player.

19. Show that the following set is infinite by setting up a one-to-one correspondence between the set and a proper subset of itself.

$$\{7, 8, 9, 10, \dots\}$$

20. Show that the following set has cardinal number $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the set.

$$\{1, 3, 5, 7, \dots\}$$

GROUP PROJECTS

Selecting a Family Pet

1. The Wilcox family is considering buying a dog. They have established several criteria for the family dog: It must be one of the breeds listed in the table, must not shed, must be less than 16 in. tall, and must be good with children.
   a) Using the information in the table,* construct a Venn diagram in which the universal set is the dogs listed. Indicate the set of dogs to be placed in each region of the Venn diagram.
   b) From the Venn diagram constructed in part (a), determine which dogs will meet the criteria set by the Wilcox family. Explain.

Breed	Sheds	Less than 16 in.	Good with children
Airedale	no	no	no
Basset hound	yes	yes	yes
Beagle	yes	yes	yes
Border terrier	no	yes	yes
Cairn terrier	no	yes	no
Cocker spaniel	yes	yes	yes
Collie	yes	no	yes
Dachshund	yes	yes	no
Poodle, miniature	no	yes	no
Schnauzer, miniature	no	yes	no
Scottish terrier	no	yes	no
Wirehaired fox terrier	no	yes	no

Classification of the Domestic Cat

2. Read the Did You Know feature on page 54. Do research and indicate the name of the following groupings to which the domestic cat belongs.

*The information is a collection of the opinions of an animal psychologist, Dr. Daniel Tortora, and a group of veterinarians.

a) Kingdom
b) Phylum
c) Class
d) Order
e) Family
f) Genus
g) Species

Who Lives Where

3. On Diplomat Row, an area of Washington, D.C., there are five houses. Each owner is a different nationality, each has a different pet, each has a different favorite food, each has a different favorite drink, and each house is painted a different color.
   The green house is directly to the right of the ivory house.
   The Senegalese has the red house.
   The dog belongs to the Spaniard.
   The Afghanistani drinks tea.
   The person who eats cheese lives next door to the fox.
   The Japanese eats fish.
   Milk is drunk in the middle house.
   Apples are eaten in the house next to the horse.
   Ale is drunk in the green house.
   The Norwegian lives in the first house.
   The peach eater drinks whiskey.
   Apples are eaten in the yellow house.
   The banana eater owns a snail.
   The Norwegian lives next door to the blue house.

   For each house find
   a) the color.
   b) the nationality of the occupant.
   c) the owner's favorite food.
   d) the owner's favorite drink.
   e) the owner's pet.
   f) Finally, the crucial question is: Does the zebra's owner drink vodka or ale?