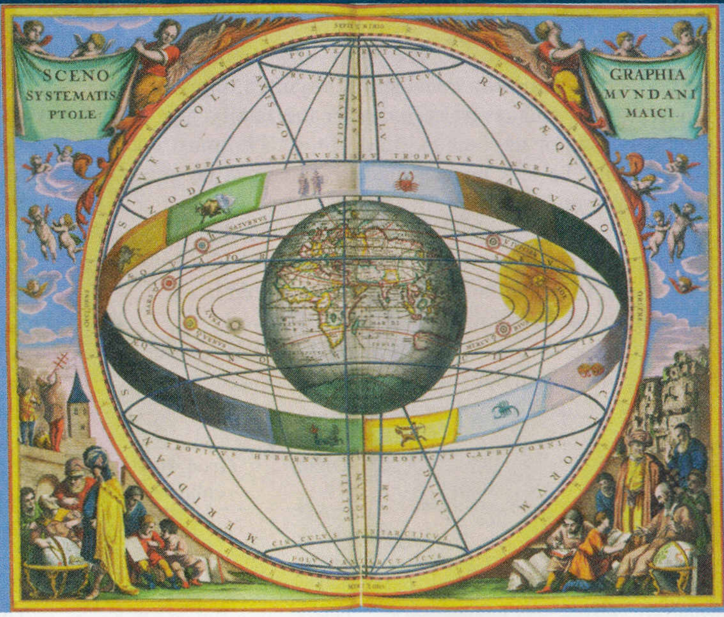
CHAPTER 3




**Logical reasoning** can tell us whether a conclusion follows from a set of premises, but not whether those premises are true. For example, Greek astronomers, using the assumption that the planets revolved around Earth, correctly predicted the positions of the planets even though their premise was false.

# LOGIC

The ancient Greeks were the first people to analyze systematically the way people think and arrive at a conclusion. Aristotle, whose study of logic is presented in a work called *Organon,* is called the father of logic. Since Aristotle's time, the study of logic has been continued by other great mathematicians.

Although most people believe that logic deals with the way people think, it does not. In the study of logic, we use deductive reasoning to analyze complicated situations and come to a reasonable conclusion from a given set of information.

If human thought does not always follow the rules of logic, then why do we study it? Logic enables us to communicate effectively, to make more convincing arguments, and to develop patterns of reasoning for decision making. The study of logic also prepares an individual to better understand other areas of mathematics, computer programming and design, and in general the thought process involved in learning any subject.

# 3.1 STATEMENTS AND LOGICAL CONNECTIVES

## History

The ancient Greeks were the first people to systematically analyze the way humans think and arrive at conclusions. Aristotle (384–322 B.C.) organized the study of logic for the first time in a work called *Organon*. As a result of his work, Aristotle is called the father of logic. The logic from this period, called ***Aristotelian logic***, has been taught and studied for more than 2000 years.

Since Aristotle's time, the study of logic has been continued by other great philosophers and mathematicians. Gottfried Wilhelm Leibniz (1646–1716) had a deep conviction that all mathematical and scientific concepts could be derived from logic. As a result, he became the first serious student of ***symbolic logic***. One difference between symbolic logic and Aristotelian logic is that in symbolic logic, as its name implies, symbols (usually letters) represent written statements. The forms of the statements in the two types of logic are different. The self-educated English mathematician George Boole (1815–1864) is considered to be the founder of symbolic logic because of his impressive work in this area. Among Boole's publications are *The Mathematical Analysis of Logic* (1847) and *An Investigation of the Law of Thought* (1854). Mathematician Charles Dodgson, better known as Lewis Carroll, incorporated many interesting ideas from logic into his books *Alice's Adventures in Wonderland* and *Through the Looking Glass* and his other children's stories.

Logic has been studied through the ages to exercise the mind's ability to reason. Understanding logic will enable you to think clearly, communicate effectively, make more convincing arguments, and develop patterns of reasoning that will help you in making decisions. It will also help you to detect the fallacies in the reasoning or arguments of others such as advertisers and politicians. Studying logic has other practical applications, such as helping you to understand wills, contracts, and other legal documents.

The study of logic is also good preparation for other areas of mathematics. If you preview Chapter 12, on probability, you will see formulas for the probability of $a$ or $b$ and the probability of $a$ and $b$, symbolized as $P(A \text{ or } B)$ and $P(A \text{ and } B)$, respectively. Special meanings of common words such as *or* and *and* apply to all areas of mathematics. The meaning of these and other special words is discussed in this chapter.

**DID YOU KNOW**

*Playing on Words*

George Boole, Augustus De Morgan, and other mathematicians of the nineteenth century were anxious to make logic an abstract science that would operate like algebra but be applicable to all fields. One of the problems logicians faced was that verbal language could be ambiguous and could easily lead to confusion and contradiction. Comedians Bud Abbott and Lou Costello had fun with the ambiguity of language in their skit about the baseball players: "Who's on first, What's on second, I Don't Know is on third—Yeah, but who's on first?" (in the film *The Naughty Nineties*, 1945).

## Logic and the English Language

In reading, writing, and speaking, we use many words such as *and, or,* and *if . . . then . . .* to connect thoughts. In logic we call these words ***connectives***. How are these words interpreted in daily communication? A judge announces to a convicted offender, "I hereby sentence you to five months of community service *and* a fine of $100." In this case, we normally interpret the word *and* to indicate that *both* events will take place. That is, the person must do community service and must also pay a fine.

Now suppose a judge states, "I sentence you to six months in prison *or* 10 months of community service." In this case, we interpret the connective *or* as meaning the convicted person must either spend the time in jail or do community service, but not both. The word *or* in this case is the ***exclusive or***. When the exclusive or is used, one or the other of the events can take place, but *not both*.

In a restaurant a waiter asks, "May I interest you in a cup of soup or a sandwich?" This question offers three possibilities: You may order soup, you may order a sandwich, or you may order both soup and a sandwich. The *or* in this case is the *inclusive or*. When the inclusive or is used, one or the other, *or both* events can take place. ***In this chapter, when we use the word* or *in a logic statement, it will mean the* inclusive or *unless stated otherwise***.

If–then statements are often used to relate two ideas, as in the bank policy statement "If the average daily balance is greater than $500, then there will be no service charge." If–then statements are also used to emphasize a point or add humor, as in the statement "If the Cubs win, then I will be a monkey's uncle."

Now let's look at logic from a mathematical point of view.

## Statements and Logical Connectives

**The Brooklyn Bridge** in New York City

A sentence that can be judged either true or false is called a ***statement***. Labeling a statement true or false is called *assigning a truth value*. Here are some examples of statements.

1. The Brooklyn Bridge goes over San Francisco Bay.
2. Disney World is in Idaho.
3. The Mississippi River is the longest river in the United States.

In each case, we can say that the sentence is either true or false. Statement 1 is false because the Brooklyn Bridge does not go over San Francisco Bay. Statement 2 is false. Disney World is in Florida. By looking at a map or reading an almanac, we can determine that the Mississippi River is the longest river in the United States, and, therefore, statement 3 is true.

The three sentences discussed above are examples of ***simple statements*** because they convey one idea. Sentences combining two or more ideas that can be assigned a truth value are called ***compound statements***. Compound statements are discussed shortly.

## Quantifiers

Sometimes it is necessary to change a statement to its opposite meaning. To do so, we use the ***negation*** of a statement. For example, the negation of the statement "Emily is at home" is "Emily is not at home." The negation of a true statement is always a false statement, and the negation of a false statement is always a true statement. We must use special caution when negating statements containing the words ***all***, ***none*** (or ***no***), and ***some***. These words are referred to as ***quantifiers***.

Consider the statement "All lakes contain fresh water." We know this statement is false because the Great Salt Lake in Utah contains salt water. Its negation must therefore be true. We may be tempted to write its negation as "No lake contains fresh water," but this statement is also false because Lake Superior contains fresh water. Therefore, "No lakes contain fresh water" is not the negation of "All lakes contain fresh water." The correct negation of "All lakes contain fresh water" is "Not all lakes contain fresh water" or "At least one lake does not contain fresh water" or "Some lakes do not contain fresh water." These statements all imply that at least one lake does not contain fresh water, which is a true statement.

Now consider the statement "No birds can swim." This statement is false, since at least one bird, the penguin, can swim. Therefore, the negation of this statement must be true. We may be tempted to write the negation as "All birds can swim," but because this statement is also false it cannot be the negation. The correct negation of the statement is "Some birds can swim" or "At least one bird can swim," which are true statements.

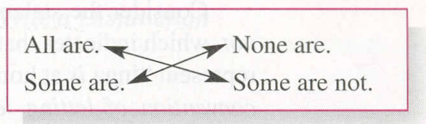
Now let's consider statements involving the quantifier *some,* as in "Some students have a driver's license." This is a true statement, meaning that at least one student has a driver's license. The negation of this statement must therefore be false. The negation is "No student has a driver's license," which is a false statement.

Consider the statement "Some students do not ride motorcycles." This statement is true because it means "At least one student does not ride a motorcycle." The negation of this statement must therefore be false. The negation is "All students ride motorcycles," which is a false statement.

The negation of quantified statements is summarized as follows:

| **Form of statement** | **Form of negation** |
|---|---|
| All are. | Some are not. |
| None are. | Some are. |
| Some are. | None are. |
| Some are not. | All are. |

The following diagram might help you to remember the statements and their negations:




The quantifiers diagonally opposite each other are the negations of each other.

**EXAMPLE 1** *Write Negations*

Write the negation of each statement.

a) Some snakes are poisonous.

b) All swimming pools are rectangular.

**SOLUTION:**

a) Since *some* means "at least one," the statement "Some snakes are poisonous" is the same as "At least one snake is poisonous." Because it is a true statement, its negation must be false. The negation is "No snakes are poisonous," which is a false statement.

b) The statement "All swimming pools are rectangular" is a false statement since some pools are circular, some are oval, and some have other shapes. Its negation must therefore be true. The negation may be written as "Some swimming pools are not rectangular" or "Not all swimming pools are rectangular" or "At least one swimming pool is not rectangular." Each of these statements is true. ▲

## Compound Statements

Statements consisting of two or more simple statements are called **compound statements**. The connectives often used to join two simple statements are

**and,** **or,** **if, . . . then . . . ,** **if and only if**

In addition, we consider a simple statement that has been negated to be a compound statement. The word *not* is generally used to negate a statement.

To reduce the amount of writing in logic, it is common to represent each simple statement with a lowercase letter. For example, suppose we are discussing the simple statement "Leland is a farmer." Instead of writing "Leland is a farmer" over and over again, we can let $p$ represent the statement "Leland is a farmer." Thereafter we can simply refer to the statement with the letter $p$. It is customary to use the letters $p$, $q$, $r$, and $s$ to represent simple statements, but other letters may be used instead. Let's now look at the connectives used to make compound statements.

## Not Statements

The negation is symbolized by $\sim$ and read "not." For example, the negation of the statement "Steve is a college student" is "Steve is not a college student." If $p$ represents the simple statement "Steve is a college student," then $\sim p$ represents the compound statement "Steve is not a college student." For any statement $p$, $\sim(\sim p) = p$. For example, the negation of the statement "Steve is not a college student" is "Steve is a college student."

Consider the statement "Inga is not at home." This statement contains the word *not*, which indicates that it is a negation. To write this statement symbolically, we let $p$ represent "Inga *is* at home." Then $\sim p$ would be "Inga is not at home." *We will use this convention of letting letters such as p, q, or r represent statements that are not negated. We will represent negated statements with the negation symbol,* $\sim$.

## And Statements

The ***conjunction*** is symbolized by $\wedge$ and read "and." The $\wedge$ looks like an A (for And) with the bar missing. Let $p$ and $q$ represent the simple statements.

$p$: You will perform 5 months of community service.
$q$: You will pay $100 fine.

Then the following is the conjunction written in symbolic form.

$$\underbrace{\text{You will perform 5 months of community service}}_{\uparrow\; p} \quad \underbrace{\text{and}}_{\uparrow\; \wedge} \quad \underbrace{\text{you will pay a \$100 fine.}}_{\uparrow\; q}$$

The conjunction is generally expressed as *and*. Other words sometimes used to express a conjunction are *but*, *however*, and *nevertheless*.

**EXAMPLE 2** *Write a Conjunction*

Write the following conjunction in symbolic form. The dish is heavy, but the dish is not hot.

**SOLUTION:** Let $d$ and $h$ represent the simple statements.

$d$: The dish is heavy.
$h$: The dish is hot.

In symbolic form, the compound statement is $d \wedge \sim h$.

In Example 2, the compound statement is "The dish is heavy, but the dish is not hot." This statement could also be repesented as "The dish is heavy, but *it* is not hot." In this problem, it should be clear that the word *it* means *the dish*. Therefore, the statement, "The dish is heavy, but it is not hot" would also be symbolized as $d \wedge \sim h$.

## Or Statements

The ***disjunction*** is symbolized by $\vee$ and read "or." The *or* we use in this book (except where indicated in the exercise sets) is the *inclusive or* described on pages 93 and 94.

### EXAMPLE 3 *Write a Disjunction*

Let

$p$: Maria will go to the circus.
$q$: Maria will go to the zoo.

Write the following statements in symbolic form.

a) Maria will go to the circus or Maria will go to the zoo.
b) Maria will go to the zoo or Maria will not go to the circus.
c) Maria will not go to the circus or Maria will not go to the zoo.

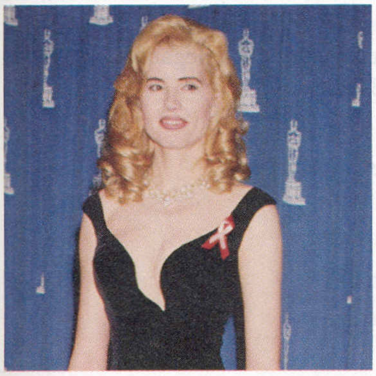
**SOLUTION:**

a) $p \vee q$   b) $q \vee \sim p$   c) $\sim p \vee \sim q$

Because *or* represents the *inclusive or,* the statement "Maria will go to the circus or Maria will go to the zoo" in Example 3(a) may mean that Maria will go to the circus, or that Maria will go to the zoo, or that Maria will go to both the circus *and* the zoo. The statement in Example 3(a) could also be written as "Maria will go to the circus or the zoo."

When a compound statement contains more that one connective, a comma can be used to indicate which simple statements are to be grouped together. When we write the compound statement symbolically, ***the simple statements on the same side of the comma are to be grouped together within parentheses.***

For example, "Pink is a singer ($p$) or Geena Davis is an actress ($g$), and Dallas is in Texas ($d$)", is written $(p \vee g) \wedge d$. Note that the $p$ and $g$ are both on the same side of the comma in the written statement. They are therefore grouped together within parentheses. The statement "Pink is a singer, or Geena Davis is an actress and Dallas is in Texas" is written $p \vee (g \wedge d)$. In this case, $g$ and $d$ are on the same side of the comma and are therefore grouped together within parentheses.



Geena Davis

## PROFILE IN MATHEMATICS

### GEORGE BOOLE

The self-taught English mathematician George Boole (1815–1864) took the operations of algebra and used them to extend Aristotelian logic. He used symbols such as $x$ and $y$ to represent particular qualities or objects in question. For example, if $x$ represents all butterflies, then $1 - x$ represents everything else except butterflies. If $y$ represents the color yellow, then $(1 - x)(1 - y)$ represents everything except butterflies and things that are yellow, or yellow butterflies. This development added a computational dimension to logic that provided a basis for twentieth-century work in the field of computing.

**EXAMPLE 4** *Understand How Commas Are Used to Group Statements*

Let

$p$: Dinner includes soup.
$q$: Dinner includes salad.
$r$: Dinner includes the vegetable of the day.

Write the following statements in symbolic form.

a) Dinner includes soup, and salad or the vegetable of the day.
b) Dinner includes soup and salad, or the vegetable of the day.

$(p \wedge q) \vee r$

**SOLUTION:**

a) The comma tells us to group the statement "Dinner includes salad" with the statement "Dinner includes the vegetable of the day." Note that both statements are on the same side of the comma. The statement in symbolic form is $p \wedge (q \vee r)$.

In mathematics, we always evaluate the information within the parentheses first. Since the conjunction, $\wedge$, is outside the parentheses and is evaluated *last*, this statement is considered a *conjunction*.

b) The comma tells us to group the statement "Dinner includes soup" with the statement "Dinner includes salad." Note that both statements are on the same side of the comma. The statement in symbolic form is $(p \wedge q) \vee r$. Since the disjunction, $\vee$, is outside the parentheses and is evaluated *last*, this statement is considered a *disjunction*. ▲

The information provided in Example 4 is summarized below.

| Statement | Symbolic representation | Type of statement |
|---|---|---|
| Dinner includes soup, and salad or the vegetable of the day. | $p \wedge (q \vee r)$ | conjunction |
| Dinner includes soup and salad, or the vegetable of the day. | $(p \wedge q) \vee r$ | disjunction |

*An important point to remember is that a negation symbol has the effect of negating only the statement that directly follows it.* To negate a compound statement, we must use parentheses. When a negation symbol is placed in front of a statement in parentheses, it negates the entire statement in parentheses. The negation symbol in this case is read, "It is not true that . . ." or "It is false that . . ."

**EXAMPLE 5** *Change Symbolic Statements to Words*

Let

$p$: Jozsef is making breakfast
$q$: Arum is setting the table

Write the following symbolic statements in words.

a) $p \wedge \sim q$ b) $\sim p \vee \sim q$ c) $\sim(p \wedge q)$

**SOLUTION:**

a) Jozsef is making breakfast and Arum is not setting the table.

b) Jozsef is not making breakfast or Arum is not setting the table.

c) It is false that Jozsef is making breakfast and Arum is setting the table. ▲

Recall that the word *but* may also be used in a conjunction. Therefore, Example 5(a) could also be written "Jozsef is making breakfast, *but* Arum is not setting the table."

Part (b) of Example 5 is a disjunction, since it can be written $(\sim p) \vee (\sim q)$. Part (c), which is $\sim(p \wedge q)$, is a negation, since the negation symbol negates the entire statement within parentheses. The similarity of these two statements is discussed in Section 3.4.

Occasionally, we come across a ***neither–nor*** statement, such as "John is neither handsome nor rich." This statement means that John is not handsome *and* John is not rich. If $p$ represents "John is handsome" and $q$ represents "John is rich," this statement is symbolized by $\sim p \wedge \sim q$.

## If–Then Statements

The ***conditional*** is symbolized by $\rightarrow$ and is read "if–then." The statement $p \rightarrow q$ is read "If $p$, then $q$."* The conditional statement consists of two parts: the part that precedes the arrow is the ***antecedent***, and the part that follows the arrow is the ***consequent***.† In the conditional statement $p \rightarrow q$, the $p$ is the antecedent and the $q$ is the consequent.

In the conditional statement $\sim(p \vee q) \rightarrow (p \wedge q)$, the antecedent is $\sim(p \vee q)$ and the consequent is $(p \wedge q)$. An example of a conditional statement is "If you drink your milk, then you will grow up to be healthy." A conditional symbol may be placed between any two statements even if the statements are not related.

Sometimes the word *then* in a conditional statement is not explicitly stated. For example, the statement "If you pass this course, I will buy you a car" is a conditional statement because it actually means "If you pass this course, then I will buy you a car."

### EXAMPLE 6 *Write Conditional Statements*

Let $p$: Jennifer goes to the library.

$q$: Jennifer will study.

Write the following statements symbolically.

a) If Jennifer goes to the library, then she will study.

b) If Jennifer does not go to the library, then she will not study.

c) It is false that if Jennifer goes to the library then she will study.

**SOLUTION:**

a) $p \rightarrow q$ b) $\sim p \rightarrow \sim q$ c) $\sim(p \rightarrow q)$ ▲

*Some books indicate that $p \rightarrow q$ may also be read "$p$ implies $q$." Many higher-level mathematics books, however, indicate that $p \rightarrow q$ may be read "$p$ implies $q$" only under certain conditions. Implications are discussed in Section 3.3.

†Some books refer to the antecedent as the hypothesis or premise and the consequent as the conclusion.

**EXAMPLE 7** *Use Commas When Writing a Symbolic Statement in Words*

Let

$p$: Jorge is enrolled in calculus.
$q$: Jorge's major is criminal justice.
$r$: Jorge's major is engineering.

Write the following symbolic statements in words and indicate whether the statement is a negation, conjunction, disjunction, or conditional.

a) $(q \to \sim p) \vee r$  b) $q \to (\sim p \vee r)$

**SOLUTION:** The parentheses indicate where to place the commas in the sentences.

a) "If Jorge's major is criminal justice then Jorge is not enrolled in calculus, or Jorge's major is engineering." This statement is a disjunction because $\vee$ is outside the parentheses.
b) "If Jorge's major is criminal justice, then Jorge is not enrolled in calculus or Jorge's major is engineering." This is a conditional statement because $\to$ is outside the parentheses.

## If and Only if Statements

The *biconditional* is symbolized by $\leftrightarrow$ and is read "if and only if." The phrase *if and only if* is sometimes abbreviated as "iff." The statement $p \leftrightarrow q$ is read "$p$ if and only if $q$."

**EXAMPLE 8** *Write Statements Using the Biconditional*

Let

$p$: The printer is working.
$q$: The ink cartridge is correctly inserted.

Write the following symbolic statements in words.

a) $q \leftrightarrow p$  b) $\sim(p \leftrightarrow \sim q)$

**SOLUTION:**

a) The ink cartridge is correctly inserted if and only if the printer is working.
b) It is false that the printer is working if and only if the ink cartridge is not correctly inserted.

You will learn later that $p \leftrightarrow q$ means the same as $(p \to q) \wedge (q \to p)$. Therefore, the statement "I will go to college if and only if I can pay the tuition" has the same logical meaning as "If I go to college then I can pay the tuition, and if I can pay the tuition then I will go to college."

The following is a summary of the connectives discussed in this section.

| Formal name | Symbol | Read | Symbolic form |
|---|---|---|---|
| Negation | $\sim$ | "Not" | $\sim p$ |
| Conjunction | $\wedge$ | "And" | $p \wedge q$ |
| Disjunction | $\vee$ | "Or" | $p \vee q$ |
| Conditional | $\rightarrow$ | "If-then" | $p \rightarrow q$ |
| Biconditional | $\leftrightarrow$ | "If and only if" | $p \leftrightarrow q$ |

## Dominance of Connectives

What is the answer to the problem $2 + 3 \times 4$? Some of you might say 20, but others might say 14. If you evaluate $2 + 3 \times 4$ on a calculator by pressing

$$2 \boxed{+} 3 \boxed{\times} 4$$

some may give you the answer 14, whereas others may give you the answer 20. Which is the correct answer? In mathematics, unless otherwise changed by parentheses or some other grouping symbol, multiplication is *always* performed before addition. Thus,

$$2 + 3 \times 4 = 2 + (3 \times 4) = 14$$

The calculators that gave the incorrect answer of 20 are basic calculators that are not programmed according to the order of operations used in mathematics.

Just as an order of operations exists in the evaluation of arithmetic expressions, a dominance of connectives is used in the evaluation of logic statements. How do we evaluate a symbolic logic statement when no parentheses are used? For example, does $p \vee q \rightarrow r$ mean $(p \vee q) \rightarrow r$, or does it mean $p \vee (q \rightarrow r)$? If we are given a symbolic logic statement for which grouping has not been indicated by parentheses or a written logic statement for which grouping has not been indicated by a comma, then we use the dominance of connectives shown in Table 3.1. Note that *the least dominant connective is the negation and the most dominant is the biconditional.*

**TABLE 3.1** Dominance of Connectives

| | | |
|---|---|---|
| Least dominant | 1. Negation, $\sim$ | Evaluate first |
| | 2. Conjunction, $\wedge$; disjunction, $\vee$ | |
| | 3. Conditional, $\rightarrow$ | |
| Most dominant | 4. Biconditional, $\leftrightarrow$ | Evaluate last |

As indicated in Table 3.1, the conjunction and disjunction have the same level of dominance. Thus, to determine whether the symbolic statement $p \wedge q \vee r$ is a conjunction or a disjunction, we have to use grouping symbols (parentheses). When evaluating a symbolic statement that does not contain parentheses, we *evaluate the least dominant connective first and the most dominant connective last*. For example,

| **Statement** | **Most dominant connective used** | **Statement means** | **Type of statement** |
|---|---|---|---|
| $\sim p \vee q$ | $\vee$ | $(\sim p) \vee q$ | Disjunction |
| $p \rightarrow q \vee r$ | $\rightarrow$ | $p \rightarrow (q \vee r)$ | Conditional |
| $p \wedge q \rightarrow r$ | $\rightarrow$ | $(p \wedge q) \rightarrow r$ | Conditional |
| $p \rightarrow q \leftrightarrow r$ | $\leftrightarrow$ | $(p \rightarrow q) \leftrightarrow r$ | Biconditional |
| $p \vee r \leftrightarrow r \rightarrow \sim p$ | $\leftrightarrow$ | $(p \vee r) \leftrightarrow (r \rightarrow \sim p)$ | Biconditional |
| $p \rightarrow r \leftrightarrow s \wedge p$ | $\leftrightarrow$ | $(p \rightarrow r) \leftrightarrow (s \wedge p)$ | Biconditional |

## EXAMPLE 9 *Use the Dominance of Connectives*

Use the dominance of connectives to add parentheses to each statement. Then indicate whether each statement is a negation, conjunction, disjunction, conditional, or biconditional.

a) $p \rightarrow q \vee r$ b) $\sim p \wedge q \leftrightarrow r \vee p$

**SOLUTION:**

a) The conditional has greater dominance than the disjunction, so we place parentheses around $q \vee r$, as follows:

$$p \rightarrow (q \vee r)$$

It is a conditional statement because the conditional symbol is outside the parentheses.

b) The biconditional has the greatest dominance, so we place parentheses as follows:

$$(\sim p \wedge q) \leftrightarrow (r \vee p)$$

It is a biconditional statement because the biconditional symbol is outside the parentheses.

## EXAMPLE 10 *Identify the Type of Statement*

Use the dominance of connectives and parentheses to write each statement symbolically. Then indicate whether each statement is a negation, conjunction, disjunction, conditional, or biconditional.

a) If you are late in paying your rent or you have damaged the apartment then you may be evicted.

b) You are late in paying your rent, or if you have damaged the apartment then you may be evicted.

**SOLUTION:**

a) Let

$p$: You are late in paying your rent.
$q$: You have damaged the apartment.
$r$: You may be evicted.

No commas appear in the sentence, so we will evaluate it by using the dominance of connectives. Because the conditional has higher dominance than the disjunction, the conditional statement will be evaluated last. Thus, the statements "You are late in paying your rent" and "You have damaged the apartment" are to be grouped together. The statement written symbolically with parentheses is

$$(p \vee q) \rightarrow r$$

This is a conditional statement.

b) A comma is used in this statement to indicate grouping, just as parentheses do in arithmetic. The placement of the comma indicates that the statements "You have damaged the apartment" and "You may be evicted" are to be grouped together. Therefore, this statement written symbolically is

$$p \vee (q \rightarrow r)$$

This statement is a disjunction. Note that the comma overrides the dominance of connectives and tells us to evaluate the conditional statement before the disjunction. ▲

## SECTION 3.1 EXERCISES

### Concept/Writing Exercises

1. a) What is a simple statement?
   b) What is a compound statement?
2. List the words identified as quantifiers.
3. Write the general form of the negation for statements of the form
   a) none are.
   b) some are not.
   c) all are.
   d) some are.
4. Represent the statement "The ink is not purple" symbolically. Explain your answer.
5. Draw the symbol used to represent the
   a) conditional.
   b) disjunction.
   c) conjunction.
   d) negation.
   e) biconditional.
6. a) When the *exclusive or* is used as a connective between two events, can both events take place? Explain.
   b) When the *inclusive or* is used as a connective between two events, can both events take place? Explain.
   c) Which *or*, the *inclusive or* or the *exclusive or*, is used in this chapter?
7. Explain how a comma is used to indicate the grouping of simple statements.
8. List the dominance of connectives from the most dominant to the least dominant.

### Practice the Skills/Problem Solving

*In Exercises 9–22, indicate whether the statement is a simple statement or a compound statement. If it is a compound statement, indicate whether it is a negation, conjunction, disjunction, conditional, or biconditional by using both the word and its appropriate symbol (for example, "a negation," ~).*

9. The sun is shining and the air is crisp.
10. The water in the lake is not drinkable.
11. The figure is a quadrilateral if and only if it has four sides.
12. If the electricity goes out then the standard telephone will still work.
13. Joni Burnette is teaching calculus or she is teaching trigonometry.
14. The book was neither a novel nor an autobiography.
15. The hurricane did $400,000 worth of damage to DeSoto County.

16. Inhibor Melendez will be admitted to law school if and only if he earns his bachelor's degree.
17. It is false that Jeffery Hilt is a high school teacher and a grade school teacher.
18. If Cathy Smith walks 4 miles today then she will be sore tomorrow.
19. Mary Jo Woo ran 4 miles today and she lifted weights for 30 minutes.
20. Nancy Wallin went to the game, but she did not eat a hot dog.
21. It is false that if John Wubben fixes your car then you will need to pay him in cash.
22. If Buddy and Evelyn Cordova are residents of Budville, then they must vote for mayor on Tuesday.

*In Exercises 23–34, write the negation of the statement.*

23. Some picnic tables are portable.
24. No stock mutual funds have guaranteed yields.
25. All chickens fly.
26. All plants contain chlorophyll.
27. Some turtles do not have claws.
28. No teachers made the roster.
29. No bicycles have three wheels.
30. All horses have manes.

31. Some pine trees do not produce pinecones.
32. No one likes asparagus.
33. Some pedestrians are in the crosswalk.
34. Some dogs with long hair do not get cold.

*In Exercises 35–40, write the statement in symbolic form. Let*

$p$: The tent is pitched.
$q$: The bonfire is burning.

35. The tent is not pitched.
36. The tent is pitched and the bonfire is burning.
37. The bonfire is not burning or the tent is not pitched.
38. The bonfire is not burning if and only if the tent is not pitched.
39. If the tent is not pitched, then the bonfire is not burning.
40. The bonfire is not burning, however the tent is pitched.

*In Exercises 41–46, write the statement in symbolic form. Let*

$p$: The charcoal is hot.
$q$: The chicken is on the grill.

41. If the chicken is not on the grill then the charcoal is not hot.
42. The chicken is not on the grill if and only if the charcoal is not hot.
43. Neither is the charcoal hot nor is the chicken on the grill.
44. The charcoal is not hot, but the chicken is on the grill.
45. It is false that if the chicken on the grill then the charcoal is not hot.
46. It is false that the charcoal is hot and the chicken is on the grill.

*In Exercises 47–56, write the compound statement in words. Let*

$p$: Firemen work hard.
$q$: Firemen wear red suspenders.

47. $\sim p$
48. $\sim q$
49. $q \vee p$
50. $p \wedge q$
51. $\sim p \leftrightarrow \sim q$
52. $\sim p \rightarrow q$
53. $\sim(q \vee p)$
54. $\sim p \vee \sim q$
55. $\sim p \wedge \sim q$
56. $\sim(p \wedge q)$

*In Exercises 57–66, write the statements in symbolic form. Let*

$p$: The temperature is 90°.
$q$: The air conditioner is working.
$r$: The apartment is hot.

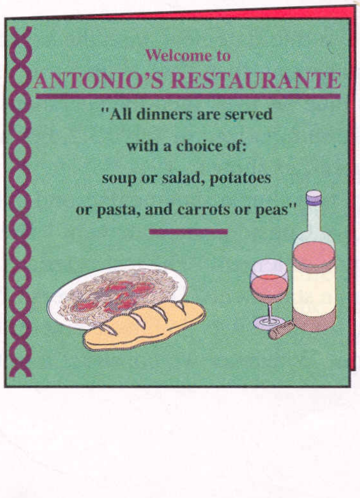
57. If the temperature is 90° or the air conditioner is not working, then the apartment is hot.
58. The apartment is hot if and only if the temperature is not 90°, or the air conditioner is not working.
59. The temperature is 90° and the air conditioner is working, or the apartment is hot.
60. If the apartment is hot and the air conditioner is working, then the temperature is 90°.
61. If the temperature is 90°, then the air conditioner is working or the apartment is not hot.
62. The temperature is not 90° if and only if the air conditioner is not working, or the apartment is not hot.
63. The apartment is hot if and only if the air conditioner is working, and the temperature is 90°.
64. It is false that if the apartment is hot then the air conditioner is not working.
65. If the air conditioner is working, then the temperature is 90° if and only if the apartment is hot.
66. The apartment is hot or the air conditioner is not working, if and only if the temperature is 90°.

*In Exercises 67–76, write each symbolic statement in words. Let*

$p$: The water is 70°.
$q$: The sun is shining.
$r$: We go swimming.

67. $(p \vee q) \wedge \sim r$
68. $(p \wedge q) \vee r$
69. $\sim p \wedge (q \vee r)$
70. $(q \rightarrow p) \vee r$
71. $\sim r \rightarrow (q \wedge p)$
72. $(q \wedge r) \rightarrow p$
73. $(q \rightarrow r) \wedge p$
74. $\sim p \rightarrow (q \vee r)$
75. $(q \leftrightarrow p) \wedge r$
76. $q \rightarrow (p \leftrightarrow r)$

**Dinner Menu** *In Exercises 77–80, use the following information to arrive at your answers. Many restaurant dinner menus include statements such as the following. All dinners are served with a choice of: Soup or Salad, and Potatoes or Pasta, and Carrots or Peas. Which of the following selections are permissible? If a selection is not permissible, explain why. See the discussion of the* exclusive or *on page 93.*




77. Soup, salad, and peas
78. Salad, pasta, and carrots
79. Soup, potatoes, pasta, and peas
80. Soup, pasta, and potatoes

*In Exercises 81–94, (a) add parentheses by using the dominance of connectives and (b) indicate whether the statement is a negation, conjunction, disjunction, conditional, or biconditional (see Example 9).*

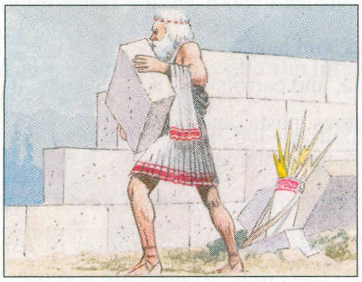
81. $\sim p \rightarrow q$
82. $\sim p \wedge r \leftrightarrow \sim q$
83. $\sim q \wedge \sim r$
84. $\sim p \vee q$
85. $p \vee q \rightarrow r$
86. $q \rightarrow p \wedge \sim r$
87. $r \rightarrow p \vee q$
88. $q \rightarrow p \leftrightarrow p \rightarrow q$
89. $\sim p \leftrightarrow \sim q \rightarrow r$
90. $\sim q \rightarrow r \wedge p$
91. $r \wedge \sim q \rightarrow q \wedge \sim p$
92. $\sim[p \rightarrow q \vee r]$
93. $\sim[p \wedge q \leftrightarrow p \vee r]$
94. $\sim[r \wedge \sim q \rightarrow q \wedge r]$

*In Exercises 95–103, (a) select letters to represent the simple statements and write each statement symbolically by using parentheses and (b) indicate whether the statement is a negation, conjunction, disjunction, conditional, or biconditional (see Example 10).*

95. Ruth Bignel retired, but she did not start her concrete business.
96. If the water level is up, then we can go canoeing or we can go rafting.
97. It is false that if your speed is below the speed limit then you will not get pulled over.
98. If dinner is ready then we can eat, or we cannot go to the restaurant.
99. If the food has fiber or the food has vitamins, then you will be healthy.
100. If Corliss is teaching then Faye is in the math lab if and only if it is not a weekend.
101. You may take this course if and only if you did not fail the previous course or you passed the placement test.
102. If the car has gas and the battery is charged then the car will start.
103. The classroom is empty if and only if it is the weekend, or it is 7 A.M.

## Challenge Problems/Group Activities

104. *An Ancient Question* If Zeus could do anything, could he build a wall that he could not jump over? Explain your answer.



*In Exercises 105 and 106, place parentheses in the statement according to the dominance of connectives. Indicate whether the statement is a negation, conjunction, disjunction, conditional, or biconditional.*

105. $\sim q \rightarrow r \vee p \leftrightarrow \sim r \wedge q$

106. $\sim[\sim r \rightarrow p \wedge q \leftrightarrow \sim p \vee r]$

107. a) We cannot place parentheses in the statement $p \vee q \wedge r$. Explain why.
b) Make up three simple statements and label them $p$, $q$, and $r$. Then write compound statements to represent $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$.
c) Do you think that the statements for $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ mean the same thing? Explain.

## Internet/Research Activities

108. *Legal Documents* Obtain a legal document such as a will or rental agreement and copy one page of the document. Circle every connective used. Then list the number of times each connective appeared. Be sure to include conditional statements from which the word *then* was omitted from the sentence. Give the page and your listing to your instructor.

109. Write a report on the life and accomplishments of George Boole, who was an important contributor to the development of logic. In your report, indicate how his work eventually led to the development of the computer. References include encyclopedias, history of mathematics books, and the Internet.

# 3.2 TRUTH TABLES FOR NEGATION, CONJUNCTION, AND DISJUNCTION

A ***truth table*** is a device used to determine when a compound statement is true or false. Five basic truth tables are used in constructing other truth tables. Three are discussed in this section (Tables 3.2, 3.4, and 3.7), and two are discussed in the next section. Section 3.5 uses truth tables in determining whether a logical argument is valid or invalid.

**TABLE 3.2 Negation**

| | $p$ | $\sim p$ |
|---|---|---|
| Case 1 | T | F |
| Case 2 | F | T |

## Negation

The first truth table is for *negation*. If $p$ is a true statement, then the negation of $p$, "not $p$," is a false statement. If $p$ is a false statement, then "not $p$" is a true statement. For example, if the statement "The shirt is blue" is true, then the statement "The shirt is not blue" is false. These relationships are summarized in Table 3.2. For a simple statement, there are exactly two true–false cases, as shown.

**TABLE 3.3**

| | $p$ | $q$ |
|---|---|---|
| Case 1 | T | T |
| Case 2 | T | F |
| Case 3 | F | T |
| Case 4 | F | F |

If a compound statement consists of two simple statements $p$ and $q$, there are four possible cases, as illustrated in Table 3.3. Consider the statement "The test is today and the test covers Chapter 5." The simple statement "The test is today" has two possible truth values, true or false. The simple statement "The test covers Chapter 5" also has two truth values, true or false. Thus, for these two simple statements there are four distinct possible true–false arrangements. Whenever we construct a truth table for a compound statement that consists of two simple statements, we begin by listing the four true–false cases shown in Table 3.3.

## Conjunction

To illustrate the conjunction, consider the following situation. You have recently purchased a new house. To decorate it, you ordered a new carpet and new furniture from the same store. You explain to the salesperson that the carpet must be delivered before the furniture. He promises that the carpet will be delivered on Thursday and that the furniture will be delivered on Friday.

**TABLE 3.4 Conjunction**

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| Case 1 | T | T | T |
| Case 2 | T | F | F |
| Case 3 | F | T | F |
| Case 4 | F | F | F |

To help determine whether the salesperson kept his promise, we assign letters to each simple statement. Let $p$ be "The carpet will be delivered on Thursday" and $q$ be "The furniture will be delivered on Friday." The salesperson's statement written in symbolic form is $p \wedge q$. There are four possible true–false situations to be considered. (Table 3.4).

**CASE 1:** $p$ is true and $q$ is true. The carpet is delivered on Thursday and the furniture is delivered on Friday. The salesperson has kept his promise and the compound statement is true. Thus, we put a T in the $p \wedge q$ column.

**CASE 2:** $p$ is true and $q$ is false. The carpet is delivered on Thursday but the furniture is not delivered on Friday. Since the furniture was not delivered as promised, the compound statement is false. Thus, we put an F in the $p \wedge q$ column.

**CASE 3:** $p$ is false and $q$ is true. The carpet is not delivered on Thursday but the furniture is delivered on Friday. Since the carpet was not delivered on Thursday as promised, the compound statement is false. Thus, we put an F in the $p \wedge q$ column.

**CASE 4:** $p$ is false and $q$ is false. The carpet is not delivered on Thursday and the furniture is not delivered on Friday. Since the carpet and furniture were not delivered as promised, the compound statement is false. Thus, we put an F in the $p \wedge q$ column.

Examining the four cases, we see that in only one case did the salesperson keep his promise: in case 1. Therefore, case 1 (T, T) is true. In cases 2, 3, and 4, the salesperson did not keep his promise and the compound statement is false. The results are summarized in Table 3.4, the truth table for the conjunction.

The **conjunction** $p \wedge q$ is true only when both $p$ and $q$ are true.

### EXAMPLE 1 *Construct a Truth Table*

Construct a truth table for $p \wedge \sim q$.

**SOLUTION:** Because there are two statements, $p$ and $q$, construct a truth table with four cases; see Table 3.5(a). Then write the truth values under the $p$ in the compound statement and label this column 1, as in Table 3.5(b). Copy these truth values directly from the $p$ column on the left. Write the corresponding truth values under the $q$ in the compound statement and call this column 2, as in Table 3.5(c). Copy the truth values for column 2 directly from the $q$ column on the left. Now find the truth values of $\sim q$ by negating the truth values in column 2 and call this column 3,

**TABLE 3.5**

(a)

| | $p$ | $q$ | $p \wedge \sim q$ |
|---|---|---|---|
| Case 1 | T | T | |
| Case 2 | T | F | |
| Case 3 | F | T | |
| Case 4 | F | F | |

(b)

| $p$ | $q$ | $p \wedge \sim q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | F |
| F | F | F |

(c)

| $p$ | $q$ | $p$ | $\wedge$ | $\sim$ | $q$ |
|---|---|---|---|---|---|
| T | T | T | | | T |
| T | F | T | | | F |
| F | T | F | | | T |
| F | F | F | | | F |
| | | 1 | | | 2 |

(d)

| $p$ | $q$ | $p$ | $\wedge$ | $\sim$ | $q$ |
|---|---|---|---|---|---|
| T | T | T | | F | T |
| T | F | T | | T | F |
| F | T | F | | F | T |
| F | F | F | | T | F |
| | | 1 | | 3 | 2 |

(e)

| $p$ | $q$ | $p$ | $\wedge$ | $\sim$ | $q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | T | T | T | F |
| F | T | F | F | F | T |
| F | F | F | F | T | F |
| | | 1 | 4 | 3 | 2 |

as in Table 3.5(d). Use the conjunction table, Table 3.4, and the entries in columns 1 and 3 to complete column 4, as in Table 3.5(e). The results in column 4 are obtained as follows:

Row 1: $T \wedge F$ is F. Row 2: $T \wedge T$ is T.

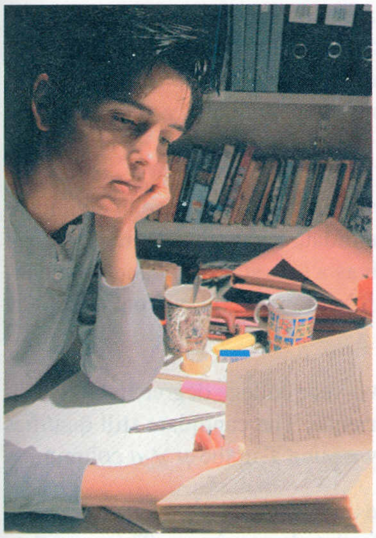
Row 3: $F \wedge F$ is F. Row 4: $F \wedge T$ is F.

The answer is always the last column completed. Columns 1, 2, and 3 are only aids in arriving at the answer in column 4.

The statement $p \wedge \sim q$ in Example 1 actually means $p \wedge (\sim q)$. In the future, instead of listing a column for $q$ and a separate column for its negation, we will make one column for $\sim q$, which will have the opposite values of those in the $q$ column on the left. Similarly, when we evaluate $\sim p$, we will use the opposite values of those in the $p$ column on the left. This procedure is illustrated in Example 2.

In Example 1, we spoke about *cases* and also *columns*. Consider Table 3.5(e). This table has four cases indicated by the four different rows of the two left hand (unnumbered) columns. The four *cases* are TT, TF, FT, and FF. In every truth table with two letters, we list the four cases (the first two columns) first. Then we complete the remaining columns in the truth table. In Table 3.5(e), after completing the two left-hand columns, we complete the remaining columns in the order indicated by the numbers below the columns. We will continue to place numbers below the columns to show the order in which the columns are completed.

**TIMELY TIP** When constructing truth tables it is very important to keep your entries in neat columns and rows. If you are using lined paper, put only one row of the table on each line. If you are not using lined paper, using a straightedge may help you correctly enter the information into the truth table's rows and columns.



## EXAMPLE 2 *Construct and Interpret a Truth Table*

a) Construct a truth table for the following statement:

I have not studied and I am not ready for the test.

b) Under which conditions will the compound statement be true?

c) Suppose "I have studied" is a false statement and "I am ready for the test" is a true statement. Is the compound statement given in part (a) true or false?

**SOLUTION:**

a) First write the simple statements in symbolic form by using simple nonnegated statements.

Let

$p$: I have studied.
$q$: I am ready for the test.

Therefore, the compound statement may be written $\sim p \wedge \sim q$. Now construct a truth table with four cases, as shown in Table 3.6

Fill in the column labeled 1 by negating the truth values under $p$ on the far left. Fill in the column labeled 2 by negating the values under $q$ in the second column from the left. Fill in the column labeled 3 by using the columns labeled 1 and 2 and the definition of conjunction.

In the first row, to determine the entry for column 3, we use false for $\sim p$ and false for $\sim q$. Since false $\wedge$ false is false (see case 4 of Table 3.4), we place an F in column 3, row 1. In the second row, we use false for $\sim p$ and true for $\sim q$. Since false $\wedge$ true is false (see case 3 of Table 3.4), we place an F in column 3, row 2. In the third row, we use true for $\sim p$ and false for $\sim q$. Since true $\wedge$ false is false (see case 2 of Table 3.4), we place an F in column 3, row 3. In the fourth row, we use true for $\sim p$ and true for $\sim q$. Since true $\wedge$ true is true (see case 1 of Table 3.4), we place a T in column 3, row 4.

**TABLE 3.6**

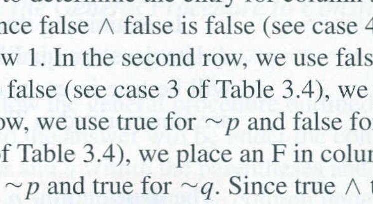
| p | q | ~p | $\wedge$ | ~q |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | F | T |
| F | T | T | F | F |
| F | F | T | T | T |
| | | 1 | 3 | 2 |

b) The compound statement in part (a) will be true only in case 4 (circled in blue) when both simple statements, $p$ and $q$, are false, that is, when I have not studied and I am not ready for the test.

c) We are told that $p$, "I have studied," is a false statement and that $q$, "I am ready for the test," is a true statement. From the truth table (Table 3.6), we can determine that when $p$ is false and $q$ is true, case 3, the compound statement, is false (circled in red).

# Disjunction

Consider the job description that contains the following requirements.


**Civil Technician**

Municipal program for redevelopment seeks on-site technician. **The applicant must have a two-year college degree in civil technology or five years of related experience.** Interested candidates please call 555-1234.

Who qualifies for the job? To help analyze the statement, translate it into symbolic form. Let $p$ be "A requirement for the job is a two-year college degree in civil technology" and $q$ be "A requirement for the job is five years of related experience." The statement in symbolic form is $p \vee q$. For the two simple statements, there are four distinct cases (see Table 3.7).

**TABLE 3.7 Disjunction**

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**CASE 1:** $p$ is true and $q$ is true. A candidate has a two-year college degree in civil technology and five years of related experience. The candidate has both requirements and qualifies for the job. Consider qualifying for the job as a true statement and not qualifying as a false statement. Since the candidate qualifies for the job, we put a T in the $p \vee q$ column.

**CASE 2:** $p$ is true and $q$ is false. A candidate has a two-year college degree in civil technology but does not have five years of related experience. The candidate still qualifies for the job with the two-year college degree. Thus, we put a T in the $p \vee q$ column.

**CASE 3:** $p$ is false and $q$ is true. The candidate does not have a two-year college degree in civil technology but does have five years of related experience. The candidate qualifies for the job with the five years of related experience. Thus, we put a T in the $p \vee q$ column.

**CASE 4:** $p$ is false and $q$ is false. The candidate does not have a two-year college degree in civil technology and does not have five years of related experience. The candidate does not meet either of the two requirements and therefore does not qualify for the job. Thus, we put an F in the $p \vee q$ column.

In examining the four cases, we see that there is only one case in which the candidate does not qualify for the job: case 4. As this example indicates, an *or* statement will be true in every case, except when both simple statements are false. The results are summarized in Table 3.7, the truth table for the disjunction.

The **disjunction**, $p \vee q$, is true when either $p$ is true, $q$ is true, or both $p$ and $q$ are true.

The disjunction $p \vee q$ is false only when $p$ and $q$ are both false.

## EXAMPLE 3 *Truth Table with a Negation*

Construct a truth table for $\sim(\sim q \wedge p)$.

**SOLUTION:** First construct the standard truth table listing the four cases. Then work within parentheses. The order to be followed is indicated by the numbers below the columns (see Table 3.8). Under $\sim q$, column 1, write the negation of the $q$ column. Then, in column 2, copy the values from the $p$ column. Next, complete the *and* column, column 3, using columns 1 and 2 and the truth table for the conjunction. The *and* column is true only when both statements are true, as in case 2. Finally, negate the values in the *and* column, column 3, and place these negated values in column 4. By examining the truth table you can see that the compound statement $\sim(\sim q \wedge p)$ is false only in case 2, that is, when $p$ is true and $q$ is false.

**TABLE 3.8**

| $p$ | $q$ | $\sim$ | $(\sim q$ | $\wedge$ | $p)$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | T | T | T |
| F | T | T | F | F | F |
| F | F | T | T | F | F |
| | | 4 | 1 | 3 | 2 |

## A General Procedure for Constructing Truth Tables

1. Study the compound statement and determine whether it is a negation, conjunction, disjunction, conditional, or biconditional statement, as was done in Section 3.1. The answer to the truth table will appear under $\sim$ if the statement is a negation, under $\wedge$ if the statement is a conjunction, under $\vee$ if the statement is a disjunction, under $\rightarrow$ if the statement is a conditional, and under $\leftrightarrow$ if the statement is a biconditional.
2. Complete the columns under the simple statements, $p$, $q$, $r$, and their negations, $\sim p$, $\sim q$, $\sim r$, within parentheses. If there are nested parentheses (one pair of parentheses within another pair), work with the innermost pair first.
3. Complete the column under the connective within the parentheses. You will use the truth values of the connective in determining the final answer in step 5.
4. Complete the column under any remaining statements and their negations.
5. Complete the column under any remaining connectives. Recall that the answer will appear under the column determined in step 1. If the statement is a conjunction, disjunction, conditional, or biconditional, you will obtain the truth values for the connective by using the last column completed on the left side and on the right side of the connective. If the statement is a negation, you will obtain the truth values by negating the truth values of the last column completed within the grouping symbols on the right side of the negation. Be sure to circle or highlight your answer column or number the columns in the order they were completed.

### EXAMPLE 4 *Use the General Procedure to Construct a Truth Table*

Construct a truth table for the statement $(\sim p \vee q) \wedge \sim p$.

**SOLUTION:** We will follow the general procedure outlined in the box. This statement is a conjunction, so the answer will be under the conjunction symbol. Complete columns under $\sim p$ and $q$ within the parentheses and call these columns 1 and 2, respectively (see Table 3.9). Complete the column under the disjunction, $\vee$, using the truth values in columns 1 and 2, and call this column 3. Next complete the column under $\sim p$, and call this column 4. The answer, column 5, is determined from the definition of the conjunction and the truth values in column 3, the last column completed on the left side of the conjunction, and column 4.

**TABLE 3.9**

| p | q | (~p | ∨ | q) | ∧ | ~p |
|---|---|---|---|---|---|---|
| T | T | F | T | T | F | F |
| T | F | F | F | F | F | F |
| F | T | T | T | T | T | T |
| F | F | T | T | F | T | T |
| | | 1 | 3 | 2 | 5 | 4 |

So far, all the truth tables we have constructed have contained at most two simple statements. Now we will explain how to construct a truth table that consists of three simple statements, such as $(p \wedge q) \wedge r$. When a compound statement consists of three simple statements, there are eight different true–false possibilities, as illustrated in Table 3.10. To begin such a truth table, write four Ts and four Fs in the column under $p$. Under the second statement, $q$, pairs of Ts alternate with pairs of Fs. Under the third statement, $r$, T alternates with F. This technique is not the only way of listing the cases, but it ensures that each case is unique and that no cases are omitted.

**TABLE 3.10**

4 2 1

| | p | q | r |
|---|---|---|---|
| Case 1 | T | T | T |
| Case 2 | T | T | F |
| Case 3 | T | F | T |
| Case 4 | T | F | F |
| Case 5 | F | T | T |
| Case 6 | F | T | F |
| Case 7 | F | F | T |
| Case 8 | F | F | F |

### EXAMPLE 5 *Construct a Truth Table with Eight Cases*

a) Construct a truth table for the statement "Santana is home and he is not at his desk, or he is sleeping."

b) Suppose that "Santana is home" is a false statement, that "Santana is at his desk" is a true statement, and that "Santana is sleeping" is a true statement. Is the compound statement in part (a) true or false?

**SOLUTION:**

a) First we will translate the statement into symbolic form. Let

$p$: Santana is home.
$q$: Santana is at his desk.
$r$: Santana is sleeping.

In symbolic form, the statement is $(p \wedge \sim q) \vee r$.

Since the statement is composed of three simple statements, there are eight cases. Begin by listing the eight cases in the three left-hand columns; see Table 3.11. By examining the statement, you can see that it is a disjunction. Therefore, the answer will be in the $\vee$ column. Fill out the truth table by working in parentheses first. Place values under $p$, column 1, and $\sim q$, column 2. Then find the conjunctions of columns 1 and 2 to obtain column 3. Place the values of $r$ in column 4. To obtain the answer, column 5, use columns 3 and 4 and your knowledge of the disjunction.

**TABLE 3.11**

| p | q | r | (p | ∧ | ~q) | ∨ | r |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T | T |
| T | T | F | T | F | F | F | F |
| T | F | T | T | T | T | T | T |
| T | F | F | T | T | T | T | F |
| F | T | T | F | F | F | T | T |
| F | T | F | F | F | F | F | F |
| F | F | T | F | F | T | T | T |
| F | F | F | F | F | T | F | F |
| | | | 1 | 3 | 2 | 5 | 4 |

b) We are given the following:

$p$: Santana is home—false.
$q$: Santana is at his desk—true.
$r$: Santana is sleeping—true.

Therefore, we need to find the truth value of the following case: false, true, true. In case 5 of the truth table, $p$, $q$, and $r$ are F, T, and T, respectively. Therefore, under these conditions, the original compound statement is true (as circled in the table). ▲

We have learned that a truth table with one simple statement has two cases, a truth table with two simple statements has four cases, and a truth table with three simple statements has eight cases. In general, *the number of distinct cases in a truth table with n distinct simple statements is* $2^n$. The compound statement $(p \vee q) \vee (r \wedge \sim s)$ has four simple statements, $p, q, r, s$. Thus, a truth table for this compound statement would have $2^4$, or 16, distinct cases.

When we construct a truth table, we determine the truth values of a compound statement for every possible case. If we want to find the truth value of the compound statement for any specific case when we know the truth values of the simple statements, we do not have to develop the entire table. For example, to determine the truth value for the statement

$$2 + 3 = 5 \quad \text{and} \quad 1 + 1 = 3$$

we let $p$ be $2 + 3 = 5$ and $q$ be $1 + 1 = 3$. Now we can write the compound statement as $p \wedge q$. We know that $p$ is a true statement and $q$ is a false statement. Thus, we can substitute T for $p$ and F for $q$ and evaluate the statement:

$$\begin{array}{c} p \wedge q \\ \text{T} \wedge \text{F} \\ \text{F} \end{array}$$

Therefore, the compound statement $2 + 3 = 5$ and $1 + 1 = 3$ is a false statement.

## EXAMPLE 6 *Determine the Truth Value of a Compound Statement*

Determine the truth value for each simple statement. Then, using these truth values, determine the truth value of the compound statement.

a) 15 is less than or equal to 9.

b) George Washington was the first U.S. president or Abraham Lincoln was the second U.S. president, but there has not been a U.S. president born in Antarctica.

**SOLUTION:**

a) Let

$p$: 15 is less than 9.

$q$: 15 is equal to 9.

The statement “15 is less than or equal to 9” means that 15 is less than 9 or 15 is equal to 9. The compound statement can be expressed as $p \vee q$. We know that both $p$ and $q$ are false statements since 15 is greater than 9. Therefore, substitute F for $p$ and F for $q$ and evaluate the statement:

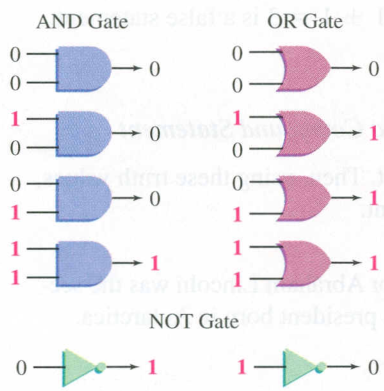
$$\begin{array}{c} p \vee q \\ \text{F} \vee \text{F} \\ \text{F} \end{array}$$

Therefore, the compound statement “15 is less than or equal to 9” is a false statement.

## DID YOU KNOW

### *Applications of Logic*

George Boole provided the key that would unlock the door to modern computing. Not until 1938, however, did Claude Shannon, in his master's thesis at MIT, propose uniting the on–off capability of electrical switches with Boole's two-value system of 0's and 1's. The operations AND, OR, and NOT and the rules of logic laid the foundation for computer gates. Such gates determine whether the current will pass. The closed switch (current flow) is represented as 1, and the open switch (no current flow) is represented as 0. For more information, see the group projects on pages 164 and 165.




As you can see from the accompanying diagrams, the gates function in essentially the same way as a truth table. The gates shown here are the simplest ones, representing simple statements. There are other gates, such as the NAND and NOR gates, that are combinations of NOT, AND, and OR gates. The microprocessing unit of a computer uses thousands of these switches.

b) Let

$p$: George Washington was the first U.S. president.
$q$: Abraham Lincoln was the second U.S. president.
$r$: There has been a U.S. president who was born in Antarctica.

The compound statement can be written in symbolic form as $(p \vee q) \wedge {\sim}r$. Recall that *but* is used to express a conjunction. We know that $p$ is a true statement and that $q$ is a false statement. We also know that $r$ is a false statement since all U.S. presidents must be born in the United States. Thus, since $r$ is a false statement, the negation, ${\sim}r$, is a true statement. So we will substitute T for $p$, F for $q$, and T for ${\sim}r$ and then evaluate the statement:

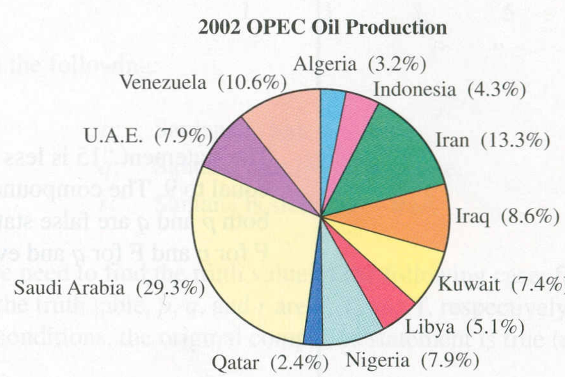
$$(p \vee q) \wedge {\sim}r$$
$$(\mathrm{T} \vee \mathrm{F}) \wedge \mathrm{T}$$
$$\mathrm{T} \wedge \mathrm{T}$$
$$\mathrm{T}$$

Therefore, the original compound statement is a true statement. ▲

## EXAMPLE 7 *OPEC Oil Production*

The Organization of Petroleum Exporting Countries (OPEC) consists of 11 developing nations whose economies are heavily reliant on oil export revenues. Figure 3.1 shows the percentage of total OPEC oil production produced by each of its member nations in 2002. Use this graph to determine the truth value of the following statement:

Saudi Arabia produces the most oil among OPEC nations and Qatar produces more oil than Venezuela, or Indonesia does not produce the least amount of oil among OPEC nations.




Source: Energy Information Administration

**Figure 3.1**

**SOLUTION:** Let

$p$: Saudi Arabia produces the most oil among OPEC nations.

$q$: Qatar produces more oil that Venezuela.

$r$: Indonesia produces the least amount of oil among OPEC nations.

The given compound statement can be written in symbolic form as $(p \wedge q) \vee \sim r$. From Fig. 3.1, we see that statement $p$ is true: Saudi Arabia does produce the most oil among OPEC nations. We also see that statement $q$ is false: Venezuela actually produces more oil than Qatar. We also see that statement $r$ is false: two OPEC nations produce less oil than Indonesia. Since $r$ is false, its negation, $\sim r$, is true. Therefore, we substitute T for $p$, F for $q$, and T for $\sim r$ and get

$$(p \wedge q) \vee \sim r$$
$$(\mathrm{T} \wedge \mathrm{F}) \vee \mathrm{T}$$
$$\mathrm{F} \vee \mathrm{T}$$
$$\mathrm{T}$$

Thus, the original compound statement is true. ▲

## SECTION 3.2 EXERCISES

1. a) How many distinct cases must be listed in a truth table that contains two simple statements?
   b) List all the cases.
2. a) How many distinct cases must be listed in a truth table that contains three simple statements?
   b) List all the cases.
3. a) Construct the truth table for the disjunction, $p \vee q$.
   b) Under what circumstances is the *or* table false?
4. a) Construct the truth table for the conjunction, $p \wedge q$.
   b) Under what circumstances is the *and* table true?

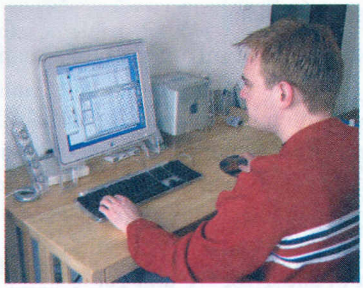
*In Exercises 5–20, construct a truth table for the statement.*

5. $p \vee \sim p$
6. $p \wedge \sim p$
7. $p \wedge \sim q$
8. $q \vee \sim p$
9. $\sim(p \vee \sim q)$
10. $\sim p \vee \sim q$
11. $\sim(p \wedge \sim q)$
12. $\sim(\sim p \wedge \sim q)$
13. $\sim q \vee (p \wedge r)$
14. $(p \vee \sim q) \wedge r$
15. $r \vee (p \wedge \sim q)$
16. $(r \wedge q) \wedge \sim p$
17. $(r \vee \sim p) \wedge \sim q$
18. $\sim p \wedge (q \vee r)$
19. $(\sim q \wedge r) \vee p$
20. $\sim r \vee (\sim p \wedge q)$

*In Exercises 21–30, write the statement in symbolic form and construct a truth table.*

21. Meetings are dull and teaching is fun.
22. The stadium is enclosed, but it is not air-conditioned.
23. Bob will get a haircut, but he will not shave his beard.
24. It is false that the class must have at least 15 students or the class will be canceled.
25. It is false that Jasper Adams is a tutor and Mark Russo is a secretary.
26. Mike made pizza and Dennis made a chef salad, but Gil burned the lemon squares.
27. The copier is out of toner, or the lens is dirty or the corona wires are broken.
28. I am hungry, and I want to eat a healthy lunch and I want to eat in a hurry.

29. The Congress must act on the bill, and the president must sign the bill or not sign the bill.
30. Gordon Langeneger likes the PowerMac G4 Cube and he likes the iBook, but he does not like the Pentium IV.



*In Exercises 31–42, determine the truth value of the statement if*

a) $p$ is true, $q$ is false, and $r$ is true.
b) $p$ is false, $q$ is true, and $r$ is true.

| P | Q | R | ~ | P | ∨ | (Q | ∧ | R) |
|---|---|---|---|---|---|---|---|---|
| T | F | T | F | T | F | F | F | T |

a. 31. $\sim p \vee (q \wedge r)$ False
32. $(\sim p \wedge r) \wedge q$
33. $(\sim q \wedge \sim p) \vee \sim r$
34. $(\sim p \vee \sim q) \vee \sim r$
35. $(p \wedge \sim q) \vee r$
36. $(p \vee \sim q) \wedge \sim (p \wedge \sim r)$
37. $(\sim r \wedge p) \vee q$
38. $\sim q \vee (r \wedge p)$
39. $(\sim q \vee \sim p) \wedge r$
40. $(\sim r \vee \sim p) \vee \sim q$
41. $(\sim p \vee \sim q) \vee (\sim r \vee q)$
42. $(\sim r \wedge \sim q) \wedge (\sim r \vee \sim p)$

*In Exercises 43–50, determine the truth value for each simple statement. Then use these truth values to determine the truth value of the compound statement. (You may have to use a reference source such as the Internet or an encyclopedia.)*

43. $3 + 5 = 4 + 4$ or $10 - 9 = 9 - 10$
44. $5 < 4$ and $4 < 5$
45. Elvis Presley was a singer or chickens can swim.

Elvis Presley

46. Alaska is the 50th state or Hawaii is a group of islands, and Atlanta is the capital of Alabama.
47. U2 is a rock band and Denzel Washington is an actor, but Jerry Seinfeld is not a comedian.
48. The city of Toronto is in Minnesota or Mexico City is in Texas, and Cairo is in Egypt.
49. Cal Ripken Jr. played football or George Bush was the prime minister of England, and Colin Powell was in the Army.
50. Holstein is a breed of cattle and collie is a breed of dogs, or beagle is not a breed of cats.

**Food Consumption** *In Exercises 51–54, use the chart to determine the truth value of each simple statement. Then determine the truth value of the compound statement.*

Annual per capita consumption in pounds:

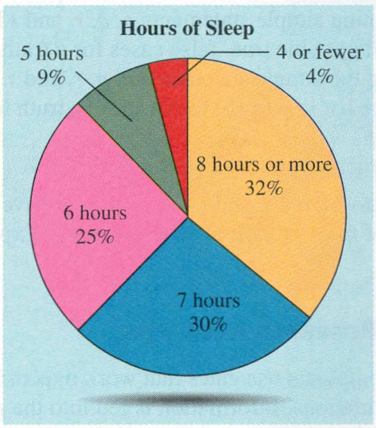
| | 1909 | 2001 |
|---|---|---|
| Red meat | 99 | 123.5 |
| Poultry | 11 | 66 |
| Fish | 11 | 15 |
| Cheese | 4 | 30 |
| Fats and oils[a] | 38 | 69 |
| Sweeteners[b] | 86 | 154 |

[a]Added fats and oils
[b]Caloric sweeteners (sugars, honey, corn syrup).
Source: U.S. Department of Agriculture

51. Thirty pounds of cheese were consumed by the average American in 1909, and the average American did not consume 154 pounds of sweeteners in 2001.
52. The per capita consumption of red meat was less for the average American in 2001 than it was in 1909 or the per capita consumption of poultry was greater for the average American in 2001 than it was in 1909.
53. The average American ate approximately the same amount of fish and poultry in 1909, but between 1909 and 2001 the per capita consumption of poultry increased at a rate higher than that of fish.

**54.** The average American ate approximately nine times as much red meat as fish in 1909, but by 2001 the average American only ate approximately eight times as much red meat as fish.

***Sleep Time*** *In Exercises 55–58, use the graph, which shows the number of hours Americans sleep, to determine the truth value of each simple statement. Then determine the truth value of the compound statement.*




**55.** It is false that 30% of Americans get 6 hours of sleep each night and 9% get 5 hours of sleep each night.

**56.** Twenty-five percent of Americans get 6 hours of sleep each night, and 30% get 7 hours of sleep each night or 9% do not get 5 hours of sleep each night.

**57.** Thirteen percent of Americans get 5 or fewer hours of sleep each night or 32% get 6 or more hours of sleep each night, and 30% get 8 or more hours of sleep each night.

**58.** Over one-half of all Americans get 7 or fewer hours of sleep each night, and over one-quarter get 6 or fewer hours of sleep each night.

*In Exercises 59–62, let*

$p$: Tanisha owns a convertible.
$q$: Joan owns a Volvo.

*Translate each statement into symbols. Then construct a truth table for each and indicate under what conditions the compound statement is true.*

**59.** Tanisha owns a convertible and Joan does not own a Volvo.

**60.** Tanisha does not own a convertible, but Joan owns a Volvo.

**61.** Tanisha owns a convertible or Joan does not own a Volvo.

**62.** Tanisha does not own a convertible or Joan does not own a Volvo.

*In Exercises 63–66, let*

$p$: The house is owned by an engineer.
$q$: The heat is solar generated.
$r$: The car is run by electric power.

*Translate each statement into symbols. Then construct a truth table for each and indicate under what conditions the compound statement is true.*

**63.** The car is run by electric power or the heat is solar generated, but the house is owned by an engineer.

**64.** The house is owned by an engineer and the heat is solar generated, or the car is run by electric power.

**65.** The heat is solar generated, or the house is owned by an engineer and the car is not run by electric power.

**66.** The house is not owned by an engineer, and the car is not run by electric power and the heat is solar generated.

***Obtaining a Loan*** *In Exercises 67 and 68, read the requirements and each applicant's qualifications for obtaining a loan.*

a) Identify which of the applicants would qualify for the loan.
b) For the applicants who do not qualify for the loan, explain why.

**67.** To qualify for a loan of \$40,000, an applicant must have a gross income of \$28,000 if single, \$46,000 combined income if married, and assets of at least \$6,000.

Mrs. Rusinek, married with three children, makes \$42,000 on her job. Mr. Rusinek does not have an income. The Rusineks have assets of \$42,000.

Mr. Duncan is not married, works in sales, and earns \$31,000. He has assets of \$9000.

Mrs. Tuttle and her husband have total assets of \$43,000. One earns \$35,000, and the other earns \$23,500.

**68.** To qualify for a loan of \$45,000, an applicant must have a gross income of \$30,000 if single, \$50,000 combined income if married, and assets of at least \$10,000.

Mr. Argento, married with two children, makes \$37,000 on his job. Mrs. Argento earns \$15,000 at a part-time job. The Argentos have assets of \$25,000.

Ms. McVey, single, has assets of \$19,000. She works in a store and earns \$25,000.

Mr. Siewert earns \$24,000 and Ms. Fox, his wife, earns \$28,000. Their assets total \$8000.

**69.** ***Airline Special Fares*** An airline advertisement states, "To get the special fare you must purchase your tickets between January 1 and February 15 and fly round trip between March 1 and April 1. You must depart on a Monday, Tuesday, or Wednesday, and return on a Tuesday, Wednesday, or Thursday, and stay over at least one Saturday."

a) Determine which of the following individuals will qualify for the special fare.
b) If the person does not qualify for the special fare, explain why.

Wing Park plans to purchase his ticket on January 15, depart on Monday, March 3, and return on Tuesday, March 18.

Gina Vela plans to purchase her ticket on February 1, depart on Wednesday, March 10, and return on Thursday, April 2.

Kara Shavo plans to purchase her ticket on February 14, depart on Tuesday, March 5, and return on Monday, March 18.

Christos Supernaw plans to purchase his ticket on January 4, depart on Monday, March 8, and return on Thursday, March 11.

Alex Chang plans to purchase his ticket on January 1, depart on Monday, March 3, and return on Monday, March 10.

## Problem Solving/Group Activities

*In Exercises 70 and 71, construct a truth table for the symbolic statement.*

70. $\sim[(\sim(p \vee q)) \vee (q \wedge r)]$
71. $[(q \wedge \sim r) \wedge (\sim p \vee \sim q)] \vee (p \vee \sim r)$
72. On page 113, we indicated that a compound statement consisting of $n$ simple statements had $2^n$ distinct true–false cases.
    a) How many distinct true–false cases does a truth table containing simple statements $p$, $q$, $r$, and $s$ have?
    b) List all possible true–false cases for a truth table containing the simple statements $p$, $q$, $r$, and $s$.
    c) Use the list in part (b) to construct a truth table for $(q \wedge p) \vee (\sim r \wedge s)$.
    d) Construct a truth table for $(\sim r \wedge \sim s) \wedge (\sim p \vee q)$.
73. Must $(p \wedge \sim q) \vee r$ and $(q \wedge \sim r) \vee p$ have the same number of trues in their answer columns? Explain.

## Internet/Research Activities

74. Digital computers use gates that work like switches to perform calculations. Information is fed into the gates and information leaves the gates, according to the type of gate. The three basic gates used in computers are the NOT gate, the AND gate, and the OR gate. Do research on the three types of gates.
    a) Explain how each gate works.
    b) Explain the relationship between each gate and the corresponding logic connectives *not, and,* and *or*.
    c) Illustrate how two or more gates can be combined to form a more complex gate.

# 3.3 TRUTH TABLES FOR THE CONDITIONAL AND BICONDITIONAL

## Conditional

In Section 3.1, we mentioned that the statement preceding the conditional symbol is called the *antecedent* and that the statement following the conditional symbol is called the *consequent*. For example, consider $(p \vee q) \rightarrow [\sim(q \wedge r)]$. In this statement, $(p \vee q)$ is the antecedent and $[\sim(q \wedge r)]$ is the consequent.

Now we will look at the truth table for the conditional. Suppose I make the following promise to you: "If you get an A in this class, then I will buy you a car." Consider the statement within the quotation marks. Assume this statement is true except when I have actually broken my promise to you.

Let

$p$: You get an A.
$q$: I buy you a car.

Translated into symbolic form, the statement becomes $p \to q$. Let's examine the four cases shown in Table 3.12.

**TABLE 3.12 Conditional**

| p | q | p → q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**CASE 1: (T, T)** You get an A, and I buy a car for you. I have met my commitment, and the statement is true.

**CASE 2: (T, F)** You get an A, and I do not buy a car for you. I have broken my promise, and the statement is false.

What happens if you don't get an A? If you don't get an A, I no longer have a commitment to you, and therefore I cannot break my promise.

**CASE 3: (F, T)** You do not get an A, and I buy you a car. I have not broken my promise, and therefore the statement is true.

**CASE 4: (F, F)** You do not get an A, and I don't buy you a car. I have not broken my promise, and therefore the statement is true.

The conditional statement is false when the antecedent is true and the consequent is false. In every other case the conditional statement is true.

> The **conditional statement** $p \to q$ is true in every case except when $p$ is a true statement and $q$ is a false statement.

## EXAMPLE 1 *A Truth Table with a Conditional*

Construct a truth table for the statement $\sim p \to q$.

**SOLUTION:** Since this is a conditional statement, the answer will lie under the $\to$. Fill out the truth table by placing the appropriate values under $\sim p$, column 1, and under $q$, column 2 (see Table 3.13). Then, using the information given in the truth table for the conditional and the truth values in columns 1 and 2, determine the solution, column 3. In row 1, the antecedent, $\sim p$, is false and the consequent, $q$, is true. Row 1 is F $\to$ T, which according to row 3 of Table 3.12, is T. Likewise, row 2 of Table 3.13 is F $\to$ F, which is T. Row 3 is T $\to$ T, which is T. Row 4 is T $\to$ F, which is F.

**TABLE 3.13**

| p | q | ~p | → | q |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | F |
| | | 1 | 3 | 2 |

## EXAMPLE 2 *A Conditional Truth Table with Three Simple Statements*

Construct a truth table for the statement $p \to (\sim q \wedge r)$.

**SOLUTION:** Since this is a conditional statement, the answer will lie under the $\to$. Work within the parentheses first. Place the truth values under $\sim q$, column 1, and $r$, column 2 (Table 3.14). Then take the conjunction of columns 1 and 2 to obtain column 3. Next, place the truth values under $p$ in column 4. To determine the answer, column 5, use columns 3 and 4 and your knowledge of the conditional statement. Column 4 represents the truth values of the antecedent, and column 3 represents the truth values of the consequent. Remember that the conditional is false only when the antecedent is true and the consequent is false, as in cases (rows) 1, 2, and 4 of column 5.

**TABLE 3.14**

| p | q | r | p | → | (~q | ∧ | r) |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | F | T |
| T | T | F | T | F | F | F | F |
| T | F | T | T | T | T | T | T |
| T | F | F | T | F | T | F | F |
| F | T | T | F | T | F | F | T |
| F | T | F | F | T | F | F | F |
| F | F | T | F | T | T | T | T |
| F | F | F | F | T | T | F | F |
| | | | 4 | 5 | 1 | 3 | 2 |

**EXAMPLE 3** *Examining an Advertisement*

An advertisement for Perky Morning coffee makes the following claim: "If you drink Perky Morning coffee, then you will not be sluggish and you will have a great day." Translate the statement into symbolic form and construct a truth table.

**SOLUTION:** Let

$p$: You drink Perky Morning coffee.
$q$: You will be sluggish.
$r$: You will have a great day.

In symbolic form, the claim is

$$p \rightarrow (\sim q \wedge r)$$

This symbolic statement is identical to the statement in Table 3.14, and the truth tables are the same. Column 3 represents the truth values of $(\sim q \wedge r)$, which corresponds to the statement "You will not be sluggish and you will have a great day." Note that column 3 is true in cases (rows) 3 and 7. In case 3, since $p$ is true, you drank Perky Morning coffee. In case 7, however, since $p$ is false, you did not drink Perky Morning coffee. From this information we can conclude that it is possible for you to not be sluggish and for you to have a great day without drinking Perky Morning coffee.

A truth table alone cannot tell us whether a statement is true or false. It can, however, be used to examine the various possibilities.

## Biconditional

The *biconditional statement*, $p \leftrightarrow q$, means that $p \rightarrow q$ and $q \rightarrow p$, or, symbolically, $(p \rightarrow q) \wedge (q \rightarrow p)$. To determine the truth table for $p \leftrightarrow q$, we will construct the truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$ (Table 3.15). Table 3.16 shows the truth values for the biconditional statement.

**TABLE 3.15**

| p | q | (p | → | q) | ∧ | (q | → | p) |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | F | T | F | F | F | F | T | T |
| F | T | F | T | T | F | T | F | F |
| F | F | F | T | F | T | F | T | F |
| | | 1 | 3 | 2 | 7 | 4 | 6 | 5 |

**TABLE 3.16 Biconditional**

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

The **biconditional statement,** $p \leftrightarrow q$, is true only when $p$ and $q$ have the same truth value, that is, when both are true or both are false.

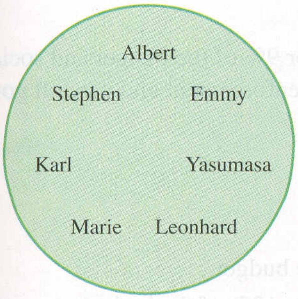
## DID YOU KNOW

### Satisfiability Problems

Suppose you are hosting a dinner party for seven people: Yasumasa, Marie, Albert, Stephen, Leonhard, Karl, and Emmy. You need to develop a seating plan around your circular dining room table that would satisfy all your guests. Albert and Emmy are great friends and must sit together. Yasumasa and Karl haven't spoken to each other in years and cannot sit by each other. Leonhard must sit by Marie or by Albert, but he cannot sit by Karl. Stephen insists he sit by Albert. Can you come up with a plan that would satisfy all your guests? Now imagine the difficulty of such a problem as the list of guests, and their demands, grows.

Problems such as this are known as *satisfiability* problems. The symbolic logic you are studying in this chapter allows computer scientists to represent these problems with symbols. Once in symbolic form, a problem can be studied with computers using algorithms. An *algorithm* is a general procedure for accomplishing a task. Even with the fastest computers, some satisfiability problems take an enormous amount of time to solve. Other problems may not even have a solution. Computer scientists and mathematicians are continually working to improve the algorithms used to solve *satisfiability problems.*

One solution to the problem posed above is as follows:




**EXAMPLE 4** *A Truth Table Using a Biconditional*

Construct a truth table for the statement $p \leftrightarrow (q \rightarrow \sim r)$.

**SOLUTION:** Since there are three letters, there must be eight cases. The parentheses indicate that the answer must be under the biconditional (Table 3.17). Use columns 3 and 4 to obtain the answer in column 5. When columns 3 and 4 have the same truth values, place a T in column 5. When columns 3 and 4 have different truth values, place an F in column 5.

**TABLE 3.17**

| p | q | r | p | ↔ | (q | → | ~r) |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | T | F | F |
| T | T | F | T | T | T | T | T |
| T | F | T | T | T | F | T | F |
| T | F | F | T | T | F | T | T |
| F | T | T | F | T | T | F | F |
| F | T | F | F | F | T | T | T |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | T | T |
| | | | 4 | 5 | 1 | 3 | 2 |

In the preceding section, we showed that finding the truth value of a compound statement for a specific case does not require constructing an entire truth table. Example 5 illustrates this technique for the conditional and the biconditional.

**EXAMPLE 5** *Determine the Truth Value of a Compound Statement*

Determine the truth value of the statement $(q \leftrightarrow r) \rightarrow (\sim p \wedge r)$ when $p$ is true, $q$ is false, and $r$ is true.

**SOLUTION:** Substitute the truth value for each simple statement:

$$(q \leftrightarrow r) \rightarrow (\sim p \wedge r)$$
$$(\mathrm{F} \leftrightarrow \mathrm{T}) \rightarrow (\mathrm{F} \wedge \mathrm{T})$$
$$\mathrm{F} \rightarrow \mathrm{F}$$
$$\mathrm{T}$$

For this specific case, the statement is true.

**EXAMPLE 6** *Determine the Truth Value of a Compound Statement*

Determine the truth value for each simple statement. Then use the truth values to determine the truth value of the compound statement.

a) If 15 is an even number, then 29 is an even number.

b) Northwestern University is in Illinois and Marquette University is in Alaska, if and only if Purdue University is in Alabama.

**SOLUTION:**

a) Let

$p$: 15 is an even number.

$q$: 29 is an even number.

Then the statement "If 15 is an even number, then 29 is an even number" can be written $p \rightarrow q$. Since 15 is not an even number, $p$ is a false statement. Also, since 29 is not an even number, $q$ is a false statement. We substitute F for $p$ and F for $q$ and evaluate the statement:

$$\begin{array}{c} p \rightarrow q \\ \mathrm{F} \rightarrow \mathrm{F} \\ \mathrm{T} \end{array}$$

Therefore, "If 15 is an even number, then 29 is an even number" is a true statement.

b) Let

$p$: Northwestern University is in Illinois.

$q$: Marquette University is in Alaska.

$r$: Purdue University is in Alabama.

The original compound statement can be written $(p \wedge q) \leftrightarrow r$. By checking the Internet or other references we can find that Northwestern University is in Illinois, Marquette University is in Wisconsin, and Purdue University is in Indiana. Therefore, $p$ is a true statement, but $q$ and $r$ are false statements. We will substitute T for $p$, F for $q$, and F for $r$ and evaluate the compound statement:

$$\begin{array}{c} (p \wedge q) \leftrightarrow r \\ (\mathrm{T} \wedge \mathrm{F}) \leftrightarrow \mathrm{F} \\ \mathrm{F} \quad \leftrightarrow \mathrm{F} \\ \mathrm{T} \end{array}$$

Therefore, the original compound statement is true.

## EXAMPLE 7 *Using Real Data in Compound Statements*

The graph in Fig. 3.2 on page 123 represents the U.S. government budget expenditures for fiscal year 2002. Use this graph to determine the truth value of the following compound statements.

a) If social programs account for 17% of the budget then interest on the national debt accounts for 12% of the budget.

b) If physical and community development account for 9% of the budget and social programs account for 37% of the budget, then law enforcement and general government account for 10% of the budget.

**SOLUTION:**

a) Let

$p$: Social programs account for 17% of the budget.

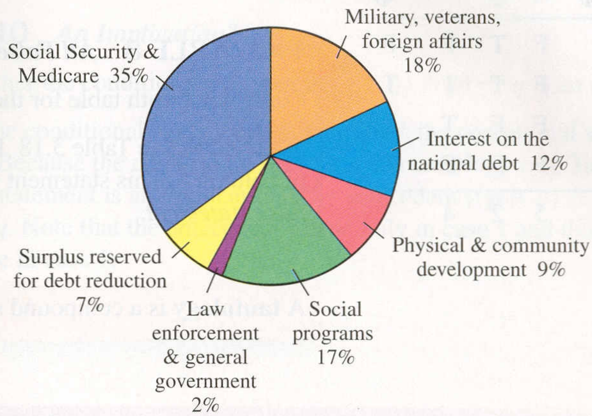
$q$: Interest on the national debt accounts for 12% of the budget.

Then the original compound statement can be written $p \rightarrow q$. We can see from Fig. 3.2 that both $p$ and $q$ are true statements. Substitute T for $p$ and T for $q$ and evaluate the statement:

$$p \rightarrow q$$
$$\mathrm{T} \rightarrow \mathrm{T}$$
$$\mathrm{T}$$

Therefore, "If social programs account for 17% of the budget then interest on the national debt accounts for 12% of the budget" is a true statement.

**2002 United States Federal Government Budget Expenditures**




Source: *United States Office of Management and Budget*

**Figure 3.2**

b) Let

*p*: Physical and community development account for 9% of the budget.
*q*: Social programs account for 37% of the budget.
*r*: Law enforcement and general government account for 10% of the budget.

Then the original compound statement can be written $(p \wedge q) \rightarrow r$. We can see from Fig. 3.2 that $p$ is true, $q$ is false, and $r$ is false. Substitute T for $p$ and F for $q$ and $r$ and evaluate the statement:

$$(p \wedge q) \rightarrow r$$
$$(\mathrm{T} \wedge \mathrm{F}) \rightarrow \mathrm{F}$$
$$\mathrm{F} \quad \rightarrow \mathrm{F}$$
$$\mathrm{T}$$

Therefore, the original compound statement, "If physical and community development account for 9% of the budget and social programs account for 37% of the budget, then law enforcement and general government account for 10% of the budget," is true. ▲

## Self-Contradictions, Tautologies, and Implications

Two special situations can occur in the truth table of a compound statement: The statement may always be false, or the statement may always be true. We give such statements special names.

A **self-contradiction** is a compound statement that is always false.

When every truth value in the answer column of the truth table is false, then the statement is a self-contradiction.

**TABLE 3.18**

| p | q | (p↔q) | ∧ | (p | ↔ | ~q) |
|---|---|---|---|---|---|---|
| T | T | T | F | T | F | F |
| T | F | F | F | T | T | T |
| F | T | F | F | F | T | F |
| F | F | T | F | F | F | T |
| | | 1 | 5 | 2 | 4 | 3 |

### EXAMPLE 8 *All Falses, a Self-Contradiction*

Construct a truth table for the statement $(p \leftrightarrow q) \wedge (p \leftrightarrow \sim q)$.

**SOLUTION:** See Table 3.18. In this example, the truth values are false in each case of column 5. This statement is an example of a self-contradiction or a *logically false statement.* ▲

A **tautology** is a compound statement that is always true.

When every truth value in the answer column of the truth table is true, the statement is a tautology.

### EXAMPLE 9 *All Trues, a Tautology*

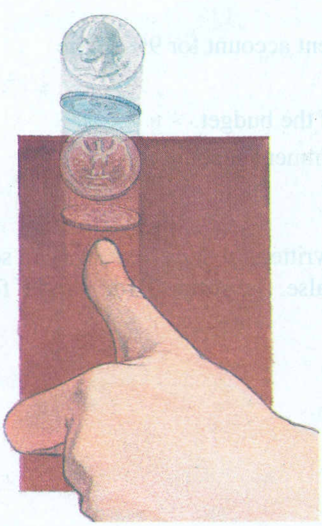
Construct a truth table for the statement $(p \wedge q) \rightarrow (p \vee r)$.

**SOLUTION:** The answer is given in column 3 of Table 3.19. The truth values are true in every case. Thus, the statement is an example of a tautology or a *logicallly true statement.*

**TABLE 3.19**

| p | q | r | (p∧q) | → | (p∨r) |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | F | T | T |
| T | F | F | F | T | T |
| F | T | T | F | T | T |
| F | T | F | F | T | F |
| F | F | T | F | T | T |
| F | F | F | F | T | F |
| | | | 1 | 3 | 2 |

▲



**"Heads I win,** tails you lose." Do you think that this statement is a tautology, self-contradiction, or neither? See Problem-Solving Exercise 81.

The conditional statement $(p \wedge q) \rightarrow (p \vee r)$ is a tautology. Conditional statements that are tautologies are called *implications.* In Example 9, we can say that $p \wedge q$ implies $p \vee r$.

An **implication** is a conditional statement that is a tautology.

In any implication the antecedent of the conditional statement implies the consequent. In other words, if the antecedent is true, then the consequent must also be true. That is, the consequent will be true whenever the antecedent is true.

**TABLE 3.20**

| p | q | [(p ∧ q) | ∧ | p] | → | q |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | F | F | T | T | F |
| F | T | F | F | F | T | T |
| F | F | F | F | F | T | F |
| | | 1 | 3 | 2 | 5 | 4 |

## EXAMPLE 10 *An Implication?*

Determine whether the conditional statement $[(p \wedge q) \wedge p] \rightarrow q$ is an implication.

**SOLUTION:** If the conditional statement is a tautology, the conditional statement is an implication. Because the conditional statement is a tautology (see Table 3.20), the conditional statement is an implication. The antecedent $[(p \wedge q) \wedge p]$ implies the consequent $q$. Note that the antecedent is true only in case 1 and that the consequent is also true in case 1.

# SECTION 3.3 EXERCISES

1. a) Construct the truth table for the conditional statement $p \rightarrow q$.
   b) Explain when the conditional statement is true and when it is false.
2. a) Construct the truth table for the biconditional statement $p \leftrightarrow q$.
   b) Explain when the biconditional statement is true and when it is false.
3. a) Explain the procedure to determine the truth value of a compound statement when specific truth values are provided for the simple statements.
   *b) Follow the procedure in part (a) and determine the truth value of the symbolic statement*

$$[(p \leftrightarrow q) \vee (\sim r \rightarrow q)] \rightarrow \sim r$$

when $p$ = true, $q$ = true, and $r$ = false.

4. What is a tautology?
5. What is a self-contradiction?
6. What is an implication?

*In Exercises 7–16, construct a truth table for the statement.*

7. $\sim q \rightarrow \sim p$
8. $p \rightarrow \sim q$
9. $\sim(q \rightarrow p)$
10. $\sim(p \leftrightarrow q)$
11. $\sim q \leftrightarrow p$
12. $(p \leftrightarrow q) \rightarrow p$
13. $p \leftrightarrow (q \vee p)$
14. $(\sim q \wedge p) \rightarrow \sim q$
15. $q \rightarrow (p \rightarrow \sim q)$
16. $(p \vee q) \leftrightarrow (p \wedge q)$

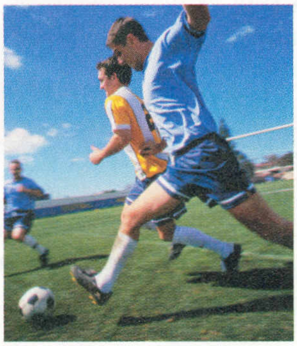
*In Exercises 17–26, construct a truth table for the statement.*

17. $r \wedge (\sim q \rightarrow p)$
18. $p \rightarrow (q \vee r)$
19. $(q \leftrightarrow p) \wedge \sim r$
20. $q \leftrightarrow (r \wedge p)$
21. $(q \vee \sim r) \leftrightarrow \sim p$
22. $(p \wedge r) \rightarrow (q \vee r)$
23. $(\sim r \vee \sim q) \rightarrow p$
24. $[r \wedge (q \vee \sim p)] \leftrightarrow \sim p$
25. $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim r)$
26. $(\sim p \leftrightarrow \sim q) \rightarrow (\sim q \leftrightarrow r)$

*In Exercises 27–32, write the statement in symbolic form. Then construct a truth table for the symbolic statement.*

27. If I drink a glass of water, then I will have a better complexion and I will sleep better.

28. The goalie will make the save if and only if the stopper is in position, or the forward cannot handle the ball.



29. The class has been canceled if and only if the teacher is not here, or we will study together in the library.
30. If the lake rises then we can go canoeing, and if the canoe has a hole in it then we cannot go canoeing.
31. If Mary Andrews does not send me an e-mail then we can call her, or we can write to Mom.
32. It is false that if Eileen Jones went to lunch, then she cannot take a message and we will have to go home.

*In Exercises 33–38, determine whether the statement is a tautology, self-contradiction, or neither.*

33. $p \to \sim q$
34. $(p \vee q) \leftrightarrow \sim p$
35. $p \wedge (q \wedge \sim p)$
36. $(p \wedge \sim q) \to q$
37. $(\sim q \to p) \vee \sim q$
38. $[(p \to q) \vee r] \leftrightarrow [(p \wedge q) \to r]$

*In Exercises 39–44, determine whether the statement is an implication.*

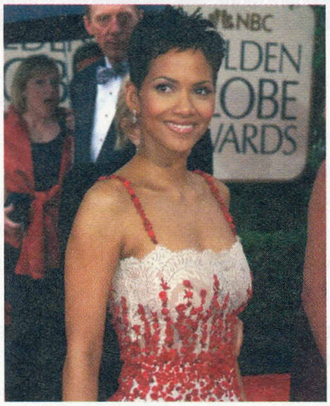
39. $p \to (p \wedge q)$
40. $(p \wedge q) \to (p \vee q)$
41. $(q \wedge p) \to (p \wedge q)$
42. $(p \vee q) \to (p \vee \sim r)$
43. $[(p \to q) \wedge (q \to p)] \to (p \leftrightarrow q)$
44. $[(p \vee q) \wedge r] \to (p \vee q)$

*In Exercises 45–56, if p is true, q is false, and r is true, find the truth value of the statement.*

45. $p \to (\sim q \wedge r)$
46. $\sim p \to (q \vee r)$
47. $(q \wedge \sim p) \leftrightarrow \sim r$
48. $p \leftrightarrow (\sim q \wedge r)$
49. $(\sim p \wedge \sim q) \vee \sim r$
50. $\sim [p \to (q \wedge r)]$
51. $(p \wedge r) \leftrightarrow (p \vee \sim q)$
52. $(\sim p \vee q) \to \sim r$
53. $(\sim p \leftrightarrow r) \vee (\sim q \leftrightarrow r)$
54. $(r \to \sim p) \wedge (q \to \sim r)$
55. $\sim [(p \vee q) \leftrightarrow (p \to \sim r)]$
56. $[(\sim r \to \sim q) \vee (p \wedge \sim r)] \to q$

*In Exercises 57–64, determine the truth value for each simple statement. Then, using the truth values, determine the truth value of the compound statement.*

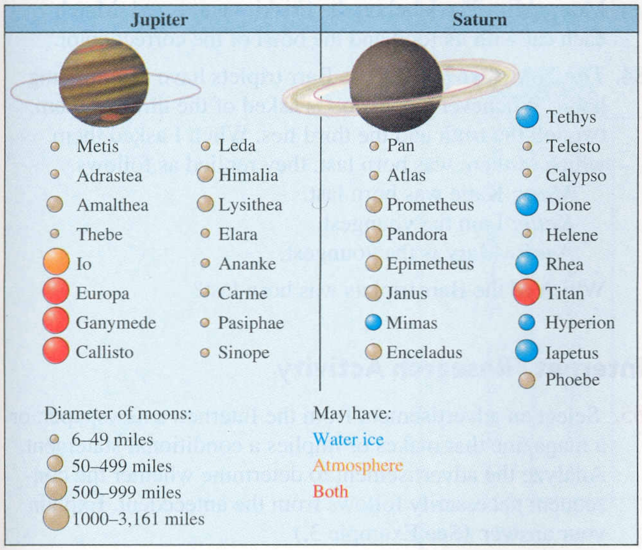
57. If $10 + 5 = 15$, then $56 \div 7 = 8$.
58. If 2 is an even number and 6 is an odd number, then 15 is an odd number.
59. A triangle has four sides or a square has three sides, and a rectangle has four sides.
60. Seattle is in Washington and Portland is in Oregon, or Boise is in California.
61. Dell makes computers, if and only if Gateway makes computers or Canon makes printers.
62. Spike Lee is a movie director, or if Halle Berry is a schoolteacher then George Clooney is a circus clown.



Halle Berry

63. Valentine's Day is in February or President's Day is in March, and Thanksgiving is in November.
64. Honda makes automobiles or Honda makes motorcycles, if and only if Toyota makes cereal.

*In Exercises 65–68, use the information provided about the moons for the planets Jupiter and Saturn on page 127 to determine the truth values of the simple statements. Then determine the truth value of the compound statement.*




Source: *Time* Magazine

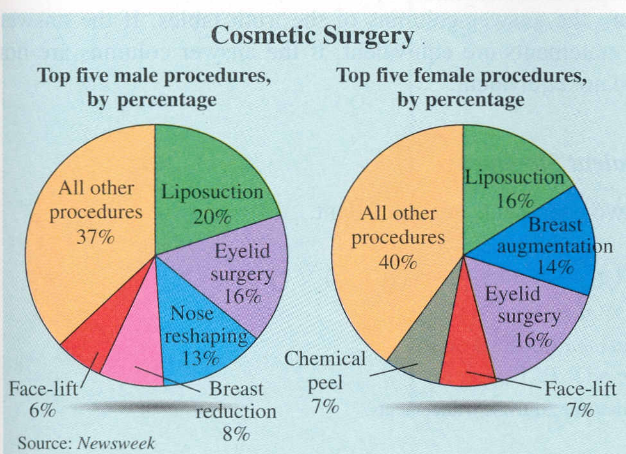
65. ***Jupiter's Moons*** Io has a diameter of 1000–3161 miles or Thebe may have water, and Io may have atmosphere.

66. ***Moons of Saturn*** Titan may have water and Titan may have atmosphere, if and only if Janus may have water.

67. ***Moon Comparisons*** Phoebe has a larger diameter than Rhea if and only if Callisto may have water ice, and Calypso has a diameter of 6–49 miles.

68. ***Moon Comparisons*** If Jupiter has 16 moons or Saturn does not have 18 moons, then Saturn has 7 moons that may have water ice.

*In Exercises 69 and 70, use the graphs to determine the truth values of each simple statement. Then determine the truth value of the compound statement.*




69. ***Most Common Cosmetic Surgery*** The most common cosmetic surgery procedure for females is liposuction or the most common procedure for males is eyelid surgery, and 20% of male cosmetic surgery is for nose reshaping.

70. ***Face-lifts and Eyelid Surgeries*** 7% of female cosmetic surgeries are for face-lifts and 10% of male cosmetic surgeries are for face-lifts, if and only if males have a higher percent of eyelid surgeries than females.

*In Exercises 71–76, suppose both of the following statements are true.*

$p$: Muhundan spoke at the teachers' conference.

$q$: Muhundan received the outstanding teacher award.

*Find the truth values of each compound statement.*

71. If Muhundan spoke at the teachers' conference, then Muhundan received the outstanding teacher award.

72. If Muhundan did not speak at the teachers' conference, then Muhundan did not receive the outstanding teacher award.

73. If Muhundan did not speak at the teachers' conference, then Muhundan received the outstanding teacher award.

74. Muhundan did not receive the outstanding teacher award if and only if Muhundan spoke at the teachers' conference.

75. Muhundan received the outstanding teacher award if and only if Muhundan spoke at the teachers' conference.

76. If Muhundan did not receive the outstanding teacher award, then Muhundan did not speak at the teachers' conference.

77. ***A New Computer*** Your parents make the following statement to your sister, "If you get straight A's this semester, then we will buy you a new computer." At the end of the semester your parents buy your sister a new computer. Can you conclude that your sister got straight A's? Explain.

78. ***Job Interview*** Consider the statement "If your interview goes well, then you will be offered the job." If you are interviewed and then offered the job, can you conclude that your interview went well? Explain.

## Problem Solving/Group Activities

*In Exercises 79 and 80, construct truth tables for the symbolic statement. Use the dominance of connectives (see Section 3.1) as needed.*

79. $p \vee q \rightarrow \sim r \leftrightarrow p \wedge \sim q$

80. $[(r \rightarrow \sim q) \rightarrow \sim p] \vee (q \leftrightarrow \sim r)$

81. Is the statement "Heads I win, tails you lose" a tautology, a self-contradiction, or neither? Explain your answer.

82. Construct a truth table for
a) $(p \vee q) \rightarrow (r \wedge s)$.
b) $(q \rightarrow \sim p) \vee (r \leftrightarrow s)$.

## Recreational Mathematics

83. *Cat Puzzle* Solve the following puzzle. The Joneses have four cats. The parents are Tiger and Boots, and the kittens are Sam and Sue. Each cat insists on eating out of its own bowl. To complicate matters, each cat will eat only its own brand of cat food. The colors of the bowls are red, yellow, green, and blue. The different types of cat food are Whiskas, Friskies, Nine Lives, and Meow Mix. Tiger will eat Meow Mix if and only if it is in a yellow bowl. If Boots is to eat her food, then it must be in a yellow bowl. Mrs. Jones knows that the label on the can containing Sam's food is the same color as his bowl. Boots eats Whiskas. Meow Mix and Nine Lives are packaged in a brown paper bag. The color of Sue's bowl is green if and only if she eats Meow Mix. The label on the Friskies can is red. Match each cat with its food and the bowl of the correct color.

84. *The Youngest Triplet* The Barr triplets have an annoying habit: Whenever a question is asked of the three of them, two tell the truth and the third lies. When I asked them which of them was born last, they replied as follows.
*Mary:* Katie was born last.
*Katie:* I am the youngest.
*Annie:* Mary is the youngest.
Which of the Barr triplets was born last?

## Internet/Research Activity

85. Select an advertisement from the Internet, a newspaper, or a magazine that makes or implies a conditional statement. Analyze the advertisement to determine whether the consequent necessarily follows from the antecedent. Explain your answer. (See Example 3.)

# 3.4 EQUIVALENT STATEMENTS

Equivalent statements are an important concept in the study of logic.

Two statements are **equivalent,** symbolized $\Leftrightarrow$,* if both statements have exactly the same truth values in the answer columns of the truth tables.

Sometimes the words *logically equivalent* are used in place of the word *equivalent.*

To determine whether two statements are equivalent, construct a truth table for each statement and compare the answer columns of the truth tables. If the answer columns are identical, the statements are equivalent. If the answer columns are not identical, the statements are not equivalent.

### EXAMPLE 1 *Equivalent Statements*

Show that the following two statements are equivalent.

$$[p \vee (q \vee r)] \qquad [(p \vee q) \vee r]$$

*The symbol $\equiv$ is also used to indicate equivalent statements.

**SOLUTION:** Construct a truth table for each statement (see Table 3.21).

**TABLE 3.21**

| p | q | r | [p | ∨ | (q ∨ r)] | [(p ∨ q) | ∨ | r] |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | F |
| T | F | T | T | T | T | T | T | T |
| T | F | F | T | T | F | T | T | F |
| F | T | T | F | T | T | T | T | T |
| F | T | F | F | T | T | T | T | F |
| F | F | T | F | T | T | F | T | T |
| F | F | F | F | F | F | F | F | F |
| | | | 1 | 3 | 2 | 1 | 3 | 2 |

Because the truth tables have the same answer (column 3 for both tables), the statements are equivalent. Thus, we can write

$$[p \vee (q \vee r)] \Leftrightarrow [(p \vee q) \vee r].$$

▲

## EXAMPLE 2 *Are the Following Equivalent Statements?*

Determine whether the following statements are equivalent.

a) If you work hard and obey all of the rules, then you will succeed in life.

b) If you do not work hard or do not obey all of the rules, then you will not succeed in life.

**SOLUTION:** First write each statement in symbolic form, then construct a truth table for each statement. If the answer columns of both truth tables are identical, then the statements are equivalent. If the answer columns are not identical, then the statements are not equivalent.

Let

$p$: You work hard.

$q$: You obey all of the rules.

$r$: You will succeed in life.

In symbolic form, the statements are

a) $(p \wedge q) \rightarrow r$. b) $(\sim p \vee \sim q) \rightarrow \sim r$.

The truth tables for these statements are given in Tables 3.22 and 3.23, respectively, on page 130. The answers in the columns labeled 5 are not identical, so the statements are not equivalent.

**TABLE 3.22**

| p | q | r | (p | ∧ | q) | → | r |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | F |
| T | F | T | T | F | F | T | T |
| T | F | F | T | F | F | T | F |
| F | T | T | F | F | T | T | T |
| F | T | F | F | F | T | T | F |
| F | F | T | F | F | F | T | T |
| F | F | F | F | F | F | T | F |
| | | | 1 | 3 | 2 | 5 | 4 |

**TABLE 3.23**

| p | q | r | (~p | ∨ | ~q) | → | ~r |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T | F |
| T | T | F | F | F | F | T | T |
| T | F | T | F | T | T | F | F |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | F | F | F |
| F | T | F | T | T | F | T | T |
| F | F | T | T | T | T | F | F |
| F | F | F | T | T | T | T | T |
| | | | 1 | 3 | 2 | 5 | 4 |

## EXAMPLE 3 *Which Statements Are Logically Equivalent?*

Determine which statement is logically equivalent to "It is not true that the tire is both out of balance and flat."

a) If the tire is not flat, then the tire is not out of balance.

b) The tire is not out of balance or the tire is not flat.

c) The tire is not flat and the tire is not out of balance.

d) If the tire is not out of balance, then the tire is not flat.

**SOLUTION:** To determine whether any of the choices are equivalent to the given statement, first write the given statements and the choices in symbolic form. Then construct truth tables and compare the answer columns of the truth tables.

Let

$p$: The tire is out of balance.

$q$: The tire is flat.

The given statement may be written "It is not true that the tire is out of balance and the tire is flat." The statement is expressed in symbolic form as $\sim(p \wedge q)$. Using $p$ and $q$ as indicated, choices (a) through (d) may be expressed symbolically as

a) $\sim q \rightarrow \sim p$. b) $\sim p \vee \sim q$. c) $\sim q \wedge \sim p$. d) $\sim p \rightarrow \sim q$.

Now construct a truth table for the given statement (Table 3.24 on page 131) and each possible choice, given in Table 3.25(a) through (d). By examining the truth tables, we see that the given statement, $\sim(p \wedge q)$, is logically equivalent to choice (b), $\sim p \vee \sim q$. Therefore, the correct answer is "The tire is not out of balance or the tire is not flat." This statement is logically equivalent to the statement "It is not true that the tire is both out of balance and flat."

**TABLE 3.24**

| $p$ | $q$ | $\sim$ | $(p$ | $\wedge$ | $q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | T | T | F | F |
| F | T | T | F | F | T |
| F | F | T | F | F | F |
| | | 4 | 1 | 3 | 2 |

**TABLE 3.25**

| $p$ | $q$ | (a) $\sim q$ | (a) $\rightarrow$ | (a) $\sim p$ | (b) $\sim p$ | (b) $\vee$ | (b) $\sim q$ | (c) $\sim q$ | (c) $\wedge$ | (c) $\sim p$ | (d) $\sim p$ | (d) $\rightarrow$ | (d) $\sim q$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T | T | F | T | F | F | F | F | F | F | F | F | T | F |
| T | F | T | F | F | F | T | T | T | F | F | F | T | T |
| F | T | F | T | T | T | T | F | F | F | T | T | F | F |
| F | F | T | T | T | T | T | T | T | T | T | T | T | T |

In the preceding section, we showed that $p \leftrightarrow q$ has the same truth table as $(p \rightarrow q) \wedge (q \rightarrow p)$. Therefore, these statements are equivalent, a useful fact for Example 4.

**EXAMPLE 4** *Write an Equivalent Biconditional Statement*

Write the following statement as an equivalent biconditional statement: "If the tree produces acorns then the tree is an oak and if the tree is an oak then the tree produces acorns."

**SOLUTION:** An equivalent statement is "The tree produces acorns if and only if the tree is an oak."

## De Morgan's Laws

Example 3 showed that a statement of the form $\sim(p \wedge q)$ is equivalent to a statement of the form $\sim p \vee \sim q$. Thus, we may write $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$. This equivalent statement is one of two special laws called De Morgan's laws. The laws, named after Augustus De Morgan, an English mathematician, were first introduced in Section 2.4, where they applied to sets.

**De Morgan's Laws**

1. $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
2. $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

You can demonstrate that De Morgan's second law is true by constructing and comparing truth tables for $\sim(p \vee q)$ and $\sim p \wedge \sim q$. Do so now.

When using De Morgan's laws, if it becomes necessary to negate an already negated statement, use the fact that $\sim(\sim p)$ is equivalent to $p$. For example, the negation of the statement "Today is not Monday" is "Today is Monday."

**EXAMPLE 5** *Use De Morgan's Laws*

Select the statement that is logically equivalent to "The sun is not shining but it is not raining."

a) It is not true that the sun is shining and it is raining.

b) It is not raining or the sun is not shining.

c) The sun is shining or it is raining.

d) It is not true that the sun is shining or it is raining.

# PROFILE IN MATHEMATICS

## CHARLES DODGSON

One of the more interesting and well-known students of logic was Charles Dodgson (1832–1898), better known to us as Lewis Carroll, the author of *Alice's Adventures in Wonderland* and *Through the Looking-Glass*. Although the books have a child's point of view, many argue that the audience best equipped to enjoy them is an adult one. Dodgson, a mathematician, logician, and photographer (among other things), uses the naïveté of a 7-year-old girl to show what can happen when the rules of logic are taken to absurd extremes.

"You should say what you mean," the March Hare went on.

"I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you know."

"Not the same thing a bit!" said the Hatter. "You might as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"

**SOLUTION:** To determine which statement is equivalent, write each statement in symbolic form.

Let

$p$: The sun is shining.
$q$: It is raining.

The statement "The sun is not shining but it is not raining" written symbolically is $\sim p \wedge \sim q$. Recall that the word *but* means the same as the word *and*. Now, write parts (a) through (d) symbolically.

a) $\sim(p \wedge q)$ b) $\sim q \vee \sim p$ c) $p \vee q$ d) $\sim(p \vee q)$

De Morgan's law shows that $\sim p \wedge \sim q$ is equivalent to $\sim(p \vee q)$. Therefore, the answer is (d): "It is not true that the sun is shining or it is raining." ▲

### EXAMPLE 6 *Using De Morgan's Laws to Write an Equivalent Statement*

Write a statement that is logically equivalent to "It is not true that tomatoes are poisonous or eating peppers cures the common cold."

**SOLUTION:** Let

$p$: Tomatoes are poisonous.
$q$: Eating peppers cures the common cold.

The given statement is of the form $\sim(p \vee q)$. Using the second of De Morgan's laws, we see that an equivalent statement in symbols is $\sim p \wedge \sim q$. Therefore, an equivalent statement in words is "Tomatoes are not poisonous and eating peppers does not cure the common cold." ▲

Consider $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$, one of De Morgan's laws. To go from $\sim(p \wedge q)$ to $\sim p \vee \sim q$, we negate both the $p$ and the $q$ within parentheses; change the conjunction, $\wedge$, to a disjunction, $\vee$; and remove the negation symbol preceding the left parentheses and the parentheses themselves. We can use a similar procedure to obtain equivalent statements. For example,

$$\sim(\sim p \wedge q) \Leftrightarrow p \vee \sim q$$
$$\sim(p \wedge \sim q) \Leftrightarrow \sim p \vee q$$

We can use a similar procedure to obtain equivalent statements when a disjunction is within parentheses. Note that

$$\sim(\sim p \vee q) \Leftrightarrow p \wedge \sim q$$
$$\sim(p \vee \sim q) \Leftrightarrow \sim p \wedge q$$

### EXAMPLE 7 *Using De Morgan's Laws to Write an Equivalent Statement*

Use De Morgan's laws to write a statement logically equivalent to "Benjamin Franklin was not a U.S. president, but he signed the Declaration of Independence."

## PROFILE IN MATHEMATICS

### AUGUSTUS DE MORGAN

Augustus De Morgan (1806–1871), the son of a member of the East India Company, was born in India and educated at Trinity College, Cambridge. One of the great reformers of logic in the nineteenth century, De Morgan made his greatest contribution to the subject by realizing that logic as it had come down from Aristotle was narrow in scope and could be applied to a wider range of arguments. His work laid the foundation for modern, symbolic logic.

**SOLUTION:** Let

$p$: Benjamin Franklin was a U.S. president.
$q$: Benjamin Franklin signed the Declaration of Independence.

The statement written symbolically is $\sim p \wedge q$. Earlier we showed that

$$\sim p \wedge q \Leftrightarrow \sim(p \vee \sim q)$$

Therefore, the statement "It is false that Benjamin Franklin was a U.S. president or Benjamin Franklin did not sign the Declaration of Independence" is logically equivalent to the given statement. ▲

There are strong similarities between the topics of sets and logic. We can see them by examining De Morgan's laws for sets and logic.

| **De Morgan's laws: set theory** | **De Morgan's laws: logic** |
|---|---|
| $(A \cap B)' = A' \cup B'$ | $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ |
| $(A \cup B)' = A' \cap B'$ | $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ |

The complement in set theory, $'$, is similar to the negation, $\sim$, in logic. The intersection, $\cap$, is similar to the conjunction, $\wedge$; and the union, $\cup$, is similar to the disjunction, $\vee$. If we were to interchange the set symbols with the logic symbols, De Morgan's laws would remain, but in a different form.

Both $'$ and $\sim$ can be interpreted as *not*.
Both $\cap$ and $\wedge$ can be interpreted as *and*.
Both $\cup$ and $\vee$ can be interpreted as *or*.

For example, the set statement $A' \cup B$ can be written as a statement in logic as $\sim a \vee b$.

Statements containing connectives other than *and* and *or* may have equivalent statements. To illustrate this point, construct truth tables for $p \to q$ and for $\sim p \vee q$. The truth tables will have the same answer columns and therefore the statements are equivalent. That is,

$$\boxed{p \to q \Leftrightarrow \sim p \vee q}$$

With these equivalent statements, we can write a conditional statement as a disjunction or a disjunction as a conditional statement. For example, the statement "If the game is polo, then you ride a horse" can be equivalently stated as "The game is not polo or you ride a horse."

To change a conditional statement to a disjunction, negate the antecedent, change the conditional symbol to a disjunction symbol, and keep the consequent the same. To change a disjunction statement to a conditional statement, negate the first statement, change the disjunction symbol to a conditional symbol, and keep the second statement the same.

**EXAMPLE 8** *Rewriting a Disjunction as a Conditional Statement*

Write a conditional statement that is logically equivalent to "The Oregon Ducks will win or the Oregon State Beavers will lose." Assume that the negation of winning is losing.

**SOLUTION:** Let

$p$: The Oregon Ducks will win.
$q$: The Oregon State Beavers will win.

The original statement may be written symbolically as $p \vee \sim q$. To write an equivalent statement, negate the first statement, $p$, change the disjunction symbol to a conditional symbol, and keep the second statement the same. Symbolically, the equivalent statement is $\sim p \rightarrow \sim q$. The equivalent statement in words is "If the Oregon Ducks lose, then the Oregon State Beavers will lose."

## Negation of the Conditional Statement

Now we will discuss how to negate a conditional statement. To negate a statement we use the fact that $p \rightarrow q \Leftrightarrow \sim p \vee q$ and De Morgan's laws. Examples 9 and 10 show the process.

**EXAMPLE 9** *The Negation of a Conditional Statement*

Determine a statement equivalent to $\sim(p \rightarrow q)$.

**SOLUTION:** Begin with $p \rightarrow q \Leftrightarrow \sim p \vee q$, negate both statements, and use De Morgan's laws.

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$
$$\sim(p \rightarrow q) \Leftrightarrow \sim(\sim p \vee q) \quad \text{Negate both statements}$$
$$\Leftrightarrow p \wedge \sim q \quad \text{De Morgan's laws}$$

Therefore, $\sim(p \rightarrow q)$ is equivalent to $p \wedge \sim q$.

**EXAMPLE 10** *Write an Equivalent Statement*

Write a statement equivalent to

"It is false that if the dog is snoring then the dog cannot sleep in our bedroom."

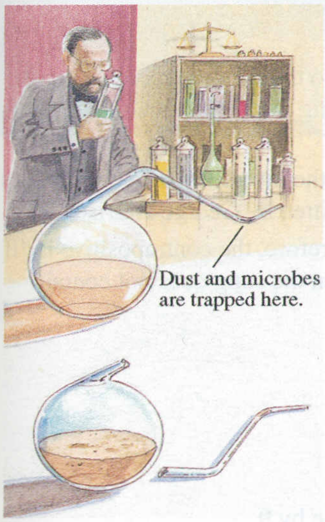
**SOLUTION:** Let

$p$ = the dog is snoring
$q$ = the dog can sleep in our room

Then the given statement can be represented symbolically as $\sim(p \rightarrow \sim q)$. Using the procedure illustrated in Example 9 we can determine that $\sim(p \rightarrow \sim q)$ is equivalent to $p \wedge q$. Verify this yourself now. Therefore, an equivalent statement is "The dog is snoring and the dog can sleep in our bedroom."

## DID YOU KNOW

### In the Service of Science




Logical reasoning is often used by scientists as they develop theories. Logical reasoning often also plays a part in developing procedures to test theories. Sometimes, their theories are supported by experimental evidence and are accepted. Other times, their theories are contradicted by the evidence and are rejected. For example, Aristotle observed that bugs appeared in spoiling meat and reasoned that life arose spontaneously from nonliving matter. With advances in technology, scientists have had more means at their disposal to test their theories. To refute Aristotle's claim of spontaneous generation of life, Louis Pasteur in 1862 conducted an experiment by isolating some meat broth in a sterile flask to demonstrate that the bugs Aristotle observed grew from microscopic life forms too small to be seen.

## Variations of the Conditional Statement

We know that $p \rightarrow q$ is equivalent to $\sim p \vee q$. Are any other statements equivalent to $p \rightarrow q$? Yes, there are many. Now let's look at the variations of the conditional statement to determine whether any are equivalent to the conditional statement. *The variations of the conditional statement are made by switching and/or negating the antecedent and the consequent of a conditional statement.* The variations of the conditional statement are the ***converse*** of the conditional, the ***inverse*** of the conditional, and the ***contrapositive*** of the conditional.

Listed here are the variations of the conditional with their symbolic form and the words we say to read each one.

**Variations of the Conditional Statement**

| Name | Symbolic form | Read |
|---|---|---|
| Conditional | $p \rightarrow q$ | "If $p$, then $q$" |
| Converse of the conditional | $q \rightarrow p$ | "If $q$, then $p$" |
| Inverse of the conditional | $\sim p \rightarrow \sim q$ | "If not $p$, then not $q$" |
| Contrapositive of the conditional | $\sim q \rightarrow \sim p$ | "If not $q$, then not $p$" |

To write the converse of the conditional statement, switch the order of the antecedent and the consequent. To write the inverse, negate both the antecedent and the consequent. To write the contrapositive, switch the order of the antecedent and the consequent and then negate both of them.

Are any of the variations of the conditional statement equivalent? To determine the answer, we can construct a truth table for each variation, as shown in Table 3.26. It reveals that the conditional statement is equivalent to the contrapositive statement and that the converse statement is equivalent to the inverse statement.

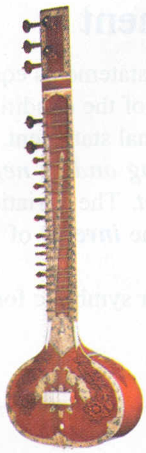
**TABLE 3.26**

| p | q | Conditional $p \rightarrow q$ | Contrapositive $\sim q \rightarrow \sim p$ | Converse $q \rightarrow p$ | Inverse $\sim p \rightarrow \sim q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | T |
| F | T | T | T | F | F |
| F | F | T | T | T | T |

**EXAMPLE 11** *The Converse, Inverse and Contrapositive*

For the conditional statement "If the song contains sitar music, then the song was written by George Harrison," write the

a) converse. b) inverse. c) contrapositive.



A sitar

**SOLUTION:**

a) Let

$p$: The song contains sitar music.

$q$: The song was written by George Harrison.

The conditional statement is of the form $p \rightarrow q$, so the converse must be of the form $q \rightarrow p$. Therefore, the converse is "If the song was written by George Harrison, then the song contains sitar music."

b) The inverse is of the form $\sim p \rightarrow \sim q$. Therefore, the inverse is "If the song does not contain sitar music, then the song was not written by George Harrison."

c) The contrapositive is of the form $\sim q \rightarrow \sim p$. Therefore, the contrapositive is "If the song was not written by George Harrison, then the song does not contain sitar music."

## EXAMPLE 12 *Determine the Truth Values*

Let

$p$: The number is divisible by 9.

$q$: The number is divisible by 3.

Write the following statements and determine which are true.

a) The conditional statement, $p \rightarrow q$

b) The converse of $p \rightarrow q$

c) The inverse of $p \rightarrow q$

d) The contrapositive of $p \rightarrow q$

**SOLUTION:**

a) *Conditional statement:* $(p \rightarrow q)$

If the number is divisible by 9, then the number is divisible by 3. This statement is true. A number divisible by 9 must also be divisible by 3, since 3 is a divisor of 9.

b) *Converse of the conditional:* $(q \rightarrow p)$

If the number is divisible by 3, then the number is divisible by 9. This statement is false. For instance, 6 is divisible by 3, but 6 is not divisible by 9.

c) *Inverse of the conditional:* $(\sim p \rightarrow \sim q)$

If the number is not divisible by 9, then the number is not divisible by 3. This statement is false. For instance, 6 is not divisible by 9, but 6 is divisible by 3.

d) *Contrapositive of the conditional:* $(\sim q \rightarrow \sim p)$

If the number is not divisible by 3, then the number is not divisible by 9. The statement is true, since any number that is divisible by 9 must be divisible by 3.

## EXAMPLE 13 *Use the Contrapositive*

Use the contrapositive to write a statement logically equivalent to "If the boat is 24 ft long, then it will not fit into the boathouse."


**SOLUTION:** Let

$p$: The boat is 24 ft long.
$q$: The boat will fit into the boathouse.

The given statement written symbolically is

$$p \to \sim q$$

The contrapositive of the statement is

$$q \to \sim p$$

Therefore, an equivalent statement is "If the boat will fit into the boathouse, then the boat is not 24 ft long."

The contrapositive of the conditional is very important in mathematics. Consider the statement "If $a^2$ is not a whole number, then $a$ is not a whole number." Is this statement true? You may find this question difficult to answer. Writing the statement's contrapositive may enable you to answer the question. The contrapositive is "If $a$ is a whole number, then $a^2$ is a whole number." Since the contrapositive is a true statement, the original statement must also be true.

## EXAMPLE 14 *Which Are Equivalent?*

Determine which, if any, of the following statements are equivalent. You may use De Morgan's laws, the fact that $p \to q \Leftrightarrow \sim p \vee q$, information from the variations of the conditional, or truth tables.

a) If you leave by 9 A.M., then you will get to your destination on time.
b) You do not leave by 9 A.M. or you will get to your destination on time.
c) It is false that you get to your destination on time or you did not leave by 9 A.M.
d) If you do not get to your destination on time, then you did not leave by 9 A.M.

**SOLUTION:** Let

$p$: You leave by 9 A.M.
$q$: You will get to your destination on time.

In symbolic form, the four statements are

a) $p \to q$. b) $\sim p \vee q$. c) $\sim(q \vee \sim p)$. d) $\sim q \to \sim p$.

Which of these statements are equivalent? Earlier in this section, you learned that $p \to q$ is equivalent to $\sim p \vee q$. Therefore, statements (a) and (b) are equivalent. Statement (d) is the contrapositive of statement (a). Therefore, statement (d) is also equivalent to statement (a) and statement (b). All these statements have the same truth table (Table 3.27 on page 138).

Now let's look at statement (c). If we use De Morgan's laws on statement (c), we get

$$\sim(q \vee \sim p) \Leftrightarrow \sim q \wedge p$$

**TABLE 3.27**

| | | (a) | (b) | (d) |
|---|---|---|---|---|
| p | q | $p \to q$ | $\sim p \vee q$ | $\sim q \to \sim p$ |
| T | T | T | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

If $\sim q \wedge p$ was one of the other statements, then $\sim(q \vee \sim p)$ would be equivalent to that statement. Because $\sim q \wedge p$ does not match any of the other choices, it does not necessarily mean that $\sim(q \vee \sim p)$ is not equivalent to the other statements. To determine whether $\sim(q \vee \sim p)$ is equivalent to the other statements, we will construct its truth table (Table 3.28) and compare the answer column with the answer columns in Table 3.27.

**TABLE 3.28** (c)

| p | q | ~ | (q | ∨ | ~p) |
|---|---|---|---|---|---|
| T | T | F | T | T | F |
| T | F | T | F | F | F |
| F | T | F | T | T | T |
| F | F | F | F | T | T |
| | | 4 | 1 | 3 | 2 |

None of the three answer columns of the truth table in Table 3.27 are the same as the answer column of the truth table in Table 3.28. Therefore $\sim(q \vee \sim p)$ is not equivalent to any of the other statements. Therefore, statements (a), (b), and (d) are equivalent to each other. ▲

## DID YOU KNOW

### *Fuzzy Logic*

Modern computers, like truth tables, work with only two values, 1 or 0 (equivalent to true or false in truth tables). This constraint prevents a computer from being able to reason as the human brain can and prevents a computer from being able to evaluate items involving vagueness or value judgments that so often occur in real-world situations. For example, a binary computer will have difficulty evaluating the subjective statement "the air is warm."

*Fuzzy logic* uses the concept: Everything is a matter of degree. Fuzzy logic manipulates vague concepts such as *bright* and *fast* by assigning values between 0 and 1 to each item. For example, suppose *bright* is assigned a value of 0.80; then *not bright* is assigned a value of $1 - 0.8 = 0.20$. As the value assigned to *bright* changes, so does the value assigned to *not bright*. Not $p$ is always $1 - p$, where $0 < p < 1$. Fuzzy logic is used to operate cameras, air conditioners, subways, and many other devices where the change in one condition changes another condition. For example, when it is bright outside, the camera's lens aperture opens less, and when it is overcast, the camera's lens aperture opens more. How many other devices can you name that may use fuzzy logic? See Problem-Solving Exercises 87 and 88.

# SECTION 3.4 EXERCISES

## Concept/Writing Exercises

1. What are equivalent statements?
2. Explain how you can determine whether two statements are equivalent.
3. Suppose two statements are connected with the biconditional and the truth table is constructed. If the answer column of the truth table has all trues, what must be true about these two statements? Explain.
4. Write De Morgan's laws for logic.
5. For a statement of the form $p \rightarrow q$, symbolically indicate the form of the
   a) converse.
   b) inverse.
   c) contrapositive.
6. Which of the following are equivalent statements?
   a) The converse
   b) The contrapositive
   c) The inverse
   d) The conditional
7. Write a disjunctive statement that is logically equivalent to $p \rightarrow q$.
8. Write a conjunction involving two conditional statements that is logically equivalent to $p \leftrightarrow q$.

## Practice the Skills

*In Exercises 9–18, use De Morgan's laws to determine whether the two statements are equivalent.*

9. $\sim p \vee \sim q, \sim(p \wedge q)$
10. $\sim(p \vee q), \sim p \wedge \sim q$
11. $\sim(p \wedge q), \sim p \wedge \sim q$
12. $\sim(p \wedge q), \sim p \wedge q$
13. $\sim(p \vee \sim q), \sim p \wedge q$
14. $\sim(p \wedge q), \sim(q \vee \sim p)$
15. $(\sim p \vee \sim q) \rightarrow r, \sim(p \wedge q) \rightarrow r$
16. $q \rightarrow \sim(p \wedge \sim r), q \rightarrow \sim p \vee r$
17. $\sim(p \rightarrow \sim q), p \wedge q$
18. $\sim(\sim p \rightarrow q), \sim p \wedge \sim q$

*In Exercises 19–30, use a truth table to determine whether the two statements are equivalent.*

19. $p \rightarrow q, \sim p \vee q$
20. $\sim p \rightarrow q, p \wedge q$
21. $(p \wedge q) \wedge r, p \wedge (q \wedge r)$
22. $p \rightarrow q, \sim q \rightarrow \sim p$
23. $(p \vee q) \vee r, p \vee (q \vee r)$
24. $p \vee (q \wedge r), \sim p \rightarrow (q \wedge r)$
25. $p \wedge (q \vee r), (p \wedge q) \vee r$
26. $\sim(q \rightarrow p) \vee r, (p \vee q) \wedge \sim r$
27. $(p \rightarrow q) \wedge (q \rightarrow r), (p \rightarrow q) \rightarrow r$
28. $\sim q \rightarrow (p \wedge r), \sim(p \vee r) \rightarrow q$
29. $(p \rightarrow q) \wedge (q \rightarrow p), (p \leftrightarrow q)$
30. $[\sim(p \rightarrow q)] \wedge [\sim(q \rightarrow p)], \sim(p \leftrightarrow q)$

## Problem Solving

*In Exercises 31–38, use De Morgan's laws to write an equivalent statement for the sentence.*

31. It is false that the Mississippi River runs through Ohio or the Ohio River runs through Mississippi.
32. It is false that the printer is out of toner and the fax machine is out of paper.
33. The snowmobile was neither an Arctic Cat nor was it a Ski-Do.

34. The pot roast is hot, but it is not well done.
35. The hotel does not have a weight room or the conference center does not have an auditorium.
36. Robert Farinelli is an authorized WedgCor dealer or he is not going to work for Prism Construction Company.
37. If we go to Cozumel, then we will go snorkeling or we will not go to Senior Frogs.

38. If Phil Murphy buys us dinner, then we will not go to the top of the CN Tower but we will be able to walk to the Red Bistro Restaurant.

*In Exercises 39–44, use the fact that $p \rightarrow q$ is equivalent to $\sim p \vee q$ to write an equivalent form of the given statement.*

39. If you drink a glass of orange juice, then you will get a full day supply of folic acid.
40. Nick-at-Nite is showing *Family Ties* or they are showing *The Facts of Life*.
41. Bob the Tomato visited the nursing home or he did not visit the Cub Scout meeting.
42. If John Peden will buy a Harley-Davidson, then he will not buy a Honda.


43. It is false that if the plumbers meet in Kansas City then the Rainmakers will provide the entertainment.
44. Mary Beth Headlee organized the conference or John Waters does not work at Sinclair Community College.

*In Exercises 45–48, use the fact that $(p \rightarrow q) \wedge (q \rightarrow p)$ is equivalent to $p \leftrightarrow q$ to write the statement in an equivalent form.*

45. If it is cloudy then the front is coming through, and if the front is coming through then it is cloudy.
46. If Model Road is closed then we use Kirkwood Road, and if we use Kirkwood Road then Model Road is closed.
47. The chemistry teacher teaches mathematics if and only if there is a shortage of mathematics teachers.
48. John Deere will hire new workers if and only if the city of Dubuque will pay to retrain the workers.

*In Exercises 49–56, write the converse, inverse, and contrapositive of the statement. (For Exercises 55 and 56, use De Morgan's laws.)*

49. If the book is interesting, then I will finish the book in 1 week.
50. If the dryer is making a loud noise, then you need to replace the blower fan.
51. If you finish your homework, then you can watch television.
52. If Bob Dylan records a new CD, then he will go on tour.
53. If that annoying paper clip shows up on my computer screen, then I will scream.
54. If the remote control is not within my reach, then I will watch the same channel all night.
55. If the sun is shining, then we will go down to the marina and we will take out the sailboat.

56. If the apple pie is baked, then we will eat a piece of pie and we will save some pie for later.

*In Exercises 57–64, write the contrapositive of the statement. Use the contrapositive to determine whether the conditional statement is true or false.*

57. If a natural number is not divisible by 5, then the natural number is not divisible by 10.
58. If the opposite sides of the quadrilateral are not parallel, then the quadrilateral is not a parallelogram.
59. If a natural number is divisible by 3, then the natural number is divisible by 6.
60. If $1/n$ is not a natural number, then $n$ is not a natural number.
61. If two lines do not intersect in at least one point, then the two lines are parallel.
62. If $\frac{m \cdot a}{m \cdot b} \neq \frac{a}{b}$, then $m$ is not a counting number.
63. If the sum of the interior angles of a polygon do not measure 360°, then the polygon is not a quadrilateral.
64. If $a$ and $b$ are not both even counting numbers, then the product of $a$ and $b$ is not an even counting number.

*In Exercises 65–80, determine which, if any, of the three statements are equivalent (see Example 14).*

**65.** a) Maria has not retired or Maria is still working.
b) If Maria is still working, then Maria has not retired.
c) If Maria has retired, then Maria is not still working.

**66.** a) If today is Monday, then tomorrow is not Wednesday.
b) It is false that today is Monday and tomorrow is not Wednesday.
c) Today is not Monday or tomorrow is Wednesday.

**67.** a) The car is not reliable and the car is noisy.
b) If the car is not reliable, then the car is not noisy.
c) It is false that the car is reliable or the car is not noisy.

**68.** a) The house is not made of wood or the shed is not made of wood.
b) If the house is made of wood, then the shed is not made of wood.
c) It is false that the shed is made of wood and the house is not made of wood.

**69.** a) Today is not Sunday or the library is open.
b) If today is Sunday, then the library is not open.
c) If the library is open, then today is not Sunday.

**70.** a) If you are fishing at 1 P.M., then you are driving a car at 1 P.M.
b) You are not fishing at 1 P.M. or you are driving a car at 1 P.M.
c) It is false that you are fishing at 1 P.M. and you are not driving a car at 1 P.M.

**71.** a) The grass grows and the trees are blooming.
b) If the trees are blooming, then the grass does not grow.
c) The trees are not blooming or the grass does not grow.

**72.** a) Johnny Patrick is chosen as department chair if and only if he is the only candidate.
b) If Johnny Patrick is chosen as department chair then he is the only candidate, and if Johnny Patrick is the only candidate then he is chosen as department chair.
c) Johnny Patrick is not chosen as department chair and he is not the only candidate.

**73.** a) It is false that if you do not drink milk then your cholesterol count will be lower.
b) Your cholesterol count will be lower if and only if you drink milk.
c) It is false that if you drink milk then your cholesterol count will not be lower.

**74.** a) Bruce Springsteen will not go on tour if and only if Clarence Clemmons does not play the saxophone in his band.
b) It is false that Bruce Springsteen will go on tour if and only if Clarence Clemmons does not play the saxophone in his band.
c) If Bruce Springsteen goes on tour, then Clarence Clemmons plays saxophone in his band.

Clarence Clemmons (left) and Bruce Springsteen

**75.** a) If the pay is good and today is Monday, then I will take the job.
b) If I do not take the job, then it is false that the pay is good or today is Monday.
c) The pay is good and today is Monday, or I will take the job.

**76.** a) If you are 18 years old and a citizen of the United States, then you can vote in the presidential election.
b) You can vote in the presidential election, if and only if you are a citizen of the United States and you are 18 years old.
c) You cannot vote in the presidential election, or you are 18 years old and you are not a citizen of the United States.

**77.** a) The package was sent by Federal Express, or the package was not sent by United Parcel Service but the package arrived on time.
b) The package arrived on time, if and only if it was sent by Federal Express or it was not sent by United Parcel Service.
c) If the package was not sent by Federal Express, then the package was not sent by United Parcel Service but the package arrived on time.

**78.** a) If we put the dog outside or we feed the dog, then the dog will not bark.
b) If the dog barks, then we did not put the dog outside and we did not feed the dog.
c) If the dog barks, then it is false that we put the dog outside or we feed the dog.

**79.** a) The car needs oil, and the car needs gas or the car is new.
b) The car needs oil, and it is false that the car does not need gas and the car is not new.
c) If the car needs oil, then the car needs gas or the car is not new.

**80.** a) The mortgage rate went down, if and only if Tim purchased the house and the down payment was 10%.
b) The down payment was 10%, and if Tim purchased the house then the mortgage rate went down.
c) If Tim purchased the house, then the mortgage rate went down and the down payment was not 10%.

81. If $p$ and $q$ represent two simple statements, and if $p \rightarrow q$ is a false statement, what must be the truth value of the converse, $q \rightarrow p$? Explain.

82. If $p$ and $q$ represent two simple statements, and if $p \rightarrow q$ is a false statement, what must be the truth value of the inverse, $\sim p \rightarrow \sim q$? Explain.

83. If $p$ and $q$ represent two simple statements, and if $p \rightarrow q$ is a false statement, what must be the truth value of the contrapositive, $\sim q \rightarrow \sim p$? Explain.

84. If $p$ and $q$ represent two simple statements, and if $p \rightarrow q$ is a true statement, what must be the truth value of the contrapositive, $\sim q \rightarrow \sim p$? Explain.

## Challenge Problems/Group Activities

85. We learned that $p \rightarrow q \Leftrightarrow \sim p \vee q$. Determine a conjunctive statement that is equivalent to $p \rightarrow q$. (*Hint:* There are many answers.)

86. Determine whether $\sim[\sim(p \vee \sim q)] \Leftrightarrow p \vee \sim q$. Explain the method(s) you used to determine your answer.

87. In an appliance or device that uses fuzzy logic, a change in one condition causes a change in a second condition. For example, in a camera, if the brightness increases, the lens aperture automatically decreases to get the proper exposure on the film. Name at least 10 appliances or devices that make use of fuzzy logic and explain how fuzzy logic is used in each appliance or device. See the Did You Know on page 138.


88. In symbolic logic, a statement is either true or false (consider true to have a value of 1 and false a value of 0). In fuzzy logic, nothing is true or false, but everything is a matter of degree. For example, consider the statement "The sun is shining." In fuzzy logic, this statement may have a value between 0 and 1 and may be constantly changing. For example, if the sun is partially blocked by clouds, the value of this statement may be 0.25. In fuzzy logic, the values of connective statements are found as follows for statements $p$ and $q$.

Not $p$ has a truth value of $1 - p$.
$p \wedge q$ has a truth value equal to the lesser of $p$ and $q$.
$p \vee q$ has a truth value equal to the greater of $p$ and $q$.
$p \rightarrow q$ has a truth value equal to the lesser of 1 and $1 - p + q$.
$p \leftrightarrow q$ has a truth value equal to $1 - |p - q|$, that is, 1 minus the absolute value* of $p$ minus $q$.

Suppose the statement "$p$: The sun is shining" has a truth value of 0.25 and the statement "$q$: Mary is getting a tan" has a truth value of 0.20. Find the truth value of

a) $\sim p$. b) $\sim q$.
c) $p \wedge q$. d) $p \vee q$.
e) $p \rightarrow q$. f) $p \leftrightarrow q$.

## Recreational Mathematics

89. Unscramble the following letters to form the names of five important terms in the study of logic.
a) ACINNLIDOOT
b) DATOONCLIBINI
c) RIENSEV
d) ROCESVEN
e) ARTSOCINVOTEPI

## Internet/Research Activities

90. Do research and write a report on fuzzy logic.

91. Read one of Lewis Carroll's books and write a report on how he used logic in the book. Give at least five specific examples.

92. Do research and write a report on the life and achievements of Augustus De Morgan. Indicate in your report his contributions to sets and logic.

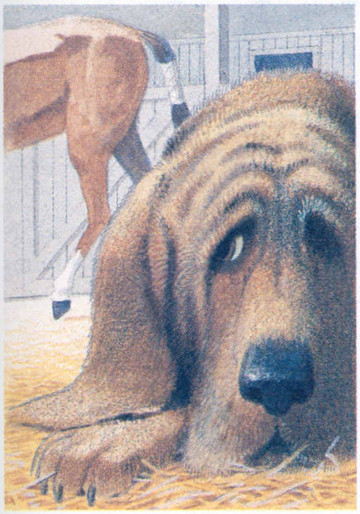
# 3.5 SYMBOLIC ARGUMENTS

In the preceding sections of this chapter, we used symbolic logic to determine the truth value of a compound statement. We now extend those basic ideas to determine whether symbolic arguments are valid or invalid.

*Absolute values are discussed in Section 13.8.

## DID YOU KNOW

### *The Dog Did Nothing*



In the case of the disappearance of the racehorse Silver Blaze, Sherlock Holmes demonstrated that sometimes the absence of a clue is itself a clue. The local police inspector asked him, "Is there any point to which you would wish to draw my attention?" Holmes replied, "To the curious incident of the dog in the nighttime." The inspector, confused, asked: "The dog did nothing in the nighttime." "That was the curious incident," remarked Sherlock Holmes. From the lack of the dog's bark, Holmes concluded that the horse had been "stolen" by a stablehand. How did Holmes reach his conclusion?

Consider the statements:

If Jason is a singer, then he is well known.

Jason is a singer.

If you accept these two statements as true, then a conclusion that necessarily follows is that

Jason is well known.

These three statements in the following form constitute a symbolic argument.

Premise 1: If Jason is a singer, then he is well known.
Premise 2: Jason is a singer.
Conclusion: Therefore, Jason is well known.

A ***symbolic argument*** consists of a set of ***premises*** and a ***conclusion***. It is called a symbolic argument because we generally write it in symbolic form to determine its validity.

> An **argument is valid** when its conclusion necessarily follows from a given set of premises.
>
> An **argument is invalid** or a **fallacy** when the conclusion does not necessarily follow from the given set of premises.

An argument that is not valid is invalid. The argument just presented is an example of a valid argument, as the conclusion necessarily follows from the premises. Now we will discuss a procedure to determine whether an argument is valid or invalid. We begin by writing the argument in symbolic form. To write the argument in symbolic form, we let $p$ and $q$ be

$p$: Jason is a singer.
$q$: Jason is well known.

Symbolically, the argument is written

Premise 1: $p \to q$
Premise 2: $p$
Conclusion: $\therefore q$ (The three-dot triangle is read "therefore.")

Write the argument in the following form.

If [*premise 1* **and** *premise 2*] **then** *conclusion*

$$[(p \to q) \quad \wedge \quad p \quad] \quad \to \quad q$$

Then construct a truth table for the statement $[(p \to q) \wedge p] \to q$ (Table 3.29 on page 144). *If the truth table answer column is true in every case, then the statement is a tautology, and the argument is valid. If the truth table is not a tautology, then the argument is invalid.* Since the statement is a tautology (see column 5), the conclusion necessarily follows from the premises and the argument is valid.

**TABLE 3.29**

| p | q | [(p → q) | ∧ | p] | → | q |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | F | F | T | T | F |
| F | T | T | F | F | T | T |
| F | F | T | F | F | T | F |
| | | 1 | 3 | 2 | 5 | 4 |

Once we have demonstrated that an argument in a particular form is valid, all arguments with exactly the same form will also be valid. In fact, many of these forms have been assigned names. The argument form just discussed,

$$\begin{array}{l} p \rightarrow q \\ \underline{p} \\ \therefore q \end{array}$$

is called the *law of detachment*, or *modus ponens*.

## EXAMPLE 1 *Determining Validity without a Truth Table*

Determine whether the following argument is valid or invalid.

If the water is warm, then the moon is made of cheese.
The water is warm.
∴ The moon is made of cheese.

**SOLUTION:** Translate the argument into symbolic form.

Let

*w*: The water is warm.
*m*: The moon is made of cheese.

In symbolic form the argument is

$$\begin{array}{l} w \rightarrow m \\ \underline{w} \\ \therefore m \end{array}$$

This argument is also the law of detachment, and therefore it is a valid argument. ▲

Note that the argument in Example 1 is valid even though the conclusion, "The moon is made of cheese," is a false statement. It is also possible to have an invalid argument in which the conclusion is a true statement. When an argument is valid, the conclusion necessarily follows from the premises. It is not necessary for the premises or the conclusion to be true statements in an argument.

### Procedure to Determine Whether an Argument Is Valid

1. Write the argument in symbolic form.
2. Compare the form of the argument with forms that are known to be valid or invalid. If there are no known forms to compare it with, or you do not remember the forms, go to step 3.
3. If the argument contains two premises, write a conditional statement of the form
$$[(\text{premise 1}) \wedge (\text{premise 2})] \to \text{conclusion}$$
4. Construct a truth table for the statement in step 3.
5. If the answer column of the truth table has all trues, the statement is a tautology, and the argument is valid. If the answer column does not have all trues, the argument is invalid.

Examples 1 through 4 contain two premises. When an argument contains more that two premises, step 3 of the procedure will change slightly, as will be explained shortly.

## EXAMPLE 2 *Determining Validity with a Truth Table*

Determine whether the following argument is valid or invalid.

If you score 90% on the final exam, then you will get an A in the course.
You will not get an A in the course.
∴ You do not score 90% on the final exam.

**SOLUTION:** We first write the argument in symbolic form.

Let

$p$: You score 90% on the final exam.
$q$: You will get an A in the course.

In symbolic form, the argument is

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

As we have not tested an argument in this form, we will construct a truth table to determine whether the argument is valid or invalid. We write the argument in the form $[(p \to q) \wedge \sim q] \to \sim p$, and construct a truth table (Table 3.30 on page 146). Since the answer, column 5, has all T's, the argument is valid.

## DID YOU KNOW

### *Guilty as Charged?*


In U.S. law, a jury must reach unanimous agreement about a verdict. Generally, less than 5% of the 120,000 jury trials conducted each year in the United States result in a "hung jury" in which a jury cannot reach a unanimous agreement. Juries use logical reasoning to determine if there are contradictions in the evidence or testimony when deciding their verdict. Most nations of the world rely solely on a judge's opinion to decide guilt or innocence.

**TABLE 3.30**

| p | q | [(p → q) | ∧ | ~q] | → | ~p |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | F |
| T | F | F | F | T | T | F |
| F | T | T | F | F | T | T |
| F | F | T | T | T | T | T |
| | | 1 | 3 | 2 | 5 | 4 |

The argument form in Example 2 is an example of the *law of contraposition*, or *modus tollens*.

## EXAMPLE 3 *Another Symbolic Argument*

Determine whether the following argument is valid or invalid.

The grass is green or the grass is full of weeds.
The grass is not green.
∴ The grass is full of weeds.

**SOLUTION:** Let

$p$: The grass is green.
$q$: The grass is full of weeds.

In symbolic form, the argument is

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore q \end{array}$$

As this form is not one of those we are familiar with, we will construct a truth table. We write the argument in the form $[(p \vee q) \wedge \sim p] \rightarrow q$. Next we construct a truth table, as shown in Table 3.31. The answer to the truth table, column 5, is true in *every case*. Therefore, the statement is a tautology, and the argument is valid.

**TABLE 3.31**

| p | q | [(p ∨ q) | ∧ | ~p] | → | q |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | F | T | F | F | T | F |
| F | T | T | T | T | T | T |
| F | F | F | F | T | T | F |
| | | 1 | 3 | 2 | 5 | 4 |

The argument form in Example 3 is an example of *disjunctive syllogism*. Other standard forms of arguments are given in the following chart.

## Standard Forms of Arguments

| **Valid Arguments** | *Law of Detachment* | *Law of Contraposition* | *Law of Syllogism* | *Disjunctive Syllogism* |
|---|---|---|---|---|
| | $p \to q$ | $p \to q$ | $p \to q$ | $p \vee q$ |
| | $p$ | $\sim q$ | $q \to r$ | $\sim p$ |
| | $\therefore q$ | $\therefore \sim p$ | $\therefore p \to r$ | $\therefore q$ |

| **Invalid Arguments** | *Fallacy of the Converse* | *Fallacy of the Inverse* |
|---|---|---|
| | $p \to q$ | $p \to q$ |
| | $q$ | $\sim p$ |
| | $\therefore p$ | $\therefore \sim q$ |

As we saw in Example 1, it is not always necessary to construct a truth table to determine whether or not an argument is valid. The next two examples will show how we can identify an argument as one of the standard arguments given in the chart above.

### EXAMPLE 4 *Identifying the Law of Syllogism in an Argument*

Determine whether the following argument is valid or invalid.

If my laptop battery is dead, then I use my home computer.
If I use my home computer, then my kids will play outside.
$\therefore$ If my laptop battery is dead, then my kids will play outside.

**SOLUTION:** Let

$p$: My laptop battery is dead.
$q$: I use my home computer.
$r$: My kids will play outside.

In symbolic form, the argument is

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

The argument is in the form of the law of syllogism. Therefore, the argument is valid, and there is no need to construct a truth table. ▲

## EXAMPLE 5 *Identifying Common Fallacies in Arguments*

Determine whether the following arguments are valid or invalid.

a)

If it is snowing, then we put salt on the driveway.
We put salt on the driveway.
∴ It is snowing.

b)

If it is snowing, then we put salt on the driveway.
It is not snowing.
∴ We do not put salt on the driveway.

**SOLUTION:**

a) Let

$p$: It is snowing.
$q$: We put salt on the driveway.

In symbolic form, the argument is

$$\begin{array}{l} p \rightarrow q \\ \underline{q} \\ \therefore p \end{array}$$

This argument is in the form of the fallacy of the converse. Therefore, the argument is a fallacy, or invalid.

b) Using the same symbols defined in the solution to part (a), in symbolic form, the argument is

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim p} \\ \therefore \sim q \end{array}$$

This argument is in the form of the fallacy of the inverse. Therefore, the argument is a fallacy, or invalid. ▲

**TIMELY TIP** If you are not sure whether an argument with two premises is one of the standard forms or if you do not remember the standard forms, you can determine whether a given argument is valid or invalid by using a truth table. To do so, follow the boxed procedure on page 145.

In Example 5b) if you did not recognize that this argument was of the same form as the Fallacy of the Inverse you could construct the truth table for the conditional statement

$$[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$$

The true-false values under the conditional column, →, would be T, T, F, T. Since the statement is not a tautology, the argument is invalid.

### MATHEMATICS *Everywhere*

#### *Freedom of Speech or Misleading Advertising?*

The first amendment to the United States Constitution guarantees that Americans have the freedom of speech. There are, however, limits to what we may say or write. For example, companies are limited in what they can say or write to advertise their products. The Federal Trade Commission (FTC) states that "ads must be truthful and not misleading; that advertisers must have evidence to back up their assertions; and that ads cannot be unfair." Although most advertisements are truthful and fair, some enter into a "gray area" of truthfulness. Some of these breeches of fairness may be found in logical fallacies either made directly or implicitly by the context of the ad's wording or artwork. One notable FTC action that began in June 2001 was Operation Cure All. This action targeted companies marketing a variety of devices, herbal products, and other dietary supplements to treat or cure cancer, HIV/AIDS, arthritis, hepatitis, Alzheimer's disease, diabetes, and many other diseases. Among the many products for which unfounded claims were being made were a DHEA hormonal supplement, St. John's wort, various multiherbal supplements, colloidal silver, and a variety of electrical therapy devices.

Now we consider an argument that has more than two premises. When an argument contains more than two premises, the statement we test, using a truth table, is formed by taking the conjunction of all the premises as the antecedent and the conclusion as the consequent. For example, if an argument is of the form

$$\begin{array}{l} p_1 \\ p_2 \\ \underline{p_3} \\ \therefore c \end{array}$$

We evaluate the truth table for $[p_1 \wedge p_2 \wedge p_3] \rightarrow c$. When we evaluate $[p_1 \wedge p_2 \wedge p_3]$, it makes no difference whether we evaluate $[(p_1 \wedge p_2) \wedge p_3]$, or $[p_1 \wedge (p_2 \wedge p_3)]$ because both give the same answer. In Example 6, we evaluate $[p_1 \wedge p_2 \wedge p_3]$ from left to right, that is, $[(p_1 \wedge p_2) \wedge p_3]$.

## EXAMPLE 6 *An Argument with Three Premises*

Use a truth table to determine whether the following argument is valid or invalid.

If Donna has a pet, then Donna owns a snail.
Donna owns a snail or Donna drives a truck.
Donna drives a truck or Donna has a pet.
∴ Donna has a pet.

**SOLUTION:** This argument contains three simple statements.

Let

$p$: Donna has a pet.
$q$: Donna owns a snail.
$r$: Donna drives a truck.

In symbolic form, the argument is

$$\begin{array}{l} p \rightarrow q \\ q \vee r \\ \underline{r \vee p} \\ \therefore p \end{array}$$

Write the argument in the form

$$[(p \rightarrow q) \wedge (q \vee r) \wedge (r \vee p)] \rightarrow p.$$

Now construct the truth table (Table 3.32 on page 150). The answer, column 7, is not true in every case. Thus, the argument is a fallacy, or invalid.

**TABLE 3.32**

| p | q | r | [(p → q) | ∧ | (q ∨ r) | ∧ | (r ∨ p)] | → | p |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T | T |
| T | F | T | F | F | T | F | T | T | T |
| T | F | F | F | F | F | F | T | T | T |
| F | T | T | T | T | T | T | T | F | F |
| F | T | F | T | T | T | F | F | T | F |
| F | F | T | T | T | T | T | T | F | F |
| F | F | F | T | F | F | F | F | T | F |
| | | | 1 | 3 | 2 | 5 | 4 | 7 | 6 |

Let's now investigate how we can arrive at a valid conclusion from a given set of premises.

## EXAMPLE 7 *Determine a Logical Conclusion*

Determine a logical conclusion that follows from the given statements. "If you own a house, then you will pay property tax. You own a house. Therefore, . . . "

**SOLUTION:** If you recognize a specific form of an argument, you can use your knowledge of that form to draw a logical conclusion.

Let

$p$: You own a house.

$q$: You will pay property tax.

The argument is of the following form.

$$\begin{array}{l} p \rightarrow q \\ \underline{p\qquad} \\ \therefore\ ? \end{array}$$

If the question mark is replaced with a $q$, this argument is of the form of the law of detachment. A logical conclusion is "Therefore, you will pay property tax."

# SECTION 3.5 EXERCISES

## Concept/Writing Exercises

1. What does it mean when an argument is valid?
2. What does it mean when an argument is a fallacy?
3. Is it possible for an argument to be valid if its conclusion is false? Explain your answer.
4. Is it possible for an argument to be invalid if the premises are all true? Explain your answer.
5. Is it possible for an argument to be valid if the premises are all false? Explain your answer.
6. Explain how to determine whether an argument with premises $p_1$ and $p_2$ and conclusion $c$ is a valid or invalid argument.

*In Exercises 7–10, (a) indicate the form of the valid argument and (b) write an original argument in words for each form.*

7. Disjunctive syllogism
8. Law of contraposition
9. Law of syllogism
10. Law of detachment

*In Exercises 11 and 12, (a) indicate the form of the fallacy, and (b) write an original argument in words for each form.*

11. Fallacy of the inverse
12. Fallacy of the converse

## Practice the Skills

*In Exercises 13–32, determine whether the argument is valid or invalid. You may compare the argument to a standard form or use a truth table.*

13. $\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$

14. $\begin{array}{l} p \to q \\ \underline{\sim p} \\ \therefore q \end{array}$

15. $\begin{array}{l} p \wedge \sim q \\ \underline{q} \\ \therefore \sim p \end{array}$

16. $\begin{array}{l} \sim p \vee q \\ \underline{q} \\ \therefore p \end{array}$

17. $\begin{array}{l} \sim p \\ \underline{p \vee q} \\ \therefore \sim q \end{array}$

18. $\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$

19. $\begin{array}{l} q \to p \\ \underline{\sim q} \\ \therefore \sim p \end{array}$

20. $\begin{array}{l} p \vee q \\ \underline{\sim q} \\ \therefore p \end{array}$

21. $\begin{array}{l} \sim p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$

22. $\begin{array}{l} q \wedge \sim p \\ \underline{\sim p} \\ \therefore \sim q \end{array}$

23. $\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$

24. $\begin{array}{l} q \wedge p \\ \underline{q} \\ \therefore \sim p \end{array}$

25. $\begin{array}{l} p \leftrightarrow q \\ \underline{q \wedge r} \\ \therefore p \vee r \end{array}$

26. $\begin{array}{l} p \leftrightarrow q \\ \underline{q \to r} \\ \therefore \sim r \to \sim p \end{array}$

27. $\begin{array}{l} r \leftrightarrow p \\ \underline{\sim p \wedge q} \\ \therefore p \wedge r \end{array}$

28. $\begin{array}{l} p \vee q \\ \underline{r \wedge p} \\ \therefore q \end{array}$

29. $\begin{array}{l} p \to q \\ q \vee r \\ \underline{r \vee p} \\ \therefore p \end{array}$

30. $\begin{array}{l} p \to q \\ q \to r \\ \underline{r \to p} \\ \therefore q \to p \end{array}$

31. $\begin{array}{l} p \to q \\ r \to \sim p \\ \underline{p \vee r} \\ \therefore q \vee \sim p \end{array}$

32. $\begin{array}{l} p \leftrightarrow q \\ p \vee r \\ \underline{q \to r} \\ \therefore q \vee r \end{array}$

## Problem Solving

*In Exercises 33–50, (a) translate the argument into symbolic form and (b) determine if the argument is valid or invalid. You may compare the argument to a standard form or use a truth table.*

33. If Will Smith wins an Academy Award, then he will retire from acting.
Will Smith did not win an Academy Award.
∴ Will Smith will not retire from acting.

Will Smith (see Exercise 33)

34. If the president of the art club resigned, then the vice president becomes president.
The vice president becomes president of the art club.
∴ The president of the art club resigned.

35. If the baby is a boy, then we will name him Alexander Martin.
The baby is a boy.
∴ We will name him Alexander Martin.

36. If I can get my child to preschool by 8:45 A.M., then I can take the 9:00 A.M. class.
If I can take the 9:00 A.M. class, then I can be done by 2:00 P.M.
∴ If I can get my child to preschool by 8:45 A.M., then I can be done by 2:00 P.M.

37. If monkeys can fly, then scarecrows can dance.
Scarecrows cannot dance.
∴ Monkeys cannot fly.

38. Rob Calcatera will go on sabbatical or Frank Cheek will teach logic.
Frank Cheek will not teach logic.
∴ Rob Calcatera will go on sabbatical.

39. If the orange was left on the tree for 1 year, then the orange is ripe.
The orange is ripe.
∴ The orange was left on the tree for 1 year.

40. If you pass general chemistry then you can take organic chemistry.
You pass general chemistry.
∴ You can take organic chemistry.

41. The X-games will be held in San Diego or they will be held in Corpus Christi.
The X-games will not be held in San Diego.
∴ The X-games will be held in Corpus Christi.

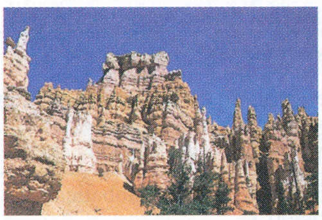
**42.** If Nicholas Thompson teaches this course, then I will get a passing grade.
I did not get a passing grade.
∴ Nicholas Thompson did not teach the course.

**43.** If it is cold, then graduation will be held indoors.
If graduation is held indoors, then the fireworks will be postponed.
∴ If it is cold, then the fireworks will be postponed.

**44.** If Miles Davis played with Louis Armstrong, then Charlie Parker played with Dizzy Gillespie.
Miles Davis did not play with Louis Armstrong.
∴ Charlie Parker did not play with Dizzy Gillespie.

**45.** If the canteen is full, then we can go for a walk.
We can go for a walk and we will not get thirsty.
∴ If we go for a walk, then the canteen is not full.

**46.** Bryce Canyon National Park is in Utah or Bryce Canyon National Park is in Arizona.
If Bryce Canyon National Park is in Arizona, then it is not in Utah.
∴ Bryce Canyon National Park is not in Arizona.



**47.** It is snowing and I am going skiing.
If I am going skiing, then I will wear a coat.
∴ If it is snowing, then I will wear a coat.

**48.** The garden has vegetables or the garden has flowers.
If the garden does not have flowers, then the garden has vegetables.
∴ The garden has flowers or the garden has vegetables.

**49.** If the house has electric heat, then the Flynns will buy the house.
If the price is not less than $100,000, then the Flynns will not buy the house.
∴ If the house has electric heat, then the price is less than $100,000.

**50.** If there is an atmosphere, then there is gravity.
If an object has weight, then there is gravity.
∴ If there is an atmosphere, then an object has weight.

*In Exercises 51–60, translate the argument into symbolic form. Then determine whether the argument is valid or invalid.*

**51.** If the prescription was called in to Walgreen's, then you can pick it up by 4:00 P.M. You cannot pick it up by 4:00 P.M. Therefore, the prescription was not called in to Walgreen's.

**52.** The printer has a clogged nozzle or the printer does not have toner. The printer has toner. Therefore, the printer has a clogged nozzle.

**53.** The television is on or the plug is not plugged in. The plug is plugged in. Therefore, the television is on.

**54.** If the cat is in the room, then the mice are hiding. The mice are not hiding. Therefore, the cat is not in the room.

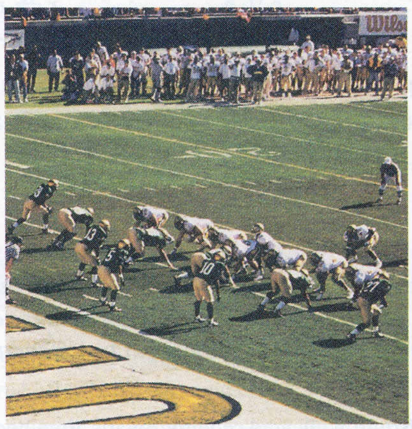
**55.** The test was easy and I received a good grade. The test was not easy or I did not receive a good grade. Therefore, the test was not easy.

**56.** If Bonnie passes the bar exam, then she will practice law. Bonnie will not practice law. Therefore, Bonnie did not pass the bar exam.

**57.** The baby is crying but the baby is not hungry. If the baby is hungry then the baby is crying. Therefore, the baby is hungry.

**58.** If the car is new, then the car has air conditioning. The car is not new and the car has air conditioning. Therefore, the car is not new.

**59.** If the football team wins the game, then Dave played quarterback. If Dave played quarterback, then the team is not in second place. Therefore, if the football team wins the game, then the team is in second place.



**60.** The engineering courses are difficult and the chemistry labs are long. If the chemistry labs are long, then the art tests are easy. Therefore, the engineering courses are difficult and the art tests are not easy.

*In Exercises 61–67, using the standard forms of arguments and other information you have learned, supply what you believe is a logical conclusion to the argument. Verify that the argument is valid for the conclusion you supplied.*

61. If you eat an entire bag of M & M's, your face will break out.
You eat an entire bag of M & M's.
Therefore, . . .

62. If the temperature hits 100°, then we will go swimming.
We did not go swimming.
Therefore, . . .

63. A tick is an insect or a tick is an arachnid.
A tick is not an insect.
Therefore, . . .

64. If Margaret Chang arranges the conference, then many people will attend the conference.
If many people attend the conference, then our picture will be in the paper.
Therefore, . . .

65. If you close the deal, then you will get a commission.
You did not get a commission.
Therefore, . . .

66. If you do not read a lot, then you will not gain knowledge.
You do not read a lot.
Therefore, . . .

67. If you do not pay off your credit card bill, then you will have to pay interest.
If you have to pay interest, then the bank makes money.
Therefore, . . .

## Challenge Problems/Group Activities

68. Determine whether the argument is valid or invalid.
If Lynn wins the contest or strikes oil, then she will be rich.
If Lynn is rich, then she will stop working.
$\therefore$ If Lynn does not stop working, she did not win the contest.

69. Is it possible for an argument to be invalid if the conjunction of the premises is false in every case of the truth table? Explain your answer.

## Recreational Mathematics

70. René Descartes was a seventeenth-century French mathematician and philosopher. One of his most memorable statements is, "I think, therefore, I am." This statement is the basis for the following joke.
Descartes walks into an inn. The innkeeper asks Descartes if he would like something to drink. Descartes replies, "I think not," and promptly vanishes into thin air!
This joke can be summarized in the following argument: If I think, then I am. I think not. Therefore, I am not.
a) Represent this argument symbolically.
b) Is this a valid argument?
c) Explain your answer using either a standard form of argument or using a truth table.

## Internet/Research Activities

71. Show how logic is used in advertising. Discuss several advertisements and show how logic is used to persuade the reader.

72. Find examples of valid (or invalid) arguments in printed matter such as newspaper or magazine articles. Explain why the arguments are valid (or invalid).

# 3.6 EULER DIAGRAMS AND SYLLOGISTIC ARGUMENTS

In the preceding section, we showed how to determine the validity of *symbolic arguments* using truth tables and comparing the arguments to standard forms. This section presents another form of argument called a ***syllogistic argument***, better known by the shorter name ***syllogism***. The validity of a syllogistic argument is determined by using Euler (pronounced "oiler") diagrams, as is explained shortly.

Syllogistic logic, a deductive process of arriving at a conclusion, was developed by Aristotle in about 350 B.C. Aristotle considered the relationships among the four types of statements that follow.

All _____ are _____.
No _____ are _____.
Some _____ are _____.
Some _____ are not _____.

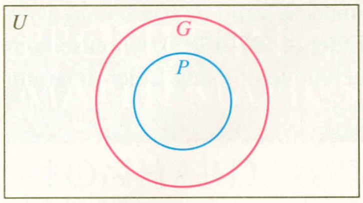



**Figure 3.5**

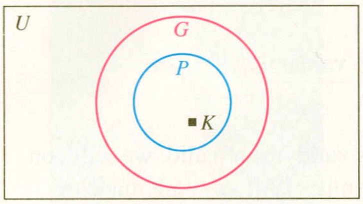



**Figure 3.6**

**SOLUTION:** The statement "All pilots have good vision" is illustrated in Fig. 3.5. The second premise, "Kaitlyn is a pilot," tells us that Kaitlyn must be placed in the inner circle (see Fig. 3.6). The Euler diagram illustrates that we must accept the conclusion "Kaitlyn has good vision" as true (when we accept the premises as true). Therefore, the argument is valid.

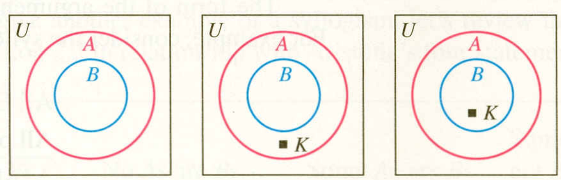
In both Example 1 and Example 2, we had no choice as to where the second premise was to be placed in the Euler diagram. In Example 1, the set of brass objects had to be placed inside the set of valuable objects. In Example 2, Kaitlyn had to be placed inside the set of people with good vision. Often when determining the truth value of a syllogism, a premise can be placed in more than one area in the diagram. *We always try to draw the Euler diagram so that the conclusion does not necessarily follow from the premises. If that can be done, then the conclusion does not necessarily follow from the premises and the argument is invalid.* If we cannot show that the argument is invalid, only then do we accept the argument as valid. We illustrate this process in Example 3.

## EXAMPLE 3 *Ballerinas and Athletes*

Determine whether the following syllogism is valid or is invalid.

All ballerinas are athletic.
Keyshawn is athletic.
∴ Keyshawn is a ballerina.

**SOLUTION:** The statement "All ballerinas are athletic" is illustrated in Fig. 3.7(a). The next premise "Keyshawn is athletic" tells us that Keyshawn must be placed in the set of athletic people. Two diagrams in which both premises are satisfied are shown in Fig. 3.7(b) and (c). By examining Fig. 3.7(b), however, we see that Keyshawn is not a ballerina. Therefore, the conclusion "Keyshawn is a ballerina" does not necessarily follow from the set of premises. Thus, the argument is invalid, or a fallacy.




(a) (b) (c)

**Figure 3.7**

## EXAMPLE 4 *Parrots and Chickens*

Determine whether the following syllogism is valid or invalid.

No parrots eat chicken.
Fletch does not eat chicken.
∴ Fletch is a parrot.

## DID YOU KNOW

### Boolean Algebra

Logic, or Boolean algebra, affects each of us daily, Nowadays, many aspects of our lives are computer related. There are computer chips in our cars, watches, radios, cameras, supermarket checkouts, and most appliances, to name just a few uses. The outcomes of these computer chips are based on the *not, and,* and *or* truth tables we have discussed. Sometimes abstract mathematics is not applied until many years after its development, which is exactly what happened with the abstract Boolean algebra concepts considered in the seventeenth and eighteenth centuries.

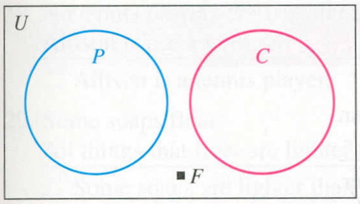



Figure 3.8

**SOLUTION:** The diagram in Fig. 3.8 satisfies the two given premises and also shows that Fletch is not a parrot. Therefore, the argument is invalid, or is a fallacy. ▲

Note that in Example 4 if we placed Fletch in circle $P$, the argument would appear to be valid. Remember that *whenever testing the validity of an argument, always try to show that the argument is invalid.* If there is any way of showing that the conclusion does not necessarily follow from the premises, then the argument is invalid.

## EXAMPLE 5 *A Syllogism Involving the Word Some*

Determine whether the following syllogism is valid or invalid.

All $A$s are $B$s.
Some $B$s are $C$s.
∴ Some $A$s are $C$s.

**SOLUTION:** The statement "All $A$s are $B$s" is illustrated in Fig. 3.9. The statement "Some $B$s are $C$s" means that there is at least one $B$ that is a $C$. We can illustrate this set of premises in four ways, as illustrated in Fig. 3.10.

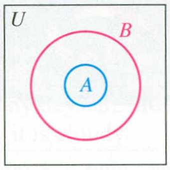



Figure 3.9

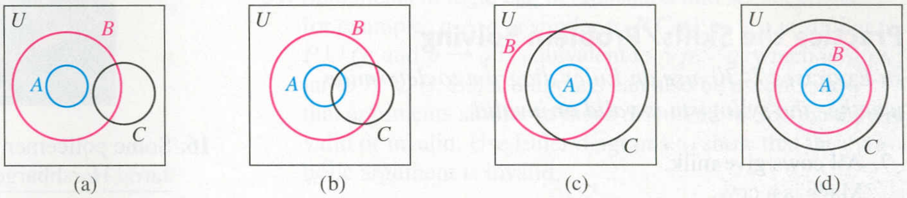



Figure 3.10

In all four illustrations, we see that (1) all $A$s are $B$s and (2) some $B$s are $C$s. The conclusion is "Some $A$s are $C$s." Since at least one of the illustrations, Fig. 3.10(a), shows that the conclusion does not necessarily follow from the given premises, the argument is invalid. ▲

## EXAMPLE 6 *Cowboys and Debutantes*

Determine whether the following syllogism is valid or invalid.

No cowboys drink lemonade.
All debutantes drink lemonade.
∴ No cowboys are debutantes.

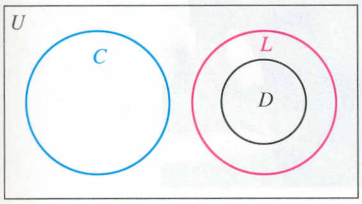



Figure 3.11

**SOLUTION:** The first premise tells us that cowboys and the people who drink lemonade are disjoint sets—that is, sets that do not intersect—as shown in Fig. 3.11. The second premise tells us that the set of debutantes is a subset of the people who drink lemonade. Therefore, the circle representing the set of debutantes must go within the circle representing the set of people who drink lemonade.

The set of debutantes and the set of cowboys cannot be made to intersect without violating a premise. Thus, no cowboys can be debutantes, and the syllogism is valid. Note that we did not say that this conclusion is true, only that the argument is valid. ▲

# SECTION 3.6 EXERCISES

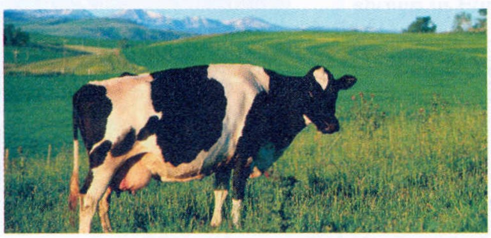
## Concept/Writing Exercises

1. If an Euler diagram can be drawn in which the conclusion does not necessarily follow from the premises, what can be said about the syllogistic argument under consideration?
2. If an Euler diagram can only be drawn in a way in which the conclusion necessarily follows from the premises, what can be said about the syllogistic argument under consideration?
3. What does it mean when we determine that an argument is valid?
4. Explain the differences between a symbolic argument and a syllogistic argument.
5. Can an argument be valid if the conclusion is a false statement? Explain your answer.
6. Can an argument be invalid if the conclusion is a true statement? Explain.

## Practice the Skills/Problem Solving

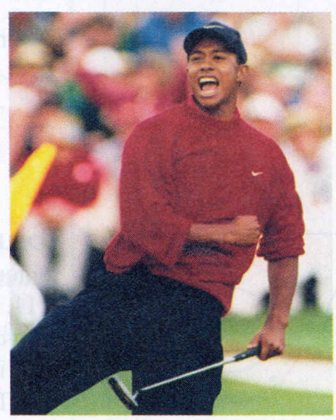
*In Exercises 7–30, use an Euler diagram to determine whether the syllogism is valid or invalid.*

7. All cows give milk.
   Mags is a cow.
   ∴ Mags gives milk.



8. All cordless telephones have antennas.
   All things with antennas are insects.
   ∴ All cordless telephones are insects.
9. No apples are citrus fruits.
   All Granny Smiths are apples.
   ∴ No Granny Smiths are citrus fruits.
10. All dolphins are mammals.
    All mammals are vertebrates.
    ∴ All dolphins are vertebrates.
11. All theme parks have walkways.
    Metropolitan Community College has walkways.
    ∴ Metropolitan Community College is a theme park.
12. All golfers have rain gear.
    John Pearse has rain gear.
    ∴ John Pearse is a golfer.
13. No horses buck.
    Palominos are horses.
    ∴ Palominos do not buck.
14. No jockeys weigh more than 200 pounds.
    Deb Otto is not a jockey.
    ∴ Deb Otto weighs more than 200 pounds.
15. Some mushrooms are poisonous.
    A morel is a mushroom.
    ∴ A morel is poisonous.

16. Some policemen are polite.
    Jarod Harshbarger is a policeman.
    ∴ Jarod Harshbarger is not polite.
17. Some farmers are politicians.
    Some politicians are senators.
    ∴ Some farmers are senators.
18. Some professional golfers give golf lessons.
    All people who belong to the PGA are professional golfers.
    ∴ All people who belong to the PGA give golf lessons.



Tiger Woods

19. No tennis players are wrestlers.
Allison is not a wrestler.
∴ Allison is a tennis player.

20. Some soaps float.
All things that float are lighter than water.
∴ Some soaps are lighter than water.

21. Some people love mathematics.
All people who love mathematics love physics.
∴ Some people love physics.

22. Some desks are made of wood.
All paper is made of wood.
∴ Some desks are made of paper.

23. No *x*s are *y*s.
No *y*s are *z*s.
∴ No *x*s are *z*s.

24. All pilots can fly.
All astronauts can fly.
∴ Some pilots are astronauts.

25. Some dogs wear glasses.
Fido wears glasses.
∴ Fido is a dog.

26. All rainy days are cloudy.
Today it is cloudy.
∴ Today is a rainy day.

27. All sweet things taste good.
All things that taste good are fattening.
All things that are fattening put on pounds.
∴ All sweet things put on pounds.

28. All books have red covers.
All books that have red covers contain 200 pages.
Some books that contain 200 pages are novels.
∴ All books that contain 200 pages are novels.

29. All country singers play the guitar.
All country singers play the drums.
Some people who play the guitar are rock singers.
∴ Some country singers are rock singers.


Bonnie Raitt

30. Some hot dogs are made of turkey.
All things made of turkey are edible.
Some things that are made of beef are edible.
∴ Some hot dogs are made of beef.

## Challenge Problem/Group Activity

31. Statements in logic can be translated into set statements: for example, $p \wedge q$ is similar to $P \cap Q$; $p \vee q$ is similar to $P \cup Q$; and $p \rightarrow q$ is equivalent to $\sim p \vee q$, which is similar to $P' \cup Q$. Euler diagrams can also be used to show that arguments similar to those discussed in Section 3.5 are valid or invalid. Use Euler diagrams to show that the symbolic argument is invalid.

$$\begin{array}{l} p \rightarrow q \\ \underline{p \vee q} \\ \therefore \sim p \end{array}$$

## Internet/Research Activity

32. Leonhard Euler is considered one of the greatest mathematicians of all time. Do research and write a report on Euler's life. Include information on his contributions to sets and to logic. Also indicate other areas of mathematics in which he made important contributions. References include encyclopedias, history of mathematics books, and the Internet.

# CHAPTER 3 SUMMARY

## IMPORTANT FACTS

### Quantifiers

| Form of Statement | Form of Negation |
|---|---|
| All are. | Some are not. |
| None are. | Some are. |
| Some are. | None are. |
| Some are not. | All are. |

### Summary of connectives

| Formal Name | Symbol | Read | Symbolic Form |
|---|---|---|---|
| Negation | $\sim$ | not | $\sim p$ |
| Conjunction | $\wedge$ | and | $p \wedge q$ |
| Disjunction | $\vee$ | or | $p \vee q$ |
| Conditional | $\rightarrow$ | if–then | $p \rightarrow q$ |
| Biconditional | $\leftrightarrow$ | if and only if | $p \leftrightarrow q$ |

### Basic truth tables

Negation

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

| $p$ | $q$ | Conjunction $p \wedge q$ | Disjunction $p \vee q$ | Conditional $p \rightarrow q$ | Biconditional $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | F | T | T | F |
| F | F | F | F | T | T |

### De Morgan's laws

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$
$$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

### Other equivalent forms

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$
$$\sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$$
$$p \leftrightarrow q \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$$

### Variations of the conditional statement

| Name | Symbolic Form | Read |
|---|---|---|
| Conditional | $p \rightarrow q$ | If $p$, then $q$. |
| Converse of the conditional | $q \rightarrow p$ | If $q$, then $p$. |
| Inverse of the conditional | $\sim p \rightarrow \sim q$ | If not $p$, then not $q$. |
| Contrapositive of the conditional | $\sim q \rightarrow \sim p$ | If not $q$, then not $p$. |

### Standard forms of arguments

#### Valid arguments

| Law of Detachment | Law of Contra-position | Law of Syllogism | Disjunctive Syllogism |
|---|---|---|---|
| $p \rightarrow q$ | $p \rightarrow q$ | $p \rightarrow q$ | $p \vee q$ |
| $p$ | $\sim q$ | $q \rightarrow r$ | $\sim p$ |
| $\therefore q$ | $\therefore \sim p$ | $\therefore p \rightarrow r$ | $\therefore q$ |

#### Invalid arguments

| Fallacy of the Converse | Fallacy of the Inverse |
|---|---|
| $p \rightarrow q$ | $p \rightarrow q$ |
| $q$ | $\sim p$ |
| $\therefore p$ | $\therefore \sim q$ |

### Symbolic argument vs. syllogistic argument

| | Words or Phrases Used | Method of Determining Validity |
|---|---|---|
| *Symbolic argument* | and, or, not, if–then, if and only if | Truth tables or by comparison with standard forms of arguments |
| *Syllogistic argument* | all are, some are, none are, some are not | Euler diagrams |

# CHAPTER 3 REVIEW EXERCISES

## 3.1

*In Exercises 1–6, write the negation of the statement.*

1. Some rock bands play ballads.
2. Some bananas are not ripe.
3. No chickens have lips.
4. All panthers are endangered.

5. All pens use ink.
6. No rabbits wear glasses.

*In Exercises 7–12, write each compound statement in words.*

$p$: The coffee is Maxwell House.
$q$: The coffee is hot.
$r$: The coffee is strong.

7. $p \vee q$
8. $\sim q \wedge r$
9. $q \rightarrow (r \wedge \sim p)$
10. $p \leftrightarrow \sim r$
11. $\sim p \leftrightarrow (r \wedge \sim q)$
12. $(p \vee \sim q) \wedge \sim r$

## 3.2

*In Exercises 13–18, use the statements for p, q, and r as in Exercises 7–12 to write the statement in symbolic form.*

13. The coffee is strong and the coffee is hot.
14. If the coffee is Maxwell House, then it is strong.
15. If the coffee is strong then the coffee is hot, or the coffee is not Maxwell House.
16. The coffee is hot if and only if the coffee is Maxwell House, and the coffee is not strong.
17. The coffee is strong and the coffee is hot, or the coffee is not Maxwell House.
18. It is false that the coffee is strong and the coffee is hot.

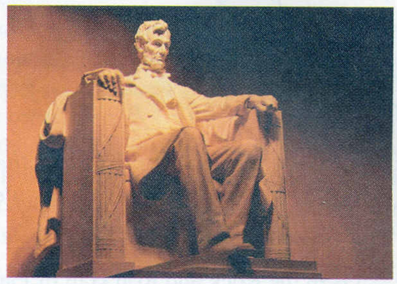
*In Exercises 19–24, construct a truth table for the statement.*

19. $(p \vee q) \wedge \sim p$
20. $q \leftrightarrow (p \vee \sim q)$
21. $(p \vee q) \leftrightarrow (p \vee r)$
22. $p \wedge (\sim q \vee r)$
23. $p \rightarrow (q \wedge \sim r)$
24. $(p \wedge q) \rightarrow \sim r$

## 3.2, 3.3

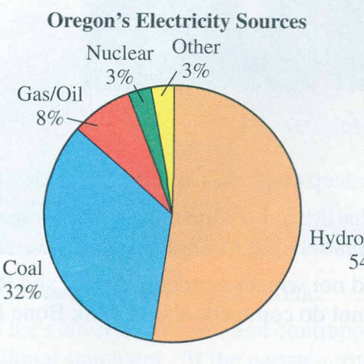
*In Exercises 25–28, determine the truth value of the statement.*

25. If 7 is an odd number, then 11 is an even number.
26. The St. Louis Arch is in St. Louis or Abraham Lincoln is buried in Grant's Tomb.



27. If Oregon borders the Pacific Ocean or California borders the Atlantic Ocean, then Minnesota is south of Texas.
28. $15 - 7 = 22$ or $4 + 9 = 13$, and $9 - 8 = 1$.

*In Exercises 29 and 30, the circle graph shows the sources and percentages of Oregon's electricity sources in 2002. Use the graph to determine the truth value of each simple statement. Then determine the truth value of the compound statement.*




Source: Renewable Northwest Project

29. Thirty-two percent of Oregon's electricity sources are coal if and only if 54% of Oregon's electricity sources are hydroelectric sources, or 38% of Oregon's electricity sources are nuclear.
30. If 3% of Oregon's electricity sources are gas/oil, then 45% of Oregon's electricity sources are coal and 3% of Oregon's electricity comes from other sources.

## 3.3

*In Exercises 31–34, determine the truth value of the statement when p is T, q is F, and r is F.*

**31.** $(p \rightarrow \sim r) \vee (p \wedge q)$ **32.** $(p \vee q) \leftrightarrow (\sim r \wedge p)$

**33.** $\sim r \leftrightarrow [(p \vee q) \leftrightarrow \sim p]$ **34.** $\sim[(q \wedge r) \rightarrow (\sim p \vee r)]$

## 3.4

*In Exercises 35–38, determine whether the pairs of statements are equivalent. You may use De Morgan's laws, the fact that $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$, truth tables, or equivalent forms of the conditional statement.*

**35.** $\sim p \vee \sim q$ $\qquad \sim p \leftrightarrow q$

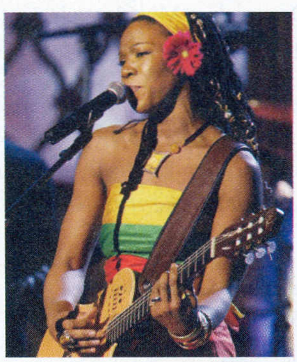
**36.** $\sim p \rightarrow \sim q$ $\qquad p \vee \sim q$

**37.** $\sim p \vee (q \wedge r)$ $\qquad (\sim p \vee q) \wedge (\sim p \vee r)$

**38.** $(\sim q \rightarrow p) \wedge p$ $\qquad \sim(\sim p \leftrightarrow q) \vee p$

*In Exercises 39–43, use De Morgan's laws or the fact that $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$ to write an equivalent statement for the given statement.*

**39.** Johnny Cash is in the Rock and Roll Hall of Fame and India Arie recorded *Acoustic Soul*.



India Arie

**40.** Her foot fell asleep or she has injured her ankle.

**41.** It is not true that Altec Lansing only produces speakers or Harman Kardon only produces stereo receivers.

**42.** Travis Tritt did not win an Academy Award and Randy Jackson does not do commercials for Milk Bone Dog Biscuits.

**43.** If the temperature is not above 32°, then we will go ice fishing at O'Leary's Lake.

*In Exercises 44–48, write the (a) converse, (b) inverse, and (c) contrapositive for the given statement.*

**44.** If you hear a beautiful songbird today, then you enjoy life.

**45.** If you followed the correct pattern, then the quilt has a uniform design.

**46.** If Maureen Gerald is not in attendance, then she is helping at the school.

**47.** If the desk is made by Winner's Only and the desk is in the Rose catalog, then we will not buy a desk at Miller's Furniture.

**48.** If you get straight A's on your report card, then I will let you attend the prom.

*In Exercises 49–52, determine which, if any, of the three statements are equivalent.*

**49. a)** If the temperature is over 80°, then the air conditioner will come on.
**b)** The temperature is not over 80° or the air conditioner will come on.
**c)** It is false that the temperature is over 80° and the air conditioner will not come on.

**50. a)** The screwdriver is on the workbench if and only if the screwdriver is not on the counter.
**b)** If the screwdriver is not on the counter, then the screwdriver is not on the workbench.
**c)** It is false that the screwdriver is on the counter and the screwdriver is not on the workbench.

**51. a)** If $2 + 3 = 6$, then $3 + 1 = 5$.
**b)** $2 + 3 = 6$ if and only if $3 + 1 \neq 5$.
**c)** If $3 + 1 \neq 5$, then $2 + 3 \neq 6$.

**52. a)** If the sale is on Tuesday and I have money, then I will go to the sale.
**b)** If I go to the sale, then the sale is on Tuesday and I have money.
**c)** I go to the sale, or the sale is on Tuesday and I have money.

## 3.5, 3.6

*In Exercises 53–58, determine whether the argument is valid or invalid.*

**53.**
$$\begin{array}{l} p \rightarrow q \\ \underline{\sim p} \\ \therefore q \end{array}$$

**54.**
$$\begin{array}{l} p \wedge q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

**55.** Nicole is in the hot tub or she is in the shower.
Nicole is in the hot tub.
∴ Nicole is not in the shower.

**56.** If the car has a sound system, then Rick will buy the car. If the price is not less than $18,000, then Rick will not buy the car. Therefore, if the car has a sound system, then the price is less than $18,000.

**57.** All plumbers wear overalls.
Some electricians wear overalls.
∴ Some electricians are plumbers.

**58.** Some submarines are yellow.
All dandelions are yellow.
∴ Some dandelions are submarines.

# CHAPTER 3 TEST

*In Exercises 1–3, write the statement in symbolic form.*

$p$: Ann is the secretary.
$q$: Dick is the vice president.
$r$: Elaine is the president.

1. Ann is the secretary but Elaine is the president, or Dick is not the vice president.
2. If Elaine is the president then Dick is the vice president, or Ann is not the secretary.
3. It is false that Elaine is the president if and only if Dick is not the vice president.

*In Exercises 4 and 5, use p, q, and r as above to write each symbolic statement in words.*

4. $p \leftrightarrow (q \wedge r)$
5. $\sim(p \rightarrow \sim r)$

*In Exercises 6 and 7, construct a truth table for the given statement.*

6. $[\sim(p \rightarrow r)] \wedge q$
7. $(q \leftrightarrow \sim r) \vee p$

*In Exercises 8 and 9, find the truth value of the statement.*

8. $2 + 6 = 8$ or $7 - 12 = 5$.
9. Scissors can cut paper or a dime has the same value as two nickels, if and only if Louisville is a city in Kentucky.

*In Exercises 10 and 11, given that p is true, q is false, and r is true, determine the truth value of the statement.*

10. $(r \vee q) \leftrightarrow (p \wedge \sim q)$
11. $[\sim(r \rightarrow \sim p)] \wedge (q \rightarrow p)$
12. Determine whether the pair of statements are equivalent.

$$\sim p \vee q, \qquad \sim(p \wedge \sim q)$$

*In Exercises 13 and 14, determine which, if any, of the three statements are equivalent.*

13. a) If the bird is red, then it is a cardinal.
    b) The bird is not red or it is a cardinal.
    c) If the bird is not red, then it is not a cardinal.
14. a) It is not true that the test is today or the concert is tonight.
    b) The test is not today and the concert is not tonight.
    c) If the test is not today, then the concert is not tonight.
15. *Translate the following argument into symbolic form. Determine whether the argument is valid or invalid by comparing the argument to a recognized form or by using a truth table.*

    If the soccer team wins the game, then Sue played fullback. If Sue played fullback, then the team is in second place. Therefore, if the soccer team wins the game, then the team is in second place.
16. *Use an Euler diagram to determine whether the syllogism is valid or is a fallacy.*

    All cars have engines.
    Some things with engines use gasoline.
    ∴ Some cars use gasoline.

*In Exercises 17 and 18, write the negation of the statement.*

17. All leopards are spotted.

18. Some jacks-in-the-box are electronic.
19. Write the converse, inverse, and contrapositive of the conditional statement, "If the garbage truck comes, then today is Saturday."
20. Is it possible for an argument to be valid when the conclusion is a false statement? Explain your answer.

# GROUP PROJECTS

## Switching Circuits

1. An application of logic is *switching circuits*. There are two basic types of electric circuits: *series circuits* and *parallel circuits*. In a series circuit, the current can flow in only one path; see Fig. 3.12. In a parallel circuit the current can flow in more than one path; see Fig. 3.13.

Series circuit

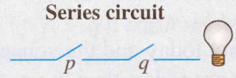



**Figure 3.12**

Parallel circuit

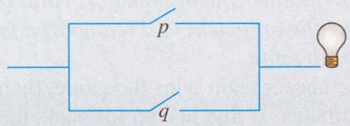



**Figure 3.13**

In Figs. 3.12 and 3.13, the $p$ and $q$ represent switches that may be opened or closed. In the series circuit in Fig. 3.12, if both switches are closed, the current will reach the bulb and the bulb will light. In the parallel circuit in Fig. 3.13, if either switch $p$ or switch $q$ is closed, or if both switches are closed, the current will reach the bulb and the bulb will light.

a) How many different open/closed arrangements of the two switches in Fig. 3.12 are possible? List all the possibilities.

b) Series circuits are represented using conjunctions. The circuit in Fig. 3.12 may be represented as $p \wedge q$. Construct a four-row truth table to represent the series circuit. Construct the table with columns for $p$, $q$, and $p \wedge q$. The statement $p \wedge q$ represents the outcome of the circuit (either the bulb lighting or the bulb not lighting). Represent a closed switch with the number 1, an open switch with the number 0, the bulb lighting with the number 1, and the bulb not lighting with the number 0. For example, if both switches are closed, the bulb will light, and so we write the first row of the truth table as

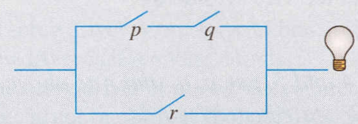
| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 1 | 1 | 1 |

c) How is the truth table determined in part (b) similar to the truth table for $p \wedge q$ discussed in earlier sections of this chapter?

d) Parallel circuits are represented using disjunctions. The circuit in Fig. 3.13 may be represented as $p \vee q$. Construct a truth table to represent the parallel circuit. Construct the table with columns for $p$, $q$, and $p \vee q$. The statement $p \vee q$ represents the outcome of the circuit (either the bulb lighting or the bulb not lighting). Use 1's and 0's as indicated in part (b).

e) How is the truth table determined in part (d) similar to the truth table for $p \vee q$ discussed in earlier sections of this chapter?

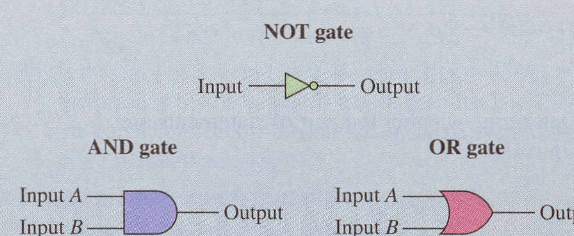
f) Represent the following circuit as a symbolic logic statement using parentheses. Explain how you determined your answer.




g) Draw a circuit to represent the logic statement $p \wedge (q \vee r)$. Explain how you determined your answer.

## Computer Gates

2. Gates in computers work on the same principles as switching circuits. The three basic types of gate are the NOT gate, the AND gate, and the OR gate. Each is illustrated along with a table that indicates current flow entering and exiting the gate. If current flows into a NOT gate, then no current exits, and vice versa. Current exits an AND gate only when both inputs have a current flow. Current exits an OR gate if current flows through either, or both, inputs. In the table, a 1 represents a current flow and a 0 indicates no current flow. For example, in the AND gate, if there is a current flow in input $A$ ($I_a$ has a value of 1) and no current flow in input $B$ ($I_b$ has a value of 0), there is no current flow in the output (O has a value of 0); see row 2 of the AND Gate table.



NOT gate

| $I$ | O |
|---|---|
| 1 | 0 |
| 0 | 1 |

AND gate

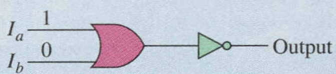
| $I_a$ | $I_b$ | O |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

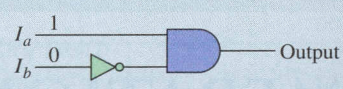
OR gate

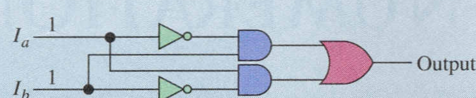
| $I_a$ | $I_b$ | O |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

a) If 1 is considered true and 0 is considered false, explain how these tables are similar to the *not, and,* and *or* truth tables.

For the inputs indicated in the following figures determine whether the output is 1 or 0.

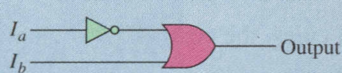
b)



c)



d)



e) What values for $I_a$ and $I_b$ will give an output of 1 in the figure in part (d)? Explain how you determined your answer.

f) Construct a truth table using 1's and 0's for the following gate. Your truth table should have columns $I_a$, $I_b$, and O and should indicate the four possible cases for the inputs and each corresponding output.




## Logic Game

3. a) Shown is a photograph of a logic game at the Ontario Science Centre. There are 12 balls on top of the game board, numbered from left to right, with ball 1 on the extreme left and ball 12 on the extreme right. On the platform in front of the players are 12 buttons, one corresponding to each of the balls. When 6 buttons are pushed, the 6 respective balls are released. When 1 or 2 balls reach an *and* gate or an *or* gate, a single ball may or may not pass through the gate. The object of the game is to select a proper combination of 6 buttons that will allow 1 ball to reach the bottom. Using your knowledge of *and* and *or*, select a combination of 6 buttons that will result in a win. (There is more than one answer.) Explain how you determined your answer.

b) Construct a game similar to this one where 15 balls are at the top and 8 must be selected to allow 1 ball to reach the bottom.

c) Indicate all solutions to the game you constructed in part (c).