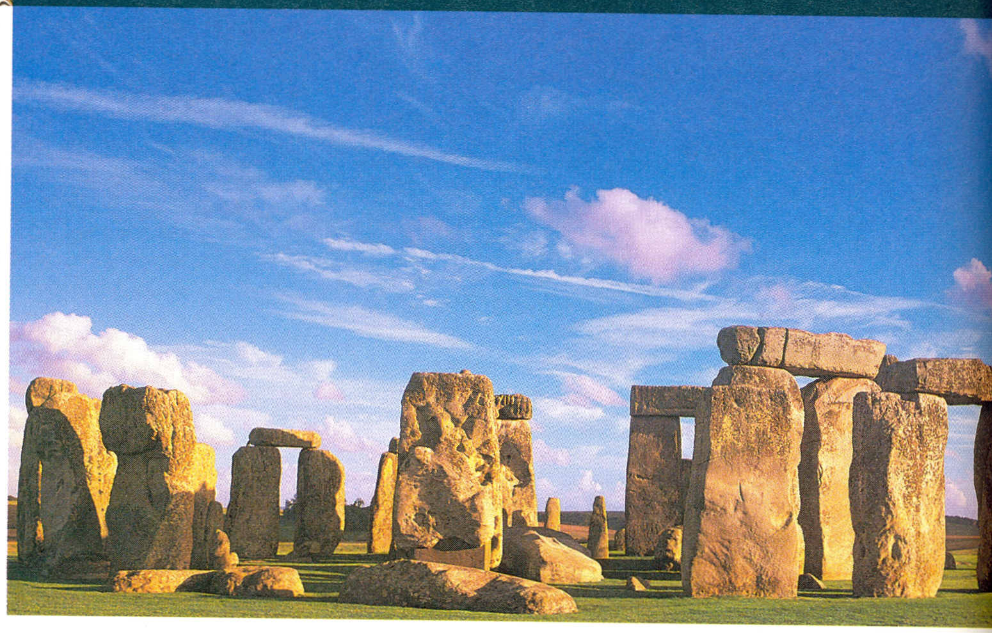
CHAPTER 4



One of the earliest reasons that human beings needed numbers was for reckoning time, marking off days in the lunar month, so the seasonal changes that dictated human activity could be anticipated. The Mayans, the Egyptians, and the ancient Britons constructed monumental stone observatories that enabled them to mark the passage of the seasons, especially the summer solstice, using the alignment of the sun as a guide. The photo is Stonehenge in England.

SYSTEMS OF NUMERATION

The number system we use—called the Hindu–Arabic system—seems to be a permanent, unchanging means of communicating quantities. However, just as languages evolve over time, so do numerical symbols that represent numbers.

Mathematics began with the practical problem of counting and record keeping. People had to count their herds, the passage of days, and objects of barter. They used physical objects—stones, shells, fingers—to represent the objects counted.

As primitive cultures grew from villages to cities, the complexity of human activities increased. Now people needed better ways of recording and communicating. It was a revolutionary step when people started using physical objects to represent not only specific objects like sheep and grain, but also the concept of pure quantity.

Through the course of human history, the evolution of numeration systems has expanded our knowledge and abilities for record keeping, communication, and computation. As a society's numeration system changes, so do the capabilities of that society. Without an understanding of the binary number system, the computer as we know it today could not exist. Without the computer, our lifestyle would not be as it is today.

4.1 ADDITIVE, MULTIPLICATIVE, AND CIPHERED SYSTEMS OF NUMERATION

Just as the first attempts to write were made long after the development of speech, the first representation of numbers by symbols came long after people had learned to count. A tally system using physical objects, such as scratch marks in the soil or on a stone, notches on a stick, pebbles, or knots on a vine, was probably the earliest method of recording numbers.

In primitive societies, such a tally system adequately served the limited need for recording livestock, agriculture, or whatever was counted. As civilization developed, however, more efficient and accurate methods of calculating and keeping records were needed. Because tally systems are impractical and inefficient, societies developed symbols to replace them. For example, the Egyptians used the symbol ∩ and the Babylonians used the symbol ≺ to represent the number we symbolize by 10.

A ***number*** is a quantity, and it answers the question "How many?" A ***numeral*** is a symbol such as ∩, ≺, or 10 used to represent the number. We think a number but write a numeral. The distinction between number and numeral will be made here only if it is helpful to the discussion.

In language, relatively few letters of the alphabet are used to construct a large number of words. Similarly, in arithmetic, a small variety of numerals can be used to represent all numbers. In general, when writing a number, we use as few numerals as possible. One of the greatest accomplishments of humankind has been the development of systems of numeration, whereby all numbers are "created" from a few symbols. Without such systems, mathematics would not have developed to its present level.

> A **system of numeration** consists of a set of numerals and a scheme or rule for combining the numerals to represent numbers.

Four types of numeration systems used by different cultures are the topic of this chapter. They are additive (or repetitive), multiplicative, ciphered, and place-value systems. You do not need to memorize all the symbols, but you should understand the principles behind each system. By the end of this chapter, we hope that you better understand the system we use, the ***Hindu–Arabic system***, and its relationship to other types of systems.

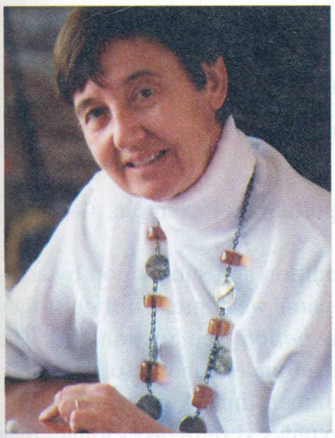
Additive Systems

An additive system is one in which the number represented by a particular set of numerals is simply the sum of the values of the numerals. The additive system of numeration is one of the oldest and most primitive types of numeration systems. One of the first additive systems, the Egyptian hieroglyphic system, dates back to about 3000 B.C. The Egyptians used symbols for the powers of 10: 10^0 or 1, 10^1 or 10, 10^2 or $10 \cdot 10$, 10^3 or $10 \cdot 10 \cdot 10$, and so on. Table 4.1 on page 168 lists the Egyptian hieroglyphic numerals with the equivalent Hindu–Arabic numerals.

To write the number 600 in Egyptian hieroglyphics, we write the numeral for 100 six times: 𓍢𓍢𓍢𓍢𓍢𓍢

DID YOU KNOW

An Important Discovery



Archaeologist Denise Schmandt-Besserat made a breakthrough discovery about early systems of numeration. She realized that the little clay geometric objects that had been found in many archaeological sites had actually been used by people to account for their goods. Later in history, these tokens were impressed on a clay tablet to represent quantities, the beginning of writing.

DID YOU KNOW

The Rhind Papyrus

Much of our knowledge of Egyptian mathematics and numeration comes from a roll of papyrus measuring 18 ft by 13 in. It was discovered in a shop in Luxor, Egypt, in 1858 by Henry Rhind, a Scottish lawyer turned archaeologist. Unlike the straightforward accounting of property and events common to Egyptian tombs, the Rhind Papyrus has inscribed on it 85 mathematical problems and solutions involving addition, subtraction, multiplication, division, and geometry.

The key to translation of the papyrus was the Rosetta Stone, which had been discovered some 60 years earlier by one of Napoleon's officers. Both of these treasures now reside at the British Museum in London.

TABLE 4.1 Egyptian Hieroglyphics

Hindu–Arabic Numerals	Egyptian Numerals	Description
1	𓏺	Staff (vertical stroke)
10	𓎆	Heel bone (arch)
100	𓍢	Scroll (coiled rope)
1,000	𓆼	Lotus flower
10,000	𓂭	Pointing finger
100,000	𓆐	Tadpole (or whale)
1,000,000	𓁨	Astonished person

EXAMPLE 1 *From Egyptian to Hindu–Arabic Numerals*

Write the following numeral as a Hindu–Arabic numeral.

𓂭𓂭𓍢𓍢𓍢𓍢𓏺𓏺

SOLUTION:

$$10{,}000 + 10{,}000 + 100 + 100 + 100 + 100 + 1 + 1 = 20{,}402$$

EXAMPLE 2 *From Hindu–Arabic to Egyptian Numerals*

Write 43,628 as an Egyptian numeral.

SOLUTION:

$$43{,}628 = 40{,}000 + 3000 + 600 + 20 + 8$$

𓂭𓂭𓂭𓂭𓆼𓆼𓆼𓍢𓍢𓍢𓍢𓍢𓍢𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺

In this system, the order of the symbols is not important. For example, 𓆐𓍢𓍢𓎆𓏺𓏺 and 𓏺𓏺𓍢𓍢𓆐𓎆 both represent 100,212.

Users of additive systems easily accomplished addition and subtraction by combining or removing symbols. Multiplication and division were more difficult; they were performed by a process called *duplation and mediation* (see Section 4.5). The Egyptians had no symbol for zero, but they did have an understanding of fractions. The symbol 𓂋 was used to take the reciprocal of a number; thus, 𓂋𓏺𓏺𓏺 meant $\frac{1}{3}$ and 𓂋𓎆𓏺 was $\frac{1}{11}$. Writing large numbers in the Egyptian Hieroglyphics system takes longer than in other systems because so many symbols have to be listed. For example, 45 symbols are needed to represent the number 99,999.

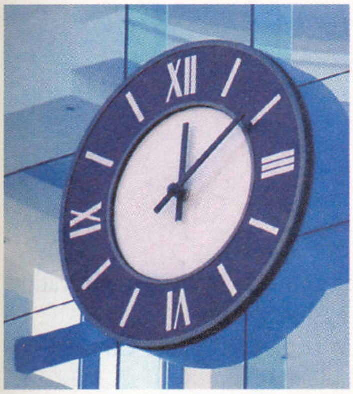
The Roman numeration system, a second example of an additive system, was developed later than the Egyptian system. Roman numerals (Table 4.2) were used in most European countries until the eighteenth century. They are still commonly seen on buildings, on clocks, and in books. Roman numerals are selected letters of the Roman alphabet.

TABLE 4.2 Roman Numerals

Roman numerals	I	V	X	L	C	D	M
Hindu–Arabic numerals	1	5	10	50	100	500	1000

DID YOU KNOW

Roman Numerals



Roman numerals remained popular on large clock faces long after their disappearance from daily transactions because they are easier to read from a distance than Hindu–Arabic numerals. The clock shown in this photo is in Vancouver, Canada.

The Roman system has two advantages over the Egyptian system. The first is that it uses the subtraction principle as well as the addition principle. Starting from the left, we add each numeral unless its value is smaller than the value of the numeral to its right. In that case, we subtract its value from the value of the numeral to its right. Only the numbers 1, 10, 100, 1000, . . . can be subtracted, and they can only be subtracted from the next two higher numbers. For example, C (100) can be subtracted only from D (500) or M (1000). The symbol DC represents 500 + 100, or 600, and CD represents 500 − 100, or 400. Similarly, MC represents 1000 + 100, or 1100, and CM represents 1000 − 100, or 900.

EXAMPLE 3 *A Roman Numeral*

Write MMCCCLXII as a Hindu–Arabic numeral.

SOLUTION: Since each numeral is larger than the one on its right, no subtraction is necessary.

$$\text{MMCCCLXII} = 1000 + 1000 + 100 + 100 + 100 + 50 + 10 + 1 + 1 = 2362$$

▲

EXAMPLE 4 *A Roman Numeral Involving a Subtraction*

Write DCXLVI as a Hindu–Arabic numeral.

SOLUTION: Checking from left to right, we see that X (10) has a smaller value than L (50). Therefore, XL represents 50 − 10, or 40.

$$\text{DCXLVI} = 500 + 100 + (50 - 10) + 5 + 1 = 646$$

▲

EXAMPLE 5 *Writing a Roman Numeral*

Write 289 as a Roman numeral.

SOLUTION:

$$289 = 200 + 80 + 9 = 100 + 100 + 50 + 10 + 10 + 10 + 9$$

(Nine is treated as 10 − 1.)

$$289 = \text{CCLXXXIX}$$

▲

In the Roman numeration system, a symbol does not have to be repeated more than three consecutive times. For example, the number 646 would be written DCXLVI instead of DCXXXXVI.

The second advantage of the Roman numeration system over the Egyptian numeration system is that it makes use of the multiplication principle for numbers over 1000. A bar above a symbol or group of symbols indicates that the symbol or symbols are to be multiplied by 1000. Thus, $\overline{\text{V}} = 5 \times 1000 = 5000$, $\overline{\text{X}} = 10 \times 1000 = 10{,}000$, and $\overline{\text{CD}} = 400 \times 1000 = 400{,}000$. Other examples are, $\overline{\text{VI}} = 6 \times 1000 = 6000$, $\overline{\text{XIX}} = 19 \times 1000 = 19{,}000$, and $\overline{\text{XCIV}} = 94 \times 1000 = 94{,}000$. This greatly reduces the number of symbols needed to write large numbers. Still, it requires 19 symbols, including the bar, to write the number 33,888 in Roman numerals. Write the number 33,888 in Roman numerals now.

Multiplicative Systems

Multiplicative numeration systems are more similar to our Hindu–Arabic system than are additive systems. The number 642 in a multiplicative system might be written (6) (100) (4) (10) (2) or

6
100
4
10
2

Note that no addition signs are needed to represent the number. From this illustration, try to formulate a rule explaining how multiplicative systems work.

The principle example of a multiplicative system is the traditional Chinese system. The numerals used in this system are given in Table 4.3.

TABLE 4.3 Traditional Chinese Numerals

Traditional Chinese numerals	零	一	二	三	四	五	六	七	八	九	十	百	千
Hindu–Arabic numerals	0	1	2	3	4	5	6	7	8	9	10	100	1000

Chinese numerals are always written vertically. The number on top will be a number from 1 to 9 inclusive. This number is to be multiplied by the power of 10 below it. The number 20 is written

二
十 } $2 \times 10 = 20$

The number 400 is written

四
百 } $4 \times 100 = 400$

EXAMPLE 6 *A Traditional Chinese Numeral*

Write 538 as a Chinese numeral.

SOLUTION:

538 =	500 =	5	五
		100	百
	30 =	3	三
		10	十
	8 =	8	八

Note that in Example 6 the units digit, the 8, is not multiplied by a power of the base. When writing Chinese numerals, there are some special cases that need to be considered. When writing a number between 11 and 19 it is not necessary to include the 1 before the 10. Thus, the number 18 would be written 十八 (vertically) rather than 一十八 (vertically). Another special case involves the use of zero.

When more than one consecutive zero occurs (except at the end of a number) you need to write a zero, but only once for two or more consecutive zeros. Zeros are not included at the end of numbers. The top two illustrations that follow show how zeros are used within a number and the bottom two show that zeros are not used at the end of a number.

406 =

四百	$4 \times 100 = 400$
零	$0 \times 10 = 0$
六	$6 = 6$

4006 =

四千	$4 \times 1000 = 4000$
零	$0 \times 100,\ 0 \times 10 = 0$
六	$6 = 6$

460 =

四百	$4 \times 100 = 400$
六十	$6 \times 10 = 60$

4600 =

四千	$4 \times 1000 = 4000$
六百	$6 \times 100 = 600$

EXAMPLE 7 *Traditional Chinese Numerals*

Write the following as traditional Chinese numerals.

a) 7080 **b)** 7008

SOLUTION: In part (a) there is one zero between the 7 and the 8. In part (b) there are two zeros between the 7 and the 8. As just mentioned, the symbol for zero is used only once in each of those numbers.

a) 7080 =

七千	7×1000
零	0×100
八十	8×10

b) 7008 =

七千	7×1000
零	$0 \times 100,\ 0 \times 10$
八	8

TIMELY TIP Notice the difference between our Hindu–Arabic numeration system, which is a positional numeration system, and the Chinese system, which is a multiplicative numeration system. Consider the number 5678. Below we show how that number would be written in Chinese numerals if the Chinese system was a positional value system similar to ours.

Multiplicative

五	5
千	1000
六	6
百	100
七	7
十	10
八	8

Positional Value

五	5
六	6
七	7
八	8

Note that the multiples of base 10 are removed when writing the number as a positional value number. We will discuss positional value systems in more detail shortly.

DID YOU KNOW

Hindu-Arabic Numerals

The Hindu–Arabic system of numeration is used throughout the world. In many countries, numbers are displayed in Hindu–Arabic numerals as well as the numerals used in the country itself.

Ciphered Systems

A ciphered numeration system is one in which there are numerals for numbers up to and including the base and for multiples of the base. The numbers represented by a particular set of numerals is the sum of the values of the numerals.

Ciphered numeration systems require the memorization of many different symbols but have the advantage that numbers can be written in a compact form. The ciphered numeration system that we discuss is the Ionic Greek system (see Table 4.4). The Ionic Greek system was developed in about 3000 B.C., and it used letters of their alphabet for numerals. Other ciphered systems include the Hebrew, Coptic, Hindu, Brahmin, Syrian, Egyptian Hieratic, and early Arabic systems.

TABLE 4.4 Ionic Greek Numerals

1	α	alpha	60	ξ	xi
2	β	beta	70	o	omicron
3	γ	gamma	80	π	pi
4	δ	delta	90	Ϙ	koph*
5	ϵ	epsilon	100	ρ	rho
6	ϛ	vau*	200	σ	sigma
7	ζ	zeta	300	τ	tau
8	η	eta	400	υ	upsilon
9	θ	theta	500	ϕ	phi
10	ι	iota	600	χ	chi
20	κ	kappa	700	ψ	psi
30	λ	lambda	800	ω	omega
40	μ	mu	900	ϡ	sampi*
50	ν	nu			

*Taken from the Phoenician alphabet.

Since the Greek alphabet contains 24 letters but 27 symbols were needed, the Greeks borrowed the symbols ϛ, Ϙ and ϡ from the Phoenician alphabet.

The number $24 = 20 + 4$. When 24 is written as a Greek numeral, the plus sign is omitted:

$$24 = \kappa\delta$$

The number 996 written as a Greek numeral is ϡϘϛ.

When a prime (′) is placed above a number, it multiplies that number by 1000. For example,

$$\beta' = 2 \times 1000 = 2000$$
$$\sigma' = 200 \times 1000 = 200{,}000$$

EXAMPLE 8 *The Ionic Greek System: A Ciphered System*

Write $\phi\,\nu\,\gamma$ as a Hindu–Arabic numeral.

SOLUTION: $\phi = 500$, $\nu = 50$, and $\gamma = 3$. Adding these numbers gives 553.

EXAMPLE 9 *Writing an Ionic Greek Numeral*

Write 9432 as an Ionic Greek numeral.

SOLUTION:

$$\begin{aligned} 9432 &= 9000 + 400 + 30 + 2 \\ &= (9 \times 1000) + 400 + 30 + 2 \\ &= \theta' \quad\quad \upsilon \quad \lambda \quad \beta \\ &= \theta'\upsilon\,\lambda\,\beta \end{aligned}$$

▲

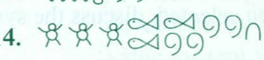
SECTION 4.1 EXERCISES

Concept/Writing Exercises

1. What is the difference between a number and a numeral?
2. List four numerals given in this section that may be used to represent the number ten.
3. What is a system of numeration?
4. List four numerals given in this section that may be used to represent the number one hundred.
5. What is the name of the system of numeration that we presently use?
6. Explain how numbers are represented in an additive numeration system.
7. Explain how numbers are represented in a multiplicative numeration system.
8. Explain how numbers are represented in a ciphered numeration system.

Practice the Skills

In Exercises 9–14, write the numeral as a Hindu–Arabic numeral.

9. 
10. 
11. 
12. 
13. 
14. 

In Exercises 15–20, write the numeral as an Egyptian numeral.

15. 634
16. 752
17. 2045
18. 1812
19. 173,845
20. 3,235,614

In Exercises 21–32, write the numeral as a Hindu–Arabic numeral.

21. XIX
22. XVI
23. DXLVII
24. DLXXV
25. MCDXCII
26. MCMXVIII
27. MMCMXLVI
28. MDCCXLVI
29. $\overline{\text{X}}$MMDCLXVI
30. $\overline{\text{L}}$MCMXLIV
31. $\overline{\text{IX}}$CDLXIV
32. $\overline{\text{V}}$MCCCXXXIII

In Exercises 33–44, write the numeral as a Roman numeral.

33. 59
34. 94
35. 134
36. 269
37. 2005
38. 4285
39. 4793
40. 6274
41. 9999
42. 14,315
43. 20,644
44. 99,999

In Exercises 45–52, write the numeral as a Hindu–Arabic numeral.

45. 七十四
46. 六十二
47. 四千零八十一
48. 三千零二十九

49. 八千五百五十

50. 三千四百八十七

51. 四千零三

52. 五千六百零二

In Exercises 53–60, write the numeral as a traditional Chinese numeral.

53. 53
54. 178
55. 378
56. 2001
57. 4260
58. 6905
59. 7056
60. 3009

In Exercises 61–66, write the numeral as a Hindu–Arabic numeral.

61. $\tau\ \mu\ \alpha$
62. ψ λ ϛ
63. $\kappa'\beta'\phi\epsilon$
64. $\rho'\ \nu'\omega\ \iota\ \gamma$
65. $\theta'\chi\ \zeta$
66. δ′ ϡ Ϙ θ

In Exercises 67–72, write the numeral as an Ionic Greek numeral.

67. 59
68. 178
69. 726
70. 2001
71. 82,704
72. 690,540

In Exercises 73–75, compare the advantages and disadvantages of a ciphered system of numeration with those of the named system.

73. An additive system
74. A multiplicative system
75. The Hindu–Arabic system

In Exercises 76–79, write the numeral as numerals in the indicated systems of numeration.

76. 𓆼∩∩| in Hindu–Arabic, Roman, traditional Chinese, and Greek
77. MCMXXXVI in Hindu–Arabic, Egyptian, Greek, and traditional Chinese
78. 五百二十七 in Hindu–Arabic, Egyptian, Roman, and Greek
79. $\upsilon\kappa\beta$ in Hindu–Arabic, Egyptian, Roman, and traditional Chinese

Challenge Problems/Group Activities

80. Write the Roman numeral for 999,999.
81. Write the Ionic Greek numeral for 999,999.
82. Make up your own additive system of numeration and indicate the symbols and rules used to represent numbers. Using your system of numeration, write
    a) your age.
    b) the year you were born.
    c) the current year.

Recreational Mathematics

83. Without using any type of writing instrument, what can you do to make the following incorrect statement a correct statement?

$$XI + I = X$$

84. Words and numbers that read the same both backward and forward are called *palindromes.* Some examples are the words CIVIC and RACECAR, and the numbers 121 and 32523. Using Roman numerals, list the last year that was a palindrome.
85. Which year in the past 2000 years required the most Roman numerals to write? Write out the year in Roman numerals.

Internet/Research Activity

86. In this section we discussed Egyptian hieroglyphics, Ionic Greek numerals, and other numeration systems. Select either Egypt or Greece.
    a) Give the current numerals used in that country.
    b) Explain how their current system of numeration works.

    If more than one numeration system is used in the country you selected, discuss the system most commonly used.

4.2 PLACE-VALUE OR POSITIONAL-VALUE NUMERATION SYSTEMS

Eighteenth-century mathematician Pierre Simon, Marquis de Laplace, speaking of the positional principle, said: "The idea is so simple that this very simplicity is the reason for our not being sufficiently aware of how much attention it deserves."

Today the most common type of numeration system is the place-value system. The Hindu–Arabic numeration system, used in the United States and many other countries, is an example of a place-value system. In a ***place-value system***, which is also called a ***positional value system***, the value of the symbol depends on its position in the representation of the number. For example, the 2 in 20 represents 2 tens, and the 2 in 200 represents 2 hundreds. A true positional-value system requires a ***base*** and a set of symbols, including a symbol for zero and one for each counting number less than the base. Although any number can be written in any base, the most common positional system is the base 10 system which is called the ***decimal number system***.

The Hindus in India are credited with the invention of zero and the other symbols used in our system. The Arabs, who traded regularly with the Hindus, also adopted the system, thus the name Hindu–Arabic. Not until the middle of the fifteenth century, however, did the Hindu–Arabic numerals take the form we know today.

The Hindu–Arabic numerals and the positional system of numeration revolutionized mathematics by making addition, subtraction, multiplication, and division much easier to learn and very practical to use. Merchants and traders no longer had to depend on the counting board or abacus. The first group of mathematicians, who computed with the Hindu–Arabic system rather than with pebbles or beads on a wire, were known as the "algorists."

In the Hindu–Arabic system, the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are called ***digits***. The base 10 system was developed from counting on fingers, and the word *digit* comes from the Latin word for fingers.

The positional values in the Hindu–Arabic system are

$$\ldots, (10)^5, (10)^4, (10)^3, (10)^2, 10, 1$$

To evaluate a number in the Hindu–Arabic system, we multiply the first digit on the right by 1. We multiply the second digit from the right by the base, 10. We multiply the third digit from the right by the base squared, 10^2 or 100. We multiply the fourth digit from the right by the base cubed, 10^3 or 1000, and so on. In general, we multiply the digit n places from the right by 10^{n-1}. Therefore, we multiply the digit eight places from the right by 10^7. Using the place-value rule, we can write a number in ***expanded form***. The number 1234 written in expanded form is

$$1234 = (1 \times 10^3) + (2 \times 10^2) + (3 \times 10) + (4 \times 1)$$

or

$$(1 \times 1000) + (2 \times 100) + (3 \times 10) + 4$$

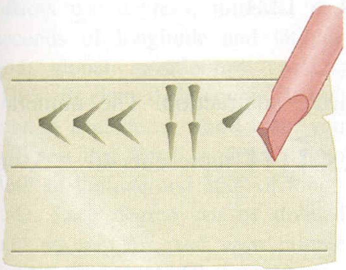
The oldest known numeration system that resembled a place-value system was developed by the Babylonians in about 2500 B.C. Their system resembled a place-value system with a base of 60, a sexagesimal system. It was not a true place-value system because it lacked a symbol for zero. The lack of a symbol for zero led to a great deal of ambiguity and confusion. Table 4.5 gives the Babylonian numerals.

The positional values in the Babylonian system are

$$\ldots, (60)^3, (60)^2, 60, 1$$

DID YOU KNOW

Babylonian Numerals



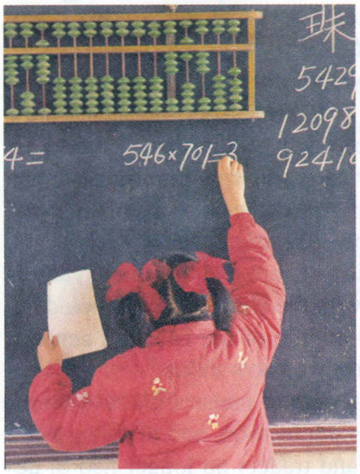
The form Babylonian numerals took is directly related to their writing materials. Babylonians used a reed (later a stylus) to make their marks in wet clay. The end could be used to make a thin wedge, 𒁹, which represents a unit, or a wider wedge, 𒌋, which represents 10 units. The clay dried quickly, so the writings tended to be short but extremely durable.

TABLE 4.5 Babylonian Numerals

Babylonian Numerals	𒁹	𒌋
Hindu–Arabic numerals	1	10

DID YOU KNOW

Counting Boards



One of the earliest counting devices, used in most ancient civilizations, was the counting board. On such a board, each column represents a positional value. The number of times a value occurs is represented by markers (beads, stones, sticks) in the column. An empty column signifies "no value." The widespread use of counting boards meant that Europeans were already long accustomed to working with positional values when they were introduced to Hindu–Arabic numerals in the fifteenth century. Some people in China, Japan, Russia, Eastern Europe, and the United States still commonly use a type of counting board known as the abacus to perform routine computations.

In a Babylonian numeral, a gap is left between the characters to distinguish between the various place values. From right to left, the sum of the first group of numerals is multiplied by 1. The sum of the second group is multiplied by 60. The sum of the third group is multiplied by $(60)^2$, and so on.

EXAMPLE 1 *The Babylonian System: A Positional Value System*

Write ◀◀ ◀◀▼▼▼▼ as a Hindu–Arabic numeral.

SOLUTION:

$$\underbrace{◀◀}_{60\text{'s}} \qquad \underbrace{◀◀▼▼▼▼}_{\text{units}}$$

$$\underbrace{10+10}_{60\text{'s}} \qquad \underbrace{10+10+1+1+1+1}_{\text{units}}$$

$$(20 \times 60) + (24 \times 1)$$
$$1200 + 24 = 1224$$

The Babylonians used the symbol ⊤ to indicate subtraction. The numeral ◀⊤▼▼ represents 10 − 2, or 8. The numeral ◀◀◀◀⊤▼▼▼ represents 40 − 3, or 37 in base 10 or decimal notation.

EXAMPLE 2 *From Babylonian to Hindu–Arabic Numerals*

Write ▼▼ ◀▼ ◀◀⊤▼▼ as a Hindu–Arabic numeral.

SOLUTION: The place value of these three groups of numerals from left to right is

	$(60)^2$,	60,	1
or	3600,	60,	1

The numeral in the group on the right has a value of 20 − 2, or 18. The numeral in the center group has a value of 10 + 1, or 11. The numeral on the left represents 1 + 1, or 2. Multiplying each group by its positional value gives

$$\begin{aligned}&(2 \times 60^2) + (11 \times 60) + (18 \times 1)\\ &= (2 \times 3600) + (11 \times 60) + (18 \times 1)\\ &= 7200 + 660 + 18\\ &= 7878\end{aligned}$$

To explain the procedure used to convert from a Hindu–Arabic numeral to a Babylonian numeral, we will consider a length of time. How can we change 9820 seconds into hours, minutes, and seconds? Since there are 3600 seconds in an hour

DID YOU KNOW

On the High Seas

Vestiges of the Babylonian sexagesimal system are still with us today, especially in navigation. Navigators use degrees, minutes, and seconds of longitude and latitude and global positioning systems (GPS) to chart their course. If you look at a globe or world map, you will see that Earth is divided into 360° of latitude and 360° of longitude. Each degree can be divided into 60 minutes, and each minute can be divided into 60 seconds.

Early explorers had to have an easy means of computing angles as they guided their ships by the stars. Base 60 easily divides into halves, thirds, fourths, fifths, sixths, tenths, twelfths, fifteenths, twentieths, and thirtieths, making such computations easier. Hence, the use of a base 60 system became popular.

(60 seconds to a minute and 60 minutes to an hour), we can find the number of hours in 9820 seconds by dividing 9820 by 60^2, or 3600.

$$
\begin{array}{r l}
2 & \leftarrow \text{Hours} \\
3600\overline{)9820} & \\
7200 & \\
\hline
2620 & \leftarrow \text{Remaining seconds}
\end{array}
$$

Now we can determine the number of minutes by dividing the remaining seconds by 60, the number of seconds in a minute.

$$
\begin{array}{r l}
43 & \leftarrow \text{Minutes} \\
60\overline{)2620} & \\
2400 & \\
\hline
220 & \\
180 & \\
\hline
40 & \leftarrow \text{Remaining seconds}
\end{array}
$$

Since the remaining number of seconds, 40, is less than the number of seconds in a minute, our task is complete.

$$9820 \text{ sec} = 2 \text{ hr, } 43 \text{ min, and } 40 \text{ sec}$$

The same procedure is used to convert a decimal (base 10) number to a Babylonian number or any number in a different base.

EXAMPLE 3 *From Hindu–Arabic to Babylonian Numerals*

Write 2519 as a Babylonian numeral.

SOLUTION: The Babylonian numeration system has positional values of

$$\ldots, 60^3, 60^2, 60, 1$$

which can be expressed as

$$\ldots, 216000, 3600, 60, 1$$

The largest positional value less than or equal to 2519 is 60. To determine how many groups of 60 are in 2519, divide 2519 by 60.

$$
\begin{array}{r l}
41 & \leftarrow \text{Groups of 60} \\
60\overline{)2519} & \\
240 & \\
\hline
119 & \\
60 & \\
\hline
59 & \leftarrow \text{Units remaining}
\end{array}
$$

Thus, $2519 \div 60 = 41$ with remainder 59. There are 41 groups of 60 and 59 units remaining. Because the remainder, 59, is less than the base, 60, no further division is necessary. The remainder represents the number of units when the number is

DID YOU KNOW

Sacred Mayan Glyphs

3

4

In addition to their base 20 numerals the Mayans had a holy numeration system used by priests to create and maintain calendars. They used a special set of hieroglyphs that consisted of pictograms of Mayan gods. For example, the number 3 was represented by the god of wind and rain, the number 4 by the god of sun.

written in expanded form. Therefore, $2519 = (41 \times 60) + (59 \times 1)$. When written as a Babylonian numeral, 2519 is

EXAMPLE 4 *Using Division to Determine a Babylonian Numeral*

Write 6270 as a Babylonian numeral.

SOLUTION: Divide 6270 by the largest positional value less than or equal to 6270. That value is 3600.

$$6270 \div 3600 = 1 \text{ with remainder } 2670$$

There is one group of 3600 in 6270. Next divide the remainder 2670 by 60 to determine the number of groups of 60 in 2670.

$$2670 \div 60 = 44 \text{ with remainder } 30$$

There are 44 groups of 60 and 30 units remaining.

$$6270 = (1 \times 60^2) + (44 \times 60) + (30 \times 1)$$

Thus, 6270 written as a Babylonian numeral is

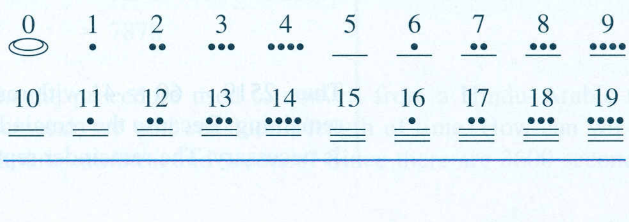
Another place-value system is the Mayan numeration system. The Mayans, who lived on the Yucatan Peninsula in present day Mexico, developed a sophisticated numeration system based on their religious and agricultural calendar. The numbers in this system are written vertically rather than horizontally, with the units position on the bottom. In the Mayan system, the number in the bottom row is to be multiplied by 1. The number in the second row from the bottom is to be multiplied by 20. The number in the third row is to be multiplied by 18×20, or 360. You probably expected the number in the third row to be multiplied by 20^2 rather than 18×20. It is believed that the Mayans used 18×20 so that their numeration system would conform to their calendar of 360 days. The positional values above 18×20 are 18×20^2, 18×20^3, and so on.

Positional values in the Mayan system

$\ldots 18 \times (20)^3$,	$18 \times (20)^2$,	18×20,	20,	1
or… 144,000,	7200,	360,	20,	1

The digits 0, 1, 2, 3, …, 19 of the Mayan systems are formed by a simple grouping of dots and lines, as shown in Table 4.6.

TABLE 4.6 Mayan Numerals



DID YOU KNOW

Numerals of the World

Although most countries presently use a place value (or positional value) numeration system with base 10, the numerals used for the digits differ by country. The photo of the artwork entitled *Numbers* by Jan Fleck shows numerals currently used in many countries of the world. For example, in Burmese, the numeral ၃ has a value of 3.

EXAMPLE 5 *The Mayan System: A Positional Value System*

Write

••••
—
••
•••
—
—

as a Hindu–Arabic numeral.

SOLUTION: In the Mayan numeration system, the first three positional values are

$$18 \times 20$$
$$20$$
$$1$$

$$\begin{aligned} \overset{\bullet\bullet\bullet\bullet}{\text{—}} &= 9 \times (18 \times 20) = 3240 \\ \bullet\bullet &= 2 \times 20 \qquad\quad\;\; = \;\;\;40 \\ \overset{\bullet\bullet\bullet}{=} &= 13 \times 1 \qquad\quad\;\; = \;\;\;\underline{13} \\ & \qquad\qquad\qquad\qquad\;\; 3293 \end{aligned}$$

EXAMPLE 6 *From Mayan to Hindu–Arabic Numerals*

Write

•••
—
•
—
—
••••

as a Hindu–Arabic numeral.

SOLUTION:

$$\begin{aligned} \overset{\bullet\bullet\bullet}{\text{—}} &= 8 \times (18 \times 20) = 2880 \\ \overset{\bullet}{=} &= 11 \times 20 \qquad\quad\; = \;\;220 \\ \bullet\bullet\bullet\bullet &= 4 \times 1 \qquad\qquad\; = \;\;\;\;\underline{4} \\ & \qquad\qquad\qquad\qquad\;\; 3104 \end{aligned}$$

EXAMPLE 7 *From Hindu–Arabic to Mayan Numerals*

Write 4025 as a Mayan numeral.

SOLUTION: To convert from a Hindu–Arabic to a Mayan numeral, we use a procedure similar to the one used to convert to a Babylonian numeral. The Mayan positional values are . . . , 7200, 360, 20, 1. The greatest positional value less than or equal to 4025 is 360. Divide 4025 by 360.

$$4025 \div 360 = 11 \text{ with remainder } 65$$

There are 11 groups of 360 in 4025. Next, divide the remainder, 65, by 20.

$$65 \div 20 = 3 \text{ with remainder } 5$$

There are 3 groups of 20 with five units remaining.

$$4025 = (11 \times 360) + (3 \times 20) + (5 \times 1)$$

4025 written as a Mayan numeral is

$$\left.\begin{aligned} 11 \times 360 \\ 3 \times 20 \\ 5 \times 1 \end{aligned}\right\} = \begin{aligned} &\overset{\bullet}{=} \\ &\bullet\bullet\bullet \\ &\text{—} \end{aligned}$$

> **TIMELY TIP** Notice that changing a number from the Babylonian or Mayan numeration System *to the Hindu–Arabic* (or decimal or base 10) system involves *multiplication*. Changing a number *from the Hindu–Arabic system* to the Babylonian or Mayan numeration system involves *division*.

SECTION 4.2 EXERCISES

Concept/Writing Exercises

1. What is the most common type of numeration system used in the world today?
2. What is another name for a place-value system?
3. Consider the numbers 40 and 400 in the Hindu–Arabic numeration system. What does the 4 represent in each number?
4. What is the most common base used for a positional value system? Explain why you believe the base you named is the most common base.
5. In a true positional-value system, what symbols are required?
6. a) What is the base in the Hindu–Arabic numeration system?
   b) What are the digits in the Hindu–Arabic numeration system?
7. Explain how to write a number in expanded form in a positional-value numeration system.
8. Why was the Babylonian system not a true place-value system?
9. a) The Babylonian system did not have a symbol for zero. Why did this lead to some confusion?
   b) Write the numbers 133 and 7980 as Babylonian numerals.
10. Consider the Babylonian number represented by . Give two numbers in Hindu–Arabic numerals this number may represent. Explain your answer.
11. List the first five positional values, starting with the units position, for the Mayan numeration system.
12. Describe two ways that the Mayan place-value system differs from the Hindu–Arabic place-value system.

Practice the Skills

In Exercises 13–24, write the Hindu–Arabic numeral in expanded form.

13. 63
14. 75
15. 359
16. 562
17. 897
18. 3769
19. 4387
20. 23,468
21. 16,402
22. 125,678
23. 346,861
24. 3,765,934

In Exercises 25–30, write the Babylonian numeral as a Hindu–Arabic numeral.

25. 
26. 
27. 
28. 
29. 
30. 

In Exercises 31–36, write the numeral as a Babylonian numeral.

31. 88
32. 97
33. 295
34. 512
35. 3685
36. 3030

In Exercises 37–42, write the Mayan numeral as a Hindu–Arabic numeral.

37. 
38.
39. 
40. 
41. 
42. 

In Exercises 43–48, write the numeral as a Mayan numeral.

43. 17
44. 257
45. 297
46. 406
47. 2163
48. 1978

49. ***Comparisons of Systems*** Compare the advantages and disadvantages of a place-value system with those of
   a) additive numeration systems.
   b) multiplicative numeration systems.
   c) ciphered numeration systems.
50. ***Your Own System*** Create your own place-value system. Write 2005 in your system.

In Exercises 51 and 52, write the numeral in the indicated systems of numeration.

51.  in Hindu–Arabic and Mayan

52.  in Hindu–Arabic and Babylonian

In Exercises 53 and 54, suppose a place-value numeration system has base *, with digits represented by the symbols*  *and* *. Write each expression in expanded form.*

53. 

54. 

Challenge Problems/Group Activities

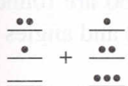
55. a) Is there a largest number in the Babylonian numeration system? Explain.
 b) Write the Babylonian numeral for 999,999.

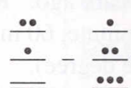
56. a) Is there a largest number in the Mayan numeration system? Explain.
 b) Write the Mayan numeral for 999,999.

In Exercises 57–60, first convert each numeral to a Hindu–Arabic numeral and then perform the indicated operation. Finally, convert the answer back to a numeral in the original numeration system.

57. 

58. 

59. 

60. 

Recreational Mathematics

61. Hidden in the box are the names of the four different types of systems of numeration we discussed in this chapter: ADDITIVE, MULTIPLICATIVE, CIPHERED, and PLACE-VALUE (without the hyphen). You find them by going box by box, but the boxes you move between must touch vertically, horizontally, or diagonally. You can use the same box more than once when spelling out a word. Find the names of the numeration systems. Make sure you understand how each system of numeration works.

M	L	P	R	E
A	I	A	D	H
V	C	T	D	P
E	A	L	I	C
A	M	U	E	V

Internet/Research Activities

62. Investigate and write a report on the development of the Hindu–Arabic system of numeration. Start with the earliest records of this system in India.

63. The Arabic numeration system currently in use is a base 10 positional-value system, which uses different symbols than the Hindu–Arabic numeration system. Write the symbols used in the Arabic system of numeration and their equivalent symbols in the Hindu–Arabic numeration system. Write 54, 607, and 2000 in Arabic numerals.

4.3 OTHER BASES

The positional values in the Hindu–Arabic numeration system are

$$\ldots, (10)^4, (10)^3, (10)^2, 10, 1$$

The positional values in the Babylonian numeration system are

$$\ldots, (60)^4, (60)^3, (60)^2, 60, 1$$

The numbers 10 and 60 are called the ***bases*** of the Hindu–Arabic and Babylonian systems, respectively.

DID YOU KNOW

Beyond Their Fingers

The Kewa people of Papua, New Guinea, have gone well beyond counting on their fingers: They use the entire upper body. Going from the little finger of one hand, down the elbow, to shoulder to head to shoulder, down the other elbow, to the little finger of the opposite hand provides them with a count of 68.

Any counting number greater than 1 may be used as a base for a positional-value numeration system. If a positional-value system has a base b, then its positional values will be

$$\ldots, b^4, b^3, b^2, b, 1$$

The positional values in a base 8 system are

$$\ldots, 8^4, 8^3, 8^2, 8, 1$$

and the positional values in a base 2 system are

$$\ldots, 2^4, 2^3, 2^2, 2, 1$$

As we indicated earlier, the Mayan numeration system is based on the number 20. It is not, however, a true base 20 positional-value system. Why not?

The reason for the almost universal acceptance of base 10 numeration systems is that most human beings have 10 fingers. Even so, there are still some positional-value numeration systems that use bases other than 10. Some societies are still using a base 2 numeration system. They include some groups of people in Australia, New Guinea, Africa, and South America. Bases 3 and 4 are also used in some areas of South America. Base 5 systems were used by some primitive tribes in Bolivia, but the tribes are now extinct. The pure base 6 system occurs only sparsely in Northwest Africa. Base 6 also occurs in other systems in combination with base 12, the *duodecimal system*.

We continue to see remains of other base systems in many countries. For example, there are 12 inches in a foot, 12 months in a year. Base 12 is also evident in the dozen, the 24-hour day, and the gross (12×12). English uses the word *score* to mean 20, as in "Four score and seven years ago." Remains of base 60 are found in measurements of time (60 seconds to a minute, 60 minutes to an hour) and angles (60 seconds to one minute, 60 minutes to one degree).

The base 2, or ***binary system***, has become very important because it is the internal language of the computer. For example, when a grocery store's cash register computer records the price of your groceries by using a scanning device, the bar codes it scans on the packages are in binary form. Computers use a two-digit "alphabet" that consists of the numerals 0 and 1. Every character on a standard keyboard can be represented by a combination of those two numerals. A single numeral such as 0 or 1 is called a ***bit***. Other bases that computers make use of are base 8 and base 16. A group of eight bits is called a ***byte***. In the American Standard Code for Information Interchange (ASCII) code, used in most computers, the byte 01000001 represents the character A, 01100001 represents the character a, 00110000 represents the character 0, and 00110001 represents the character 1.

A place-value system with base b must have b distinct symbols, one for zero and one for each number less than the base. A base 6 system must have symbols for the numbers 0, 1, 2, 3, 4, and 5. All numbers in base 6 are constructed from these 6 symbols. A base 8 system must have symbols for 0, 1, 2, 3, 4, 5, 6, and 7. All numbers in base 8 are constructed from these 8 symbols, and so on.

A number in a base other than base 10 will be indicated by a subscript to the right of the number. Thus, 123_5 represents a number in base 5. The number 123_6 represents a number in base 6. The value of 123_5 is not the same as the value of 123_{10}, and the value of 123_6 is not the same as the value 123_{10}. A base 10 number may be written without a subscript. For example 123 means 123_{10} and 456 means 456_{10}. For clarity in certain problems, we will use the subscript 10 to indicate a number in base 10.

Remember the symbols that represent the base itself, in any base b, are 10_b. For example, in base 5, the symbols 10_5 represent the number 5. Note that $10_5 = 1 \times 5 + 0 \times 1 = 5 + 0 = 5_{10}$, or the number 5 in base 10. The symbols 10_5 mean one group of 5 and no units. In base 6, the symbols 10_6 represent the number 6. The symbols 10_6 represent one group of 6 and no units, and so on.

To change a number in a base other than 10 to a base 10 number, we follow the same procedure we used in Section 4.2 to change the Babylonian and Mayan numbers to base 10 numbers. Multiply each digit in the number by its respective positional value. Then find the sum of the products.

EXAMPLE 1 *Converting from Base 6 to Base 10*

Convert 453_6 to base 10.

SOLUTION: In base 6, the positional values are $\ldots, 6^3, 6^2, 6, 1$. In expanded form,

$$\begin{aligned} 453_6 &= (4 \times 6^2) + (5 \times 6) + (3 \times 1) \\ &= (4 \times 36) + (5 \times 6) + (3 \times 1) \\ &= 144 + 30 + 3 \\ &= 177 \end{aligned}$$

In Example 1, the units digit in 453_6 is 3. Notice that 3_6 has the same value as 3_{10} since both are equal to 3 units. That is, $3_6 = 3_{10}$. If n is a digit less than the base b, and the base b is less than or equal to 10, then $n_b = n_{10}$.

EXAMPLE 2 *Converting from Base 8 to Base 10*

Convert 3615_8 to base 10.

SOLUTION:

$$\begin{aligned} 3615_8 &= (3 \times 8^3) + (6 \times 8^2) + (1 \times 8) + (5 \times 1) \\ &= (3 \times 512) + (6 \times 64) + (1 \times 8) + (5 \times 1) \\ &= 1536 + 384 + 8 + 5 \\ &= 1933 \end{aligned}$$

A base 12 system must have 12 distinct symbols. In this text, we use the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, and E, where T represents ten and E represents eleven. Why will the numerals 10_{12} and 11_{12} have different meanings than 10 and 11? The number 10_{12} represents 1 group of twelve plus 0 units, or twelve. The number 11_{12} represents 1 group of twelve plus 1 unit, or 13.

EXAMPLE 3 *Converting from Base 12 to Base 10*

Convert $12T6_{12}$ to base 10.

SOLUTION:

$$\begin{aligned} 12T6_{12} &= (1 \times 12^3) + (2 \times 12^2) + (T \times 12) + (6 \times 1) \\ &= (1 \times 1728) + (2 \times 144) + (10 \times 12) + (6 \times 1) \\ &= 1728 + 288 + 120 + 6 \\ &= 2142 \end{aligned}$$

EXAMPLE 4 *Converting from Base 2*

Convert 101101_2 to base 10.

SOLUTION:

$$101101_2 = (1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (0 \times 2) + (1 \times 1)$$
$$= 32 + 0 + 8 + 4 + 0 + 1$$
$$= 45$$

To change a number from a base 10 system to a different base, we will use a procedure similar to the one we used to convert base 10 numbers to Babylonian and Mayan numbers, as was explained in Section 4.2. Divide the base 10 number by the highest power of the new base that is less than or equal to the given number. Record this quotient. Then divide the remainder by the next smaller power of the new base and record this quotient. Repeat this procedure until the remainder is a number less than the new base. The answer is the set of quotients listed from left to right, with the remainder on the far right. This procedure is illustrated in Examples 5 through 7.

EXAMPLE 5 *Convert to Base 8*

Convert 486 to base 8.

SOLUTION: We are converting a number in base 10 to a number in base 8. The positional values in the base 8 system are ..., 8^3, 8^2, 8, 1, or ..., 512, 64, 8, 1. The highest power of 8 that is less than or equal to 486 is 8^2, or 64. Divide 486 by 64.

First digit in answer

$$486 \div 64 = 7 \text{ with remainder } 38$$

Therefore, there are 7 groups of 8^2 in 486. Next divide the remainder, 38, by 8.

Second digit in answer

$$38 \div 8 = 4 \text{ with remainder } 6$$

Third digit in answer

There are 4 groups of 8 in 38 and 6 units remaining. Since the remainder, 6, is less than the base, 8, no further division is required.

$$486 = (7 \times 64) + (4 \times 8) + (6 \times 1)$$
$$= (7 \times 8^2) + (4 \times 8) + (6 \times 1)$$
$$= 746_8$$

Notice that we placed the subscript 8 to the right of 746 to show that it is a base 8 number.

EXAMPLE 6 *Convert to Base 3*

Convert 273 to base 3.

SOLUTION: The place values in the base 3 system are ..., 3^6, 3^5, 3^4, 3^3, 3^2, 3, 1, or ..., 729, 243, 81, 27, 9, 3, 1. The highest power of the base that is less than or equal to 273 is 3^5, or 243. Successive divisions by the powers of the base give the following result.

$$273 \div 243 = 1 \text{ with remainder } 30$$
$$30 \div 81 = 0 \text{ with remainder } 30$$
$$30 \div 27 = 1 \text{ with remainder } 3$$
$$3 \div 9 = 0 \text{ with remainder } 3$$
$$3 \div 3 = 1 \text{ with remainder } 0$$

The remainder, 0, is less than the base, 3, so no further division is necessary. To obtain the answer, list the quotients from top to bottom followed by the remainder in the last division.

The number 273 can be represented as one group of 243, no groups of 81, one group of 27, no groups of 9, one group of 3, and no units.

$$\begin{aligned} 273 &= (1 \times 243) + (0 \times 81) + (1 \times 27) + (0 \times 9) + (1 \times 3) + (0 \times 1) \\ &= (1 \times 3^5) + (0 \times 3^4) + (1 \times 3^3) + (0 \times 3^2) + (1 \times 3) + (0 \times 1) \\ &= 101010_3 \end{aligned}$$

▲

EXAMPLE 7 *Convert to Base 12*

Convert 558 to base 12.

SOLUTION: The place values in base 12 are ..., 12^3, 12^2, 12, 1, or ..., 1728, 144, 12, 1. The highest power of the base that is less than or equal to 558 is 12^2, or 144.

$$558 \div 144 = 3 \text{ with remainder } 126$$
$$126 \div 12 = \text{T with remainder } 6$$

(Remember that T is used to represent 10 in base 12.)

$$558 = (3 \times 12^2) + (\text{T} \times 12) + (6 \times 1) = 3\text{T}6_{12}$$

▲

TIMELY TIP It is important to remember the following items presented in this section.

- If a number is shown without a base, we assume the number is a base 10 number, and
- When converting a base 10 number to a different base, your answer should never contain a digit greater than or equal to that different base.

You should also remember that changing a number given in a base other than 10 to a number in base 10 involves multiplication. Changing a base 10 number to a number in a different base involves division.

MATHEMATICS *Everywhere*

We use our place-value system daily without thinking of its complexity. The place-value system has come a long way from the first and oldest type of numeration system, the additive numeration system. The changes in numeration systems evolved slowly. Most countries now use a place-value system. For example, the numeration system used in China today is different from the traditional system discussed in this chapter. The present-day system in China is a positional-value system rather than a multiplicative system, and in some areas of China, 0 is used as the numeral for zero. Often when you travel to ethnic areas of cities or to foreign countries, numbers are indicated using both that country's numerals and Hindu–Arabic numerals, as shown in the photo.

It is likely that when new numeration systems are presented they appear abstract to the public. Abstract mathematics often becomes the basis for important discoveries and inventions in the future. That is the case with the binary numeration system. When the binary numeration system was first introduced, nobody could have dreamed that it would form the basis for our computers. Today, computers are found everywhere: in our cars, cameras, watches, calculators, sewing machines; at checkout counters; and in hundreds of other places. When a number or letter is entered into a computer it is converted internally into a binary number. The computations within a computer are made using binary arithmetic, and the binary answer is converted back to a decimal number for us to read. Computers also use octal (base 8) and hexadecimal (base 16) numeration systems.

Computers make use of three numeration systems: the ***binary*** (base 2), ***octal*** (base 8), and ***hexadecimal*** (base 16) numeration systems. Computers and calculators use the binary system to perform their internal computations. The binary number system contains only two digits, 0 and 1. All numbers we enter into a computer are converted internally into a series of 0's and 1's. When a computer performs a calculation it treats 0 as an "off" switch and 1 as an "on" switch. Using these electronic switches, the computer performs calculations using binary numbers, and then the internal result is converted back to a decimal number for us to view.

The octal system is used by computer programmers who work with internal computer codes. In a computer, the central processing unit (CPU) often uses the hexadecimal system to convey information to the printer and other output devices.

We have already given examples of converting numbers given in the binary system and the octal system to numbers in the decimal system, and vice versa. Now let's work an example using the hexadecimal system. Since a hexadecimal system contains 16 symbols, it is treated similarly to a base 12 system in that we need to use additional symbols for numerals. See Examples 3 and 7.

EXAMPLE 8 *Convert to and from Base 16*

In this example, let the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F represent the numerals in a base 16 system where A through F represent ten through fifteen, respectively.

a) Convert $7DE_{16}$ to base 10. b) Convert 6713 to base 16.

SOLUTION:

a) In a base 16 system the positional values are . . . , 16^3, 16^2, 16, 1 or . . . 4096, 256, 16, 1. Since D has a value of 13 and E has a value of 14, we perform the following calculation.

$$\begin{aligned} 7DE_{16} &= (7 \times 16^2) + (D \times 16) + (E \times 1) \\ &= (7 \times 256) + (13 \times 16) + (14 \times 1) \\ &= 1792 + 208 + 14 \\ &= 2014 \end{aligned}$$

b) The highest power of base 16 less than or equal to 6713 is 16^3, or 4096. If we obtain a quotient greater than nine but less than sixteen, we will use the corresponding letter A through F.

$$\begin{aligned} 6713 \div 4096 &= 1 \text{ with remainder } 2617 \\ 2617 \div 256 &= A \text{ with remainder } 57 \\ 57 \div 16 &= 3 \text{ with remainder } 9 \end{aligned}$$

Note that A has a value of ten

Thus, $6713 = 1A39_{16}$.

SECTION 4.3 EXERCISES

Concept/Writing Exercises

1. In your own words, explain how to change a number in a base other than base 10 to base 10.
2. In your own words, explain how to change a number in base 10 to a base other than base 10.

Practice the Skills

In Exercises 3–20, convert the numeral to a numeral in base 10.

3. 5_6
4. 60_7
5. 42_5
6. 101_2
7. 1011_2
8. 1101_2
9. 84_{12}
10. 21021_3
11. 565_8
12. 654_7
13. 20432_5
14. 101111_2
15. 4003_6
16. $123E_{12}$
17. 123_8
18. 2043_8
19. 14705_8
20. 67342_9

In Exercises 21–36, convert the base 10 numeral to a numeral in the base indicated.

21. 8 to base 2
22. 16 to base 2
23. 23 to base 2
24. 243 to base 6
25. 635 to base 6
26. 908 to base 4
27. 2061 to base 12
28. 200 to base 4
29. 529 to base 8
30. 81 to base 3
31. 2867 to base 12
32. 4312 to base 6
33. 1011 to base 2
34. 1589 to base 7
35. 2307 to base 8
36. 13,469 to base 8

In Exercises 37–40, assume that a base 16 positional-value system uses the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F, where A through F represent ten through fifteen, respectively. Convert the numeral to a numeral in base 10. See Example 8.

37. 735_{16}
38. 581_{16}
39. $6D3B7_{16}$
40. $24FEA_{16}$

In Exercises 41–44, convert the numeral to a numeral in base 16. See Example 8.

41. 573
42. 349
43. 5478
44. 34,721

In Exercises 45–50, convert 2005 to a numeral in the base indicated.

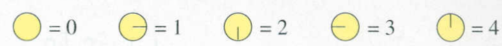
45. 2
46. 3
47. 5
48. 7
49. 12
50. 16

In Exercises 51–56, if any numerals are written incorrectly, explain why.

51. 5013_5
52. 1203_3
53. 674_8
54. 1206_{12}
55. 4086_7
56. 3004_5

Problem Solving

In Exercises 57–60, assume the numerals given are in a base 5 numeration system. The numerals in this system and their equivalent Hindu–Arabic numerals are



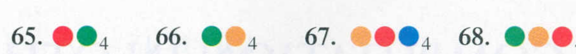
Write the Hindu–Arabic numerals equivalent to each of the following.

In Exercises 61–64, write the Hindu–Arabic numerals in the numeration system discussed in Exercises 57–60.

61. 19
62. 23
63. 74
64. 85

In Exercises 65–68, suppose colors as indicated below represent numerals in a base 4 numeration system.

Write the Hindu–Arabic numerals equivalent to each of the following.



In Exercises 69–72, write the Hindu–Arabic numerals in the base 4 numeration system discussed in Exercises 65–68. You will need to use the colors indicated above to write the answer.

69. 10
70. 15
71. 60
72. 56

73. ***Another Conversion Method*** There is an alternative method for changing a number in base 10 to a different base. This method will be used to convert 328 to base 5.

Dividing 328 by 5 gives a quotient of 65 and a remainder of 3. Write the quotient below the dividend and the remainder on the right, as shown.

$$\begin{array}{r|l} 5 & 328 \quad \text{remainder} \\ & 65 \qquad 3 \end{array}$$

Continue this process of division by 5.

		remainder
5	328	
5	65	3
5	13	0
5	2	3
	0	2

(Answer read upward ↑)

(In the last division, since the dividend, 2, is smaller than the divisor, 5, the quotient is 0 and the remainder is 2.)

Note that the division continues until the quotient is zero. The answer is read from the bottom number to the top number in the remainder column. Thus, $328 = 2303_5$.

a) Explain why this procedure results in the proper answer.

b) Convert 683 to base 5 by this method.

c) Convert 763 to base 8 by this method.

Challenge Problems/Group Activities

74. a) Use the numerals 0, 1, and 2 to write the first 20 numbers in the base 3 numeration system.
b) What is the next number after 222_3?

75. a) ***Your Own Numeration System*** Make up your own base 20 positional-value numeration system. Indicate the 20 numerals you will use to represent the 20 numbers less than the base.
b) Write the numbers 523 and 5293 in your base 20 numeration system.

76. ***Computer Code*** The ASCII code used by most computers uses the last seven positions of an eight-bit byte to represent all the characters on a standard keyboard. How many different orderings of 0's and 1's (or how many different characters) can be made by using the last seven positions of an eight-bit byte?

Recreational Mathematics

77. Find b if $111_b = 43$.

78. Find d if $ddd_5 = 124$.

79. Suppose a base 4 place-value system has its digits represented by colors as follows:

● = 0 ● = 1 ● = 2 ● = 3

a) Determine the value of $●●●●●_4$ in base 10.

b) Write 177 in the base 4 system using only the four colors given in the exercise.

Internet/Research Activities

80. Write a report on how digital computers use the binary number system.

81. We mention at the beginning of this section that some societies still use a base 2 and base 3 numeration system. These societies are in Australia, New Guinea, Africa, and South America. Write a report on these societies, covering the symbols they use and how they combine these symbols to represent numbers in their numeration system.

4.4 COMPUTATION IN OTHER BASES

Addition

When computers perform calculations, they do so in base 2, the binary system. In this section, we explain how to perform calculations in base 2 and other bases.

In a base 2 system, the only digits are 0 and 1, and the place values are

$$\ldots,\ 2^4,\ 2^3,\ 2^2,\ 2,\ 1$$

$$\text{or} \quad \ldots,\ 16,\ 8,\ 4,\ 2,\ 1$$

Suppose we want to add $1_2 + 1_2$. The subscript 2 indicates that we are adding in base 2. Remember the answer to $1_2 + 1_2$ must be written using only the digits 0 and 1. The sum of $1_2 + 1_2$ is 10_2, which represents 1 group of two and 0 units in base 2. Recall that 10_2 means $1(2) + 0(1)$.

If we wanted to find the sum of $10_2 + 1_2$, we would add the digits in the right-hand, or units, column. Since $0_2 + 1_2 = 1_2$, the sum of $10_2 + 1_2 = 11_2$.

We are going to work additional examples and exercises in base 2, so rather than performing individual calculations in every problem, we can construct and use an addition table, Table 4.7, for base 2 (just as we used an addition table in base 10 when we first learned to add in base 10).

TABLE 4.7 Base 2 Addition Table

+	0	1
0	0	1
1	1	10

EXAMPLE 1 *Adding in Base 2*

Add

$$\begin{array}{r} 1101_2 \\ \underline{111_2} \end{array}$$

SOLUTION: Begin by adding the numbers in the right-hand, or units, column. From previous discussion, and as can be seen in Table 4.7, $1_2 + 1_2 = 10_2$. Place the 0 under the units column and carry the 1 to the 2's column, the second column from the right.

Place value of columns

$$\begin{array}{cccc} 2^3 & 2^2 & 2 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & {}^{1}0 & 1 \\ & 1 & 1 & 1 \\ \hline & & & 0_2 \end{array}$$

Now add the three digits in the 2's column, $1_2 + 0_2 + 1_2$. Treat this as $(1_2 + 0_2) + 1_2$. Therefore, add $1_2 + 0_2$ to get 1_2, then add $1_2 + 1_2$ to get 10_2. Place the 0 under the 2's column and carry the 1 to the 2^2 column (the third column from the right).

$$\begin{array}{cccc} 1 & {}^{1}1 & {}^{1}0 & 1 \\ & 1 & 1 & 1 \\ \hline & & 0 & 0_2 \end{array}$$

Now add the three 1's in the 2^2 column to get $(1_2 + 1_2) + 1_2 = 10_2 + 1_2 = 11_2$. Place the 1 under the 2^2 column and carry the 1 to the 2^3 column (the fourth column from the right).

$$\begin{array}{cccc} {}^{1}1 & {}^{1}1 & {}^{1}0 & 1 \\ & 1 & 1 & 1 \\ \hline & 1 & 0 & 0_2 \end{array}$$

Now add the two 1's in the 2^3 column, $1_2 + 1_2 = 10_2$. Place the 10 as follows.

$$\begin{array}{ccccc} & {}^{1}1 & {}^{1}1 & {}^{1}0 & 1 \\ & & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0_2 \end{array}$$

Therefore, the sum is 10100_2. ▲

TABLE 4.8 Base 5 Addition Table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Let's now look at addition in a base 5 system. In base 5, the only digits are 0, 1, 2, 3, and 4, and the positional values are

$$\ldots, 5^4, 5^3, 5^2, 5, 1$$
or
$$\ldots, 625, 125, 25, 5, 1$$

What is the sum of $4_5 + 3_5$? We can consider this to mean $(1 + 1 + 1 + 1) + (1 + 1 + 1)$. We can regroup the seven 1's into one group of five and two units as $(1 + 1 + 1 + 1 + 1) + (1 + 1)$. Thus, the sum of $4_5 + 3_5 = 12_5$ (circled in Table 4.8). Recall that 12_5 means $1(5) + 2(1)$. We can use this same procedure in obtaining the remaining values in the base 5 addition table.

EXAMPLE 2 *Use the Base 5 Addition Table*

Add 42_5
$\underline{33_5}$

SOLUTION: First determine that $2_5 + 3_5$ is 10_5 from Table 4.8. Record the 0 and carry the 1 to the 5's column.

$$\begin{array}{r} {}^{1}4\ 2_5 \\ \underline{3\ 3_5} \\ 0_5 \end{array}$$

Add the numbers in the second column, $(1_5 + 4_5) + 3_5 = 10_5 + 3_5 = 13_5$. Record the 13.

$$\begin{array}{r} {}^{1}4\ 2_5 \\ \underline{3\ 3_5} \\ 1\ 3\ 0_5 \end{array}$$

The sum is 130_5. ▲

EXAMPLE 3 *Add in Base 5*

Add 1234_5
$\underline{2042_5}$

SOLUTION:

$$\begin{array}{r} 1\ {}^{1}2\ {}^{1}3\ 4_5 \\ \underline{2\ 0\ 4\ 2_5} \\ 3\ 3\ 3\ 1_5 \end{array}$$

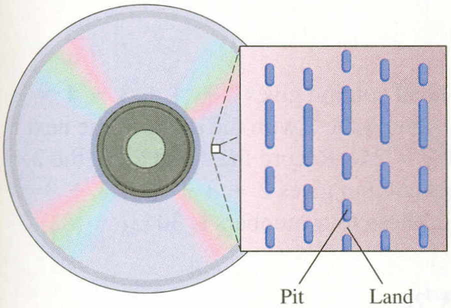
▲

You can develop an addition table for any base and use it to add in that base. As you get more comfortable with addition in other bases, however, you may prefer to add numbers in other bases by using mental arithmetic. To do so, convert the sum of the numbers being added from the given base to base 10 and then convert the base 10 number back into the given base. You must clearly understand how to convert from base 10 to the given base, as discussed in Section 4.3. For example, to add $7_9 + 8_9$, add 7 + 8 in base 10 to get 15_{10} and then mentally convert 15_{10} to 16_9 using the procedure given earlier. Remember, 16_9 when converted to base 10 becomes $1(9) + 6(1)$, or 15. Addition using this procedure is illustrated in Examples 4 and 5.

DID YOU KNOW

Speaking to Machines

For the past 600 years, we have used the Hindu–Arabic system of numeration without change. Our base 10 numeration system seems so obvious to us, perhaps because of our 10 fingers and 10 toes, but it would be rash to think that numbers in other bases are not useful. In fact, one of the most significant numeration systems is the binary system, or base 2. This system, with its elemental simplicity, is what is used by computers to process information and "talk" to one another. When a computer receives a command or data, every character in the command or data must first be converted into a binary number for the computer to understand and use it. Because of the ever-expanding number of computers in use, the users of the binary number system may soon outnumber the users of base 10.



On digital video discs (DVDs) and compact discs (CDs), video and sound are digitally encoded on the underside of the disc in a binary system of pits and "lands" (nonpits). To play the disc, a laser beam tracks along the spiral and is reflected when it hits a land (signal sent = 1), but it is not reflected by the pits (no signal = 0). The binary sequence is then converted into video images and music.

Almost all packaged goods we buy today are marked with a universal product code (UPC), a black-and-white bar code. An optical scanner "reads" the pattern of black and white, thick and thin, and converts it to a binary code that is sent to the scanner's computer, which then calls up the appropriate price and records the sale for inventory purposes.

EXAMPLE 4 *Adding in Base 10; Converting to Base 3*

Add 1022_3
$\underline{2121_3}$

SOLUTION: To solve this problem, make the necessary conversions by using mental arithmetic. $2 + 1 = 3_{10} = 10_3$. Record the 0 and carry the 1.

$$\begin{array}{r} 1\;0\;{}^{1}2\;2_3 \\ \underline{2\;1\;\;2\;1_3} \\ 0_3 \end{array}$$

$1 + 2 + 2 = 5_{10} = 12_3$. Record the 2 and carry the 1.

$$\begin{array}{r} 1\;{}^{1}0\;{}^{1}2\;2_3 \\ \underline{2\;\;1\;\;2\;1_3} \\ 2\;0_3 \end{array}$$

$1 + 0 + 1 = 2_{10} = 2_3$. Record the 2.

$$\begin{array}{r} 1\;{}^{1}0\;{}^{1}2\;2_3 \\ \underline{2\;\;1\;\;2\;1_3} \\ 2\;\;2\;0_3 \end{array}$$

$1 + 2 = 3_{10} = 10_3$. Record the 10.

$$\begin{array}{r} 1022_3 \\ \underline{2121_3} \\ 10220_3 \end{array}$$

The sum is 10220_3.

EXAMPLE 5 *Adding in Base 10; Converting to Base 5*

Add 332_5
344_5
443_5
$\underline{314_5}$

SOLUTION: Adding the digits in the right-hand column gives $2 + 4 + 3 + 4 = 13_{10} = 23_5$. Record the 3 and carry the 2. Adding the 2 with the digits in the next column yields $2 + 3 + 4 + 4 + 1 = 14_{10} = 24_5$. Record the 4 and carry the 2. Adding the 2 with the digits in the left-hand column gives $2 + 3 + 3 + 4 + 3 = 15_{10} = 30_5$. Record both digits. The sum of these four numbers is 3043_5.

$$\begin{array}{rrrr} & {}^{2}3 & {}^{2}3 & 2_5 \\ & 3 & 4 & 4_5 \\ & 4 & 4 & 3_5 \\ & 3 & 1 & 4_5 \\ \hline 3 & 0 & 4 & 3_5 \end{array}$$

Subtraction

Subtraction can also be performed in other bases. Always remember that when you "borrow," you borrow the amount of the base given in the subtraction problem. For example, if subtracting in base 5, when you borrow, you borrow 5. If subtracting in base 12, when you borrow, you borrow 12.

EXAMPLE 6 *Subtracting in Base 5*

Subtract 3032_5
$\underline{-1004_5}$

SOLUTION: We will perform the subtraction in base 10 and convert the results to base 5. Since 4 is greater than 2, we must borrow one group of 5 from the preceding column. This action gives a sum of 5 + 2, or 7 in base 10. Now we subtract 4 from 7; the difference is 3. We complete the problem in the usual manner. The 3 in the second column becomes a 2, 2 − 0 = 2, 0 − 0 = 0, and 3 − 1 = 2.

$$\begin{array}{r} 3032_5 \\ \underline{-1004_5} \\ 2023_5 \end{array}$$

EXAMPLE 7 *Subtracting in Base 12*

Subtract $\begin{array}{r} 468_{12} \\ -295_{12} \end{array}$

SOLUTION: $8 - 5 = 3$. Next we must subtract 9 from 6. Since 9 is greater than 6, borrowing is necessary. We must borrow one group of 12 from the preceding column. We then have a sum of $12 + 6 = 18$ in base 10. Now we subtract 9 from 18, and the difference is 9. The 4 in the left column becomes 3, and $3 - 2 = 1$.

$$\begin{array}{r} 468_{12} \\ -295_{12} \\ \hline 193_{12} \end{array}$$

Multiplication

Multiplication can also be performed in other bases. Doing so is helped by forming a multiplication table for the base desired. Suppose we want to determine the product of $4_5 \times 3_5$. In base 10, 4×3 means there are four groups of three units. Similarly, in a base 5 system, $4_5 \times 3_5$ means there are four groups of three units, or

$$(1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1)$$

Regrouping the 12 units above into groups of five gives

$$(1 + 1 + 1 + 1 + 1) + (1 + 1 + 1 + 1 + 1) + (1 + 1)$$

or two groups of five, and two units. Thus, $4_5 \times 3_5 = 22_5$.

We can construct other values in the base 5 multiplication table in the same way. You may, however, find it easier to multiply the values in the base 10 system and then change the product to base 5 by using the procedure discussed in Section 4.3. Multiplying 4×3 in base 10 gives 12, and converting 12 from base 10 to base 5 gives 22_5.

The product of $4_5 \times 3_5$ is circled in Table 4.9, the base 5 multiplication table. The other values in the table may be found by either method discussed.

TABLE 4.9 Base 5 Multiplication Table

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

EXAMPLE 8 *Using the Base 5 Multiplication Table*

Multiply $\begin{array}{r} 13_5 \\ \times 3_5 \end{array}$

SOLUTION: Multiply as you would in base 10, but use the base 5 multiplication table to find the products. When the product consists of two digits, record the right digit and carry the left digit. Multiplying gives $3_5 \times 3_5 = 14_5$. Record the 4 and carry the 1.

$$\begin{array}{r} {}^{1}13_5 \\ \times\ \ 3_5 \\ \hline 4 \end{array}$$

$(3_5 \times 1_5) + 1_5 = 4_5$. Record the 4.

$$\begin{array}{r} {}^{1}13_5 \\ \times\ 3_5 \\ \hline 44_5 \end{array}$$

The product is 44_5.

Constructing a multiplication table is often tedious, especially when the base is large. To multiply in a given base without the use of a table, multiply in base 10 and convert the products to the appropriate base number before recording them. This procedure is illustrated in Example 9.

EXAMPLE 9 *Multiplying in Base 7*

Multiply

$$\begin{array}{r} 43_7 \\ \times\ 25_7 \\ \hline \end{array}$$

SOLUTION: $5 \times 3 = 15_{10} = 2(7) + 1(1) = 21_7$. Record the 1 and carry the 2.

$$\begin{array}{r} {}^{2}43_7 \\ \times\ 25_7 \\ \hline 1 \end{array}$$

$(5 \times 4) + 2 = 20 + 2 = 22_{10} = 3(7) + 1(1) = 31_7$. Record the 31.

$$\begin{array}{r} {}^{2}43_7 \\ \times\ 25_7 \\ \hline 311 \end{array}$$

$2 \times 3 = 6_{10} = 6_7$. Record the 6.

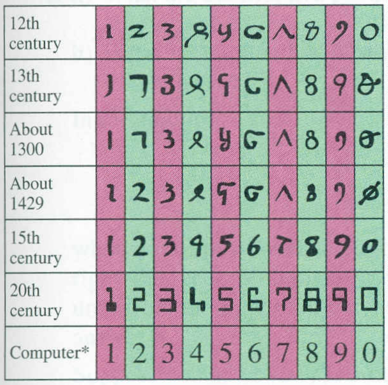
$$\begin{array}{r} {}^{2}43_7 \\ \times\ 25_7 \\ \hline 311 \\ 6 \end{array}$$

$2 \times 4 = 8_{10} = 1(7) + 1(1) = 11_7$. Record the 11. Now add in base 7 to determine the answer. Remember, in base 7, there are no digits greater than 6.

$$\begin{array}{r} {}^{2}43_7 \\ \times\ 25_7 \\ \hline 311 \\ 116 \\ \hline 1501_7 \end{array}$$

DID YOU KNOW

Ever-Changing Numbers



* Times New Roman font

During the Middle Ages, Western Europeans were reluctant to give up Roman numerals in favor of Hindu–Arabic numerals. The rapid expansion of trade and commerce during the fifteenth century, however, caused the need for quicker systems of calculation. The invention of movable type in 1450 also ensured a certain consistency in the way numerals were depicted, yet we still find ways to alter them.

Division

Division is performed in much the same manner as long division in base 10. A detailed example of a division in base 5 is illustrated in Example 10. The same procedure is used for division in any other base.

EXAMPLE 10 *Dividing in Base 5*

Divide $2_5\overline{)143_5}$.

SOLUTION: Using the multiplication table for base 5, Table 4.9 on page 193, we list the multiples of the divisor, 2.

$$2_5 \times 1_5 = 2_5$$
$$2_5 \times 2_5 = 4_5$$
$$2_5 \times 3_5 = 11_5$$
$$2_5 \times 4_5 = 13_5$$

Since $2_5 \times 4_5 = 13_5$, which is the largest product less than 14_5, 2_5 divides into 14_5 four times.

$$\begin{array}{r} 4 \\ 2_5\overline{)143_5} \\ \underline{13} \\ 1 \end{array}$$

Subtract 13_5 from 14_5. The difference is 1_5. Record the 1. Now bring down the 3 as when dividing in base 10.

$$\begin{array}{r} 4 \\ 2_5\overline{)143_5} \\ \underline{13} \\ 13 \end{array}$$

We see that $2_5 \times 4_5 = 13_5$. Use this information to complete the problem.

$$\begin{array}{r} 44_5 \\ 2_5\overline{)143_5} \\ \underline{13} \\ 13 \\ \underline{13} \\ 0 \end{array}$$

Therefore, $143_5 \div 2_5 = 44_5$ with remainder 0_5.

A division problem can be checked by multiplication. If the division was performed correctly, (quotient × divisor) + remainder = dividend. We can check Example 10 as follows.

$$(44_5 \times 2_5) + 0_5 = 143_5$$

$$\begin{array}{r} 44_5 \\ \times \quad 2_5 \\ \hline 143_5 \end{array} \quad \text{Check}$$

EXAMPLE 11 *Dividing in Base 6*

Divide $4_6\overline{)2430_6}$.

SOLUTION: The multiples of 4 in base 6 are

$$4_6 \times 1_6 = 4_6$$
$$4_6 \times 2_6 = 12_6$$
$$4_6 \times 3_6 = 20_6$$
$$4_6 \times 4_6 = 24_6$$
$$4_6 \times 5_6 = 32_6$$

$$\begin{array}{r} 404_6 \\ 4_6\overline{)2430_6} \\ \underline{24} \\ 03 \\ \underline{00} \\ 30 \\ \underline{24} \\ 2 \end{array}$$

Thus, the quotient is 404_6, with remainder 2_6.

Be careful when subtracting! When subtracting 4 from 0, you will need to borrow. Remember that you borow 10_6, which is the same as 6 in base 10.

Check: Does $(404_6 \times 4_6) + 2_6 = 2430_6$?

$$\begin{array}{r} 404_6 \\ \times \quad 4_6 \\ \hline 2424_6 \end{array} + 2_6 = 2430_6 \quad \text{True}$$

SECTION 4.4 EXERCISES

Concept/Writing Exercises

1. a) What are the first five positional values, from right to left, in base *b*?
   b) What are the first five positional values, from right to left, in base 6?
2. In the addition

$$\begin{array}{r} 367_8 \\ +24_8 \\ \hline \end{array}$$

what are the positional values of the first column on the right, the second column from the right, and the third column from the right? Explain how you determined your answer.

3. Suppose you add two base 5 numbers and you obtain an answer of 463_5. Can your answer be correct? Explain.
4. Suppose you add two base 3 numbers and you obtain an answer of 2032_3. Can your answer be correct? Explain.
5. In your own words, explain how to add two numbers in a given base. In your explanation, answer the question, "What happens when the sum of the numbers in a column is greater than the base?"
6. In your own words, explain how to subtract two numbers in a given base. Include in your explanation what you do when, in one column, you must subtract a larger number from a smaller number.

Practice the Skills

In Exercises 7–18, add in the indicated base.

7. $43_5 + 41_5$ 8. $33_8 + 65_8$ 9. $2303_4 + 232_4$

10. $101_2 + 11_2$ 11. $799_{12} + 218_{12}$ 12. $222_3 + 22_3$

13. $1112_3 + 1011_3$ 14. $470_{12} + 347_{12}$ 15. $14631_7 + 6040_7$

16. $1341_8 + 341_8$ 17. $1110_2 + 110_2$ 18.* $43A_{16} + 496_{16}$

In Exercises 19–30, subtract in the indicated base.

19. $322_4 - 103_4$ 20. $526_7 - 145_7$ 21. $2342_5 - 1442_5$

22. $1011_2 - 101_2$ 23. $782_{12} - 13T_{12}$ 24. $1221_3 - 202_3$

25. $1001_2 - 110_2$ 26. $2T34_{12} - 345_{12}$ 27. $4223_7 - 304_7$

28. $4232_5 - 2341_5$ 29. $2100_3 - 1012_3$ 30.* $4E7_{16} - 189_{16}$

In Exercises 31–42, multiply in the indicated base.

31. $33_5 \times 2_5$ 32. $323_6 \times 4_6$ 33. $342_7 \times 5_7$

34. $101_2 \times 11_2$ 35. $512_6 \times 23_6$ 36. $124_{12} \times 6_{12}$

37. $436_9 \times 25_9$ 38. $6T3_{12} \times 24_{12}$ 39. $111_2 \times 101_2$

40. $584_9 \times 24_9$ 41. $316_7 \times 16_7$ 42. $8T_{12} \times 2T_{12}$

In Exercises 43–54, divide in the indicated base.

43. $1_2\overline{)110_2}$ 44. $4_6\overline{)231_6}$ 45. $3_5\overline{)143_5}$

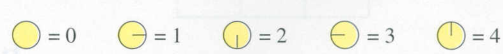
46. $7_8\overline{)335_8}$ 47. $2_4\overline{)312_4}$ 48. $6_{12}\overline{)431_{12}}$

49. $2_4\overline{)213_4}$ 50. $5_6\overline{)214_6}$ 51. $3_5\overline{)224_5}$

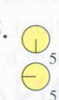
52. $4_6\overline{)210_6}$ 53. $6_7\overline{)404_7}$ 54. $3_7\overline{)2101_7}$

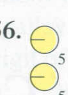
Problem Solving

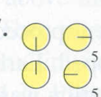
In Exercises 55–58, the numerals in a base 5 numeration system are as illustrated with their equivalent Hindu–Arabic numerals.

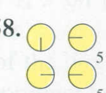


Add the following base 5 numbers.

55. 

56. 

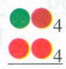
57. 

58. 

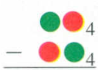
*For Exercises 18 and 30, see Exercises 37–40 in Section 4.3.

In Exercises 59–66, assume the numerals given are in a base 4 numeration system. In this system, suppose colors are used as numerals, as indicated below.

Add the following base 4 numbers. Your answers will contain a variety of the colors indicated.

59.  60.  61.  62. 

Subtract the following in base 4. Your answer will contain a variety of the colors indicated.

63.  64.  65.  66. 

For Exercises 67 and 68, study the pattern in the boxes. The number in the bottom row of each box represents the value of each dot in the box directly above it. For example, the following box represents $(3 \times 7^2) + (2 \times 7) + (4 \times 1)$, *or the number* 324_7. *This number in base 10 is 165.*

• • •	• •	• • • •
7^2	7	1

67. Determine the base 5 number represented by the dots in the top row of the boxes. Then convert the base 5 number to a number in base 10.

• •	• • •		• •
5^3	5^2	5	1

68. Fill in the correct amount of dots in the columns above the base values if the number represented by the dots is to equal 327 in base 10.

9^2	9	1

Challenge Problems/Group Activities

Divide in the indicated base.

69. $14_5\overline{)242_5}$ 70. $20_4\overline{)223_4}$

71. Consider the multiplication

$$\begin{array}{r} 462_8 \\ \times\ 35_8 \\ \hline \end{array}$$

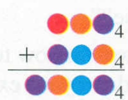
a) Multiply the numbers in base 8.
b) Convert 462_8 and 35_8 to base 10.
c) Multiply the base 10 numbers determined in part (b).
d) Convert the answer obtained in base 8 in part (a) to base 10.
e) Are the answers obtained in parts (c) and (d) the same? Why or why not?

Recreational Mathematics

72. Determine b, by trial and error, if $1304_b = 204$.

73. In a base 4 system, each of the four numerals is represented by one of the following colors:

Determine the value of each color if the following addition is true in base 4.



Internet/Research Activities

74. Investigate and write a report on the use of the duodecimal (base 12) system as a system of numeration. You might contact the Dozenal Society (formerly the Duodecimal Society), Nassau Community College, Garden City, NY 11530 or use their website www.polar.sunynassau.edu/~dozenal/ for information.

75. One method used by computers to perform subtraction is the "end around carry method." Do research and write a report explaining, with specific examples, how a computer performs subtraction by using the end around carry method.

4.5 EARLY COMPUTATIONAL METHODS

Our present procedures for multiplying and dividing numbers are the most recent to be developed. Early civilizations used various methods for multiplying and dividing. Multiplication was performed by *duplation and mediation*, by the *galley method*, and by *Napier rods*. Following is an explanation of each method.

Duplation and Mediation

EXAMPLE 1 ***A Pairing Technique for Multiplying***

Multiply 17 × 30 using duplation and mediation.

SOLUTION: Write 17 and 30 with a dash between to separate them. Divide the number on the left, 17, in half, drop the remainder, and place the quotient, 8, under the 17. Double the number on the right, 30, obtaining 60, and place it under the 30. You will then have the following paired lines.

17—30
8—60

Continue this process, taking one-half the number in the left-hand column, disregarding the remainder, and doubling the number in the right-hand column, as shown below. When a 1 appears in the left-hand column, stop.

17—30
8—60
4—120
2—240
1—480

Cross out all the even numbers in the left-hand column and the corresponding numbers in the right-hand column.

17—30
~~8—60~~
~~4—120~~
~~2—240~~
1—480

Now add the remaining numbers in the right-hand column, obtaining 30 + 480 = 510, which is the product you want. If you check, you will find that 17 × 30 = 510. ▲

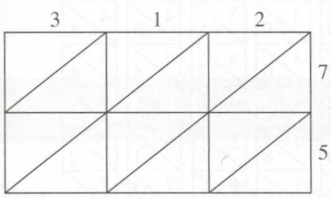


Figure 4.1

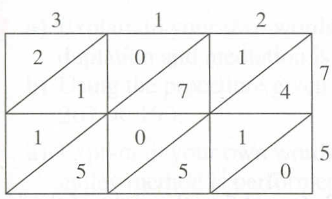


Figure 4.2

The Galley Method

The galley method (sometimes referred to as the Gelosia method) was developed after duplation and mediation. To multiply 312 × 75 using the galley method, you construct a rectangle consisting of three columns (one for each digit of 312) and two rows (one for each digit of 75).

Place the digits 3, 1, 2 above the boxes and the digits 7, 5 on the right of the boxes, as shown in Fig. 4.1. Then place a diagonal in each box.

Complete each box by multiplying the number on top of the box by the number to the right of the box (Fig. 4.2). Place the units digit of the product below the diagonal and the tens digit of the product above the diagonal.

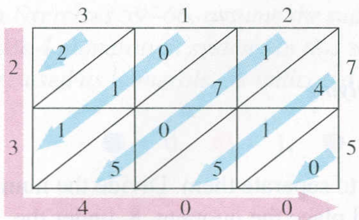


Figure 4.3

Add the numbers along the diagonals, as shown with the blue shaded arrows in Fig. 4.3, starting with the bottom right diagonal. If the sum in a diagonal is 10 or greater, record the units digit below the rectangle and carry the tens digit to the next diagonal to the left.

For example, when adding 4, 1, and 5 (along the second blue diagonal from the right), the sum is 10. Record the 0 below the rectangle and carry the 1 to the next blue diagonal. The sum of $1 + 1 + 7 + 0 + 5$ is 14. Record the 4 and carry the 1. The sum of the numbers in the next blue diagonal is $1 + 0 + 1 + 1$ or 3.

The answer is read down the left-hand column and along the bottom, as shown by the purple arrow in Fig. 4.3. The answer is 23,400.

PROFILE IN MATHEMATICS

JOHN NAPIER

During the seventeenth century, the growth of scientific fields such as astronomy required the ability to perform often unwieldy calculations. The English mathematician John Napier (1550–1617) made great contributions toward solving the problem of computing these numbers. His inventions include simple calculating machines and a device for performing multiplication and division known as Napier rods. Napier also developed the theory of logarithms.

Napier Rods

The third method used to multiply numbers was developed from the galley method by John Napier in the seventeenth century. His method of multiplication, known as Napier rods, proved to be one of the forerunners of the modern-day computer. Napier developed a system of separate rods numbered from 0 through 9 and an additional strip for an index, numbered vertically 1 through 9 (Fig. 4.4). Each rod is divided into 10 blocks. Each block below the first block contains a multiple of the number in the first block, with a diagonal separating the digits. The units digits are placed to the right of the diagonals and the tens digits to the left. Example 2 explains how Napier rods are used to multiply numbers.

INDEX	0	1	2	3	4	5	6	7	8	9
1	0/0	0/1	0/2	0/3	0/4	0/5	0/6	0/7	0/8	0/9
2	0/0	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8
3	0/0	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7
4	0/0	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
5	0/0	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
6	0/0	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
7	0/0	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
8	0/0	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
9	0/0	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

Figure 4.4

EXAMPLE 2 *Using Napier Rods*

Multiply 8×365, using Napier rods.

SOLUTION: To multiply 8×365, line up the rods 3, 6, and 5 to the right of the index, as shown in Fig. 4.5 on page 201. Below the 3, 6, and 5 place the blocks that

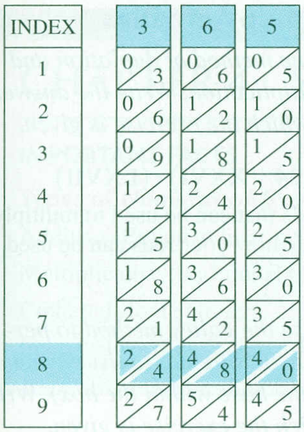
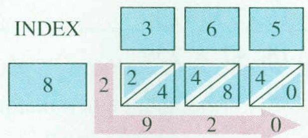


Figure 4.5

contain the products of 8×3, 8×6, and 8×5, respectively. To obtain the answer, add along the diagonals as in the galley method.



Thus, $8 \times 365 = 2920$.

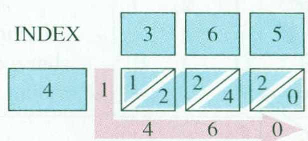
Example 3 illustrates the procedure to follow to multiply numbers containing more than one digit, using Napier rods.

EXAMPLE 3 *Using Napier Rods to Multiply Two- and Three-Digit Numbers*

Multiply 48×365, using Napier rods.

SOLUTION: $48 \times 365 = (40 + 8) \times 365$

Write $(40 + 8) \times 365 = (40 \times 365) + (8 \times 365)$. To find 40×365, determine 4×365 and multiply the product by 10. To evaluate 4×365, set up Napier rods for 3, 6, and 5 with index 4, and then evaluate along the diagonals, as indicated.



Therefore, $4 \times 365 = 1460$. Then $40 \times 365 = 1460 \times 10 = 14{,}600$.

$$\begin{aligned} 48 \times 365 &= (40 \times 365) + (8 \times 365) \\ &= 14{,}600 + 2920 \\ &= 17{,}520 \end{aligned}$$

$8 \times 365 = 2920$ from Example 2

SECTION 4.5 EXERCISES

Concept/Writing Exercises

1. What are the three early computational methods discussed in this section?
2. a) Explain in your own words how multiplication by duplation and mediation is performed.
   b) Using the procedure given in part (a), multiply 267×193.
3. a) Explain in your own words how multiplication by the galley method is performed.
   b) Using the procedure given in part (a), multiply 362×29.
4. a) Explain in your own words how multiplication using Napier rods is performed.
   b) Using the procedure given in part (a), multiply 25×6.

Practice the Skills

In Exercises 5–12, multiply using duplation and mediation.

5. 23×31
6. 35×23
7. 9×162
8. 175×86
9. 35×236
10. 96×53
11. 93×93
12. 49×124

In Exercises 13–20, multiply using the galley method.

13. 6 × 375 **14.** 8 × 365
15. 4 × 583 **16.** 7 × 125
17. 75 × 12 **18.** 47 × 259
19. 314 × 652 **20.** 634 × 832

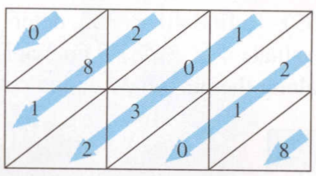
In Exercises 21–28, multiply using Napier rods.

21. 8 × 63 **22.** 7 × 63
23. 7 × 58 **24.** 7 × 125
25. 5 × 125 **26.** 75 × 125
27. 9 × 6742 **28.** 7 × 3456

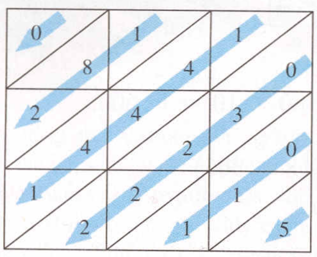
Problem Solving

In Exercises 29 and 30, we show multiplications using the galley method. (a) Determine the numbers being multiplied. Explain how you determined your answer. (b) Find the product.

29.

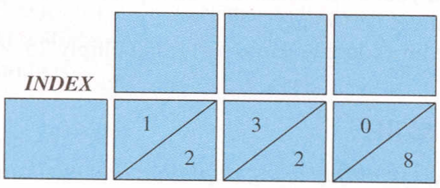


30.

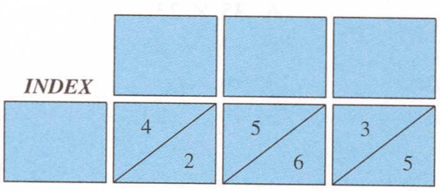


In Exercises 31 and 32, we solve a multiplication problem using Napier rods. (a) Determine the numbers being multiplied. Each empty box contains a single digit. Explain how you determined your answer. (b) Find the product.

31.



32.



Challenge Problems/Group Activities

In Exercises 33 and 34, use the method of duplation and mediation to perform the multiplication. Write the answer in the numeration system in which the exercise is given.

33. (∩|||) • (∩∩||) **34.** (XXVI) · (LXVII)

35. Develop a set of Napier rods that can be used to multiply numbers in base 5. Illustrate how your rods can be used to multiply $3_5 \times 21_5$.

In Exercises 36 and 37, (a) use the galley method to perform the multiplication. (Hint: Be sure not to list any number greater than or equal to the base within the box). Write the answer in the base in which the exercise is given. (b) Multiply the numbers as explained in Section 4.4. If you do not obtain the results obtained in part (a), explain why.

36. $21_3 \times 21_3$ **37.** $24_5 \times 234_5$

Recreational Mathematics

38. Obtain a clean U.S. $1 bill. On the back side of the bill is a circle containing a pyramid. On the base of the pyramid is a Roman numeral. a) Determine the value of that Roman numeral in our Hindu–Arabic system.

While you have that bill out, let's consider something else. For many people the number 13 is considered an unlucky number. In fact, many hotels do not have a thirteenth floor because many guests refuse to stay on the thirteenth floor of a hotel. Yet, if you look at the back of a $1 bill, you will find

13 steps on the pyramid.
13 letters in the Latin words *Annuit Coeptis.*
13 letters in "E Pluribus Unum".
13 stars above the eagle.
13 plum feathers on each span of the eagle's wing.
13 bars on the eagle's shield.
13 leaves on the olive branch.
13 fruits.
13 arrows.

In addition, the U.S. flag has 13 stripes, there were 13 signers of the Declaration of Independence, there were 13 original colonies, and don't forget the important Thirteenth Amendment that abolished slavery. b) So why, in your opinion, do we as a society fear the number 13? By the way, the official name for the fear of the number 13 is *triskaidekaphobia.*

Internet/Research Activities

39. In addition to Napier rods, John Napier is credited with making other important contributions to mathematics. Write a report on John Napier and his contributions to mathematics.

40. Write a paper explaining why the duplation and mediation method works.

CHAPTER 4 SUMMARY

IMPORTANT FACTS

Types of Numeration Systems

Additive (Egyptian hieroglyphics, Roman)

Multiplicative (traditional Chinese)

Ciphered (Ionic Greek)

Place-value (Babylonian, Mayan, Hindu–Arabic)

Early Computational Methods

Duplation and mediation

The galley method

Napier rods

CHAPTER 4 REVIEW EXERCISES

4.1, 4.2

In Exercises 1–6, assume an additive numeration system in which $a = 1$, $b = 10$, $c = 100$, and $d = 1000$. Find the value of the numeral.

1. *dddcaaa* 2. *ccbda* 3. *bcccad*
4. *cbdadaaa* 5. *ddcccbaaaa* 6. *ccbaddac*

In Exercises 7–12, assume the same additive numeration system as in Exercises 1–6. Write the numeral in terms of a, b, c, and d.

7. 56 8. 125 9. 293
10. 2005 11. 6851 12. 2314

In Exercises 13–18, assume a multiplicative numeration system in which $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 6$, $g = 7$, $h = 8$, $i = 9$, $x = 10$, $y = 100$, and $z = 1000$. Find the value of the numeral.

13. *dxc* 14. *bxg*
15. *gydxi* 16. *dzfxh*
17. *ezfydxh* 18. *fziye*

In Exercises 19–24, assume the same multiplicative numeration system as in Exercises 13–18. Write the Hindu–Arabic numeral in that system.

19. 82 20. 295 21. 862
22. 3094 23. 6004 24. 2001

In Exercises 25–36, use the following ciphered numeration system.

Decimal	1	2	3	4	5	6	7	8	9
Units	a	b	c	d	e	f	g	h	i
Tens	j	k	l	m	n	o	p	q	r
Hundreds	s	t	u	v	w	x	y	z	A
Thousands	B	C	D	E	F	G	H	I	J
Ten thousands	K	L	M	N	O	P	Q	R	S

Convert the numeral to a Hindu–Arabic numeral.

25. me 26. uh 27. woh
28. NGzqc 29. PEvqa 30. Pwki

Write the numeral in the ciphered numeration system.

31. 85 32. 372 33. 493
34. 1997 35. 53,467 36. 75,496

In Exercises 37–42, convert 1462 to a numeral in the indicated numeration system.

37. Egyptian 38. Roman 39. Chinese
40. Ionic Greek 41. Babylonian 42. Mayan

In Exercises 43–48, convert the numeral to a Hindu–Arabic numeral.

43. 

44. 八千二百五十四

45. $\chi\pi\epsilon$

46. MCMXCI

47. 

48. 

4.3

In Exercises 49–54, convert the numeral to a Hindu–Arabic numeral.

49. 47_8
50. 101_2
51. 130_4
52. 3425_7
53. TOE_{12}
54. 20220_3

In Exercises 55–60, convert 463 to a numeral in the base indicated.

55. base 4
56. base 3
57. base 2
58. base 5
59. base 12
60. base 8

4.4

In Exercises 61–66, add in the base indicated.

61. $52_7 + 55_7$
62. $10110_2 + 11001_2$
63. $TE_{12} + 87_{12}$
64. $234_7 + 456_7$
65. $3024_5 + 4023_5$
66. $3407_8 + 7014_8$

In Exercises 67–72, subtract in the base indicated.

67. $4032_7 - 321_7$
68. $1001_2 - 101_2$
69. $3TT_{12} - E7_{12}$
70. $4321_5 - 442_5$
71. $1713_8 - 1243_8$
72. $2021_3 - 212_3$

In Exercises 73–78, multiply in the base indicated.

73. $32_6 \times 4_6$
74. $34_5 \times 21_5$
75. $126_{12} \times 47_{12}$
76. $221_3 \times 22_3$
77. $1011_2 \times 101_2$
78. $476_8 \times 23_8$

In Exercises 79–84, divide in the base indicated.

79. $1_2\overline{)1011_2}$
80. $2_4\overline{)320_4}$
81. $3_5\overline{)130_5}$
82. $4_6\overline{)3020_6}$
83. $3_6\overline{)2034_6}$
84. $6_8\overline{)5072_8}$

4.5

85. Multiply 142×24, using the duplation and mediation method.
86. Multiply 142×24, using the galley method.
87. Multiply 142×24, using Napier rods.

CHAPTER 4 TEST

1. Explain the difference between a numeral and a number.

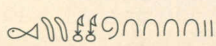
In Exercises 2–7, convert the numeral to a Hindu–Arabic numeral.

2. MMMDCXLVI

3. 

4. 八 千 零 九 十

5. 

6. 

7. $\theta' \pi\ Q\ \theta$

In Exercises 8–12, convert the number written in base 10 to a numeral in the numeration system indicated.

8. 463 to Egyptian
9. 2476 to Ionic Greek
10. 1434 to Mayan
11. 1596 to Babylonian
12. 2378 to Roman

In Exercises 13–16, describe briefly each of the systems of numeration. Explain how each type of numeration system is used to represent numbers.

13. Additive system
14. Multiplicative system
15. Ciphered system
16. Place-value system

In Exercises 17–20, convert the numeral to a numeral in base 10.

17. 56_7 **18.** 403_5

19. 101101_2 **20.** 368_9

In Exercises 21–24, convert the base 10 numeral to a numeral in the base indicated.

21. 36 to base 2 **22.** 93 to base 5

23. 2356 to base 12 **24.** 2938 to base 7

In Exercises 25–28, perform the indicated operations.

25. $133_5 + 434_5$

26. $324_6 - 142_6$

27. $45_6 \times 23_6$

28. $3_5\overline{)1210_5}$

29. Multiply 35 × 28, using duplation and mediation.

30. Multiply 43 × 196, using the galley method.

GROUP PROJECTS

U.S. Postal Service Bar Codes

Wherever we look nowadays, we see bar codes. We find them on items we buy at grocery stores and department stores and on many pieces of mail we receive. There are various types of bar codes, but each can be considered a type of numeration system. Although bar codes may vary in design, most are made up of a series of long and short bars. (New bar codes now being developed use a variety of shapes.) In this group project, we explain how postal codes are used.

The U.S. Postal Service introduced a bar coding system for zip codes in 1976. The system became known as Postnet (*pos*tal *n*umeric *e*ncoding *t*echniqe), and it has been refined over the years. Our basic zip code consists of five digits. The post office would like us to use the basic zip code followed by a hyphen and four additional digits. The post office refers to this nine-digit zip code as "zip + 4."

The Postnet bar code uses a series of long and short bars. A bar code may contain either 52 or 62 bars. The code designates the location to which the letter is being sent. The following bar code, with 52 bars, is for an address in Pittsburgh, Pennsylvania.

15250-7406 (Pittsburgh, PA)

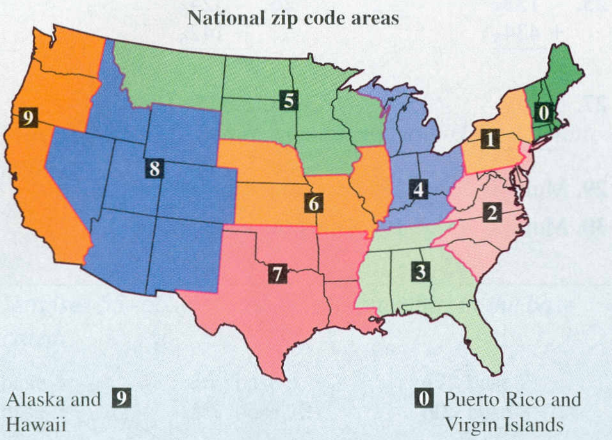
In bar codes, each short bar represents 0 and each long bar represents 1. Each code starts and ends with a long bar that is *not* used in determining the zip + 4. If the code contains 52 bars, the code represents the zip + 4 and an extra digit referred to as a check digit. If the code contains 62 bars, it contains the zip + 4, the last two digits of the address number, and a check digit. If the code contains 52 bars, the sum of the zip + 4 and the check digit must equal a number that is divisible by 10. If the code contains 62 bars, the sum of the zip + 4, the last two digits of the address number, and the check digit must equal a number that is divisible by 10. The check digit is added to make each sum divisible by 10.

In a postal bar code, each of the digits 0 through 9 is represented by a series of five digits containing zeros and ones:

11000 (0)	00011 (1)	00101 (2)	00110 (3)	01001 (4)
01010 (5)	01100 (6)	10001 (7)	10010 (8)	10100 (9)

Consider the postal code from Pittsburgh given earlier. If you disregard the bar on the left, the next five bars are . Since each small bar represents a 0 and each large bar represents a 1, these five bars can be represented as 00011. From the chart, we see that this represents the number 1. The first five bars (after the bar on the far left has been excluded) tell the region of the country in which the address is located on the map shown on the next page.

Notice that Pennsylvania is located, along with New York, in the region marked 1 on the map.



The second five lines in the bar code, , represent 01010 and have a value of 5. The other digits in the zip + 4 are determined in a similar manner. This code has 52 bars. The 45 bars, after the first bar, give the zip + 4. If you add the digits in the zip + 4, you get $1 + 5 + 2 + 5 + 0 + 7 + 4 + 0 + 6 = 30$. Since 30 is divisible by 10, the five bars to the left of the bar on the very right should be 0. Note that is represented as 11000 and has a value of 0. If, for example, the sum of the nine digits in the zip + 4 were 36, then the last five digits would need to represent the number 4, to make the sum of the digits divisible by 10. The five bars to the left of the last bar on the right are always used as a check.

Now let's work some problems.

a) For the Postnet code

determine the zip + 4 and the check digit. Then check by adding the zip + 4 and the check digit. Is the sum divisible by 10?

b) For each of the following Postnet codes, determine the zip + 4, the last two numbers of the address number (if applicable), and the check digit.

i)

ii)

c) Construct the Postnet code of long and short bars for each of the numbers. The numbers represent the zip + 4 and the last two digits of the address number. Do not forget the check digit.

i) 32226-8600-34 ii) 20794-1063-50

d) Construct the 52-bar Postnet code for your college's zip + 4. Don't forget to include the check digit.