CHAPTER 6

When planning a trip, knowledge of a coordinate system is helpful.

ALGEBRA, GRAPHS, AND FUNCTIONS

Algebra, and, in particular, word problems: The very mention of them is enough to frighten many people, and yet algebra is one of the most practical tools for solving everyday problems. You probably use algebra in your daily life without realizing it.

For example, you use a coordinate system when you consult your car map to find directions to a new destination. You solve simple equations when you change a recipe to increase or decrease the number of servings. To evaluate how much interest you will earn on a savings account or to figure out how long it will take you to travel a given distance, you use common formulas that are algebraic equations.

The symbolic language of algebra makes it an excellent tool for solving problems. Symbolism has three advantages. First, it allows us to write lengthy expressions in compact form. Second, symbolic language is clear—each symbol has a precise meaning. Finally, symbolism allows us to consider a large or infinite number of separate cases with a common property.

The English philosopher Alfred North Whitehead explained the power of algebra when he stated, "By relieving the brain of all unnecessary work, a good notation sets the mind free to concentrate on more advanced problems and in effect increases the mental power of the race."

6.1 ORDER OF OPERATIONS

Algebra is a generalized form of arithmetic. The word *algebra* is derived from the Arabic word *al-jabr* (meaning "reunion of broken parts"), which was the title of a book written by the mathematician Muhammed ibn-Musa al Khwarizmi in about A.D. 825.

Why study algebra? You can solve many problems in everyday life by using arithmetic or by trial and error, but with a knowledge of algebra you can find the solutions with less effort. You can solve other problems, like some we will present in this chapter, only by using algebra.

Algebra uses letters of the alphabet called ***variables*** to represent numbers. Often the letters x and y are used to represent variables. However, any letter may be used as a variable. A symbol that represents a specific quantity is called a ***constant***.

Multiplication of numbers and variables may be represented in several different ways in algebra. Since the "times" sign might be confused with the variable x, a dot between two numbers or variables indicates multiplication. Thus, $3 \cdot 4$ means 3 times 4, and $x \cdot y$ means x times y. Placing two letters or a number and a letter next to one another, with or without parentheses, also indicates multiplication. Thus, $3x$ means 3 times x, xy means x times y, and $(x)(y)$ means x times y.

An ***algebraic expression*** (or simply an ***expression***) is a collection of variables, numbers, parentheses, and operation symbols. Some examples of algebraic expressions are

$$x, \qquad x + 2, \qquad 3(2x + 3), \qquad \frac{3x + 1}{2x - 3}, \qquad \text{and} \qquad x^2 + 7x + 3$$

Two algebraic expressions joined by an equal sign form an ***equation***. Some examples of equations are

$$x + 2 = 4, \qquad 3x + 4 = 1, \qquad \text{and} \qquad x + 3 = 2x$$

The ***solution to an equation*** is the number or numbers that replace the variable to make the equation a true statement. For example, the solution to the equation $x + 3 = 4$ is $x = 1$. When we find the solution to an equation, we ***solve the equation***.

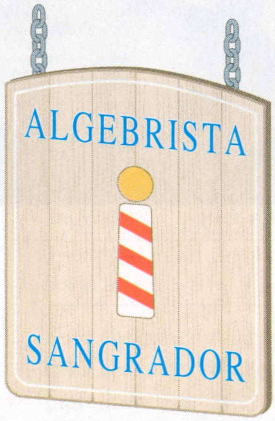
We can determine if any number is a solution to an equation by ***checking the solution***. To check the solution, we substitute the number for the variable in the equation. If the resulting statement is a true statement, that number is a solution to the equation. If the resulting statement is a false statement, the number is not a solution to the equation. To check the number $x = 1$ in the equation $x + 3 = 4$, we do the following.

$$x + 3 = 4$$
$$1 + 3 = 4 \qquad \text{Substitute 1 for } x$$
$$4 = 4 \qquad \text{True}$$

The same number is obtained on both sides of the equal sign, so the solution is correct. For the equation $x + 3 = 4$, the only solution is $x = 1$. Any other value of x would result in the check being a false statement.

DID YOU KNOW

Broken Bones



Imagine yourself walking down a street in Spain during the Middle Ages. You see a sign over a door: "*Algebrista y Sangrador.*" Inside, you would find a person more ready to give you a haircut than help you with your algebra. The sign translates into "Bonesetter and Bloodletter," relatively simple medical treatments administered in barbershops of the day.

The root word *al-jabr*, which the Muslims (Moors) brought to Spain along with some concepts of algebra, suggests the restoring of broken parts. The parts might be bones, or they might be mathematical expressions that are broken into separate parts and the parts moved from one side of an equation to the other and reunited in such a way as to make a solution more obvious.

To *evaluate an expression* means to find the value of the expression for a given value of the variable. To evaluate expressions and solve equations, you must have an understanding of exponents. Exponents (Section 5.6) are used to abbreviate repeated multiplication. For example, the expressions 5^2 means $5 \cdot 5$. The 2 in the expression 5^2 is the *exponent*, and the 5 is the *base*. We read 5^2 as "5 to the second power" or "5 squared," and 5^2 means $5 \cdot 5$ or 25.

In general, the number b to the nth power, written b^n, means

$$\overset{\text{Exponent}}{\underset{\text{Base}}{b}}{}^{n} = \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{n \text{ factors of } b}$$

An exponent refers only to its base. In the expression -5^2, the base is 5. In the expression $(-5)^2$, the base is -5.

$$-5^2 = -(5)^2 = -1(5)^2 = -1(5)(5) = -25$$

$$(-5)^2 = (-5)(-5) = 25$$

Note that $-5^2 \neq (-5)^2$ since $-25 \neq 25$.

Order of Operations

To evaluate an expression or to check the solution to an equation, we need to know the *order of operations* to follow. For example, suppose we want to evaluate the expression $2 + 3x$ when $x = 4$. Substituting 4 for x, we obtain $2 + 3 \cdot 4$. What is the value of $2 + 3 \cdot 4$? Does it equal 20, or does it equal 14? Some standard rules, called the order of operations, have been developed to ensure that there is only one correct answer. In mathematics, unless parentheses indicate otherwise, always perform multiplication before addition. Thus, the correct answer is 14.

$$2 + 3 \cdot 4 = 2 + (3 \cdot 4) = 2 + 12 = 14$$

The order of operations for evaluating an expression is as follows.

Order of Operations

1. First, perform all operations within parentheses or other grouping symbols (according to the following order).
2. Next, perform all exponential operations (that is, raising to powers or finding roots).
3. Next, perform all multiplications and divisions from left to right.
4. Finally, perform all additions and subtractions from left to right.

DID YOU KNOW

An Important Breakthrough

This painting by Charles Demuth, called *I Saw the Figure 5 in Gold*, depicts the abstract nature of numbers. Mathematician and philosopher Bertrand Russell observed in 1919 that it must have required "many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2." The discovery that numbers could be used not only to count objects such as the number of birds but also to represent abstract quantities represented a breakthrough in the development of algebra.

TIMELY TIP Some students use the phrase, "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally," or the word "PEMDAS" (**P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction) to remind them of the order of operations. Remember: Multiplication and division are of the same order, and addition and subtraction are of the same order.

EXAMPLE 1 *Evaluating an Expression*

Evaluate the expression $-x^2 + 3x + 20$ for $x = 4$.

SOLUTION: Substitute 4 for each x and use the order of operations to evaluate the expression.

$$\begin{aligned} &-x^2 + 3x + 20 \\ &= -(4)^2 + 3(4) + 20 \\ &= -16 + 12 + 20 \\ &= -4 + 20 \\ &= 16 \end{aligned}$$

EXAMPLE 2 *Finding the Height*

A ball is thrown upward off a bridge 40 feet above ground. Its height, in feet, above ground, t seconds after it is thrown, can be determined by the expression $-16t^2 + 30t + 40$. Find the height of the ball, above ground, 2 seconds after it is thrown.

SOLUTION: Substitute 2 for each t.

$$\begin{aligned} &-16t^2 + 30t + 40 \\ &= -16(2)^2 + 30(2) + 40 \\ &= -16(4) + 30(2) + 40 \\ &= -64 + 60 + 40 \\ &= -4 + 40 \\ &= 36 \end{aligned}$$

The ball is 36 feet above ground 2 seconds after it is thrown.

EXAMPLE 3 *Substituting for Two Variables*

Evaluate $-3x^2 + 2xy - 2y^2$ when $x = 2$ and $y = 3$.

SOLUTION: Substitute 2 for each x and 3 for each y; then evaluate using the order of operations.

$$\begin{aligned} &-3x^2 + 2xy - 2y^2 \\ &= -3(2)^2 + 2(2)(3) - 2(3)^2 \\ &= -3(4) + 2(2)(3) - 2(9) \\ &= -12 + 12 - 18 \\ &= 0 - 18 \\ &= -18 \end{aligned}$$

EXAMPLE 4 *Is 3 a Solution?*

Determine whether 3 is a solution to the equation $2x^2 + 4x - 9 = 21$.

SOLUTION: To determine whether 3 is a solution to the equation, substitute 3 for each x in the equation. Then evaluate the left-hand side of the equation using the order of operations. If this leads to a 21 on the left-hand side of the equal sign, then both sides of the equation have the same value, and 3 is a solution. In checking the solution, we use $\stackrel{?}{=}$ which means we are not sure if the statement is true.

$$
\begin{aligned}
2x^2 + 4x - 9 &\stackrel{?}{=} 21 \\
2(3)^2 + 4(3) - 9 &\stackrel{?}{=} 21 \\
2(9) + 12 - 9 &\stackrel{?}{=} 21 \\
18 + 12 - 9 &\stackrel{?}{=} 21 \\
30 - 9 &\stackrel{?}{=} 21 \\
21 &= 21 \quad \text{True}
\end{aligned}
$$

Because 3 makes the equation a true statement, 3 is a solution to the equation. ▲

SECTION 6.1 EXERCISES

Concept/Writing Exercises

1. What is a *variable*?
2. What is a *constant*?
3. What does it mean when we state that *a number is a solution to an equation*?
4. What is an algebraic expression? Illustrate an algebraic expression with an example.
5. a) For the term 4^5, identify the base and the exponent.
   b) In your own words, explain how to evaluate 4^5.
6. In your own words, explain the order of operations.
7. Evaluate $8 + 16 \div 4$ using the order of operations.
8. Evaluate $9 + 6 \cdot 3$ using the order of operations.

Practice the Skills

In Exercises 9–28, evaluate the expression for the given value(s) of the variable(s).

9. $x^2, \quad x = 7$
10. $x^2, \quad x = -8$
11. $-x^2, \quad x = -3$
12. $-x^2, \quad x = -5$
13. $-2x^3, \quad x = -7$
14. $-x^3, \quad x = -4$
15. $x - 7, \quad x = 4$
16. $8x - 3, \quad x = \frac{5}{2}$
17. $-7x + 4, \quad x = -2$
18. $x^2 - 3x + 8, \quad x = 5$
19. $-x^2 + 5x - 13, \quad x = -2$
20. $5x^2 + 7x - 11, \quad x = -1$
21. $\frac{1}{2}x^2 - 5x + 2, \quad x = \frac{2}{3}$
22. $\frac{2}{3}x^2 + x - 1, \quad x = \frac{1}{2}$
23. $8x^3 - 4x^2 + 7, \quad x = \frac{1}{2}$
24. $-x^2 + 4xy, \quad x = 2, \quad y = 3$
25. $2x^2 + xy + 3y^2, \quad x = -2, \quad y = 1$
26. $3x^2 + \frac{2}{5}xy - \frac{1}{5}y^2, \quad x = 2, \quad y = 5$
27. $4x^2 - 12xy + 9y^2, \quad x = 3, \quad y = 2$
28. $(x + 3y)^2, \quad x = 4, \quad y = -3$

In Exercises 29–38, determine whether the value(s) is (are) a solution to the equation.

29. $7x + 3 = 23, \quad x = 3$
30. $5x - 7 = -27, \quad x = -4$
31. $x - 3y = 0, \quad x = 6, \quad y = 3$
32. $4x + 2y = -2, \quad x = -2, \quad y = 3$
33. $x^2 + 3x - 4 = 5, \quad x = 2$
34. $2x^2 - x - 5 = 0, \quad x = 3$
35. $2x^2 + x = 28, \quad x = -4$
36. $y = x^2 + 3x - 5, \quad x = 1, \quad y = -1$
37. $y = -x^2 + 3x - 1, \quad x = 3, \quad y = -1$
38. $y = x^3 - 3x^2 + 1, \quad x = 2, \quad y = -3$

Problem Solving

39. *Sales Tax* If the sales tax on an item is 7%, the sales tax, in dollars, on an item costing d dollars can be found by using the expression $0.07d$. Determine the sales tax on a telescope costing $175.

40. *Radius of a Circle* A pebble is dropped into a calm pond causing ripples in the form of concentric circles (one circle inside another circle). The radius of the outer circle, in feet, can be determined by the expression $0.5t$, where t is the time in seconds after the pebble strikes the water. Find the radius of the outer circle 3 sec after the pebble strikes the water.

41. *Cost of a Tour* The cost, in dollars, for Crescent City Tours to provide a tour for x people can be determined by the expression $220 + 2.75x$. Determine the cost for Crescent City Tours to provide a tour for 75 people.

42. *Orange Orchard* The number of baskets of oranges that are produced by x trees in a small orchard can be approximated by the expression $25x - 0.2x^2$ (assuming x is no more than 100). Find the number of baskets of oranges produced by 60 trees.

43. *8 Trillion Calculations* If a computer can do a calculation in 0.000002 sec, the time required to do n calculations can be determined by the expression $0.000002n$. Determine the number of seconds needed for the computer to do 8 trillion (8,000,000,000,000) calculations.

44. *Drying Time* The time, in minutes, needed for clothes hanging on a line outdoors to dry, at a specific temperature and wind speed, depends on the humidity, h. The time can be approximated by the expression $2h^2 + 80h + 40$, where h is the percent humidity expressed as a decimal number. Find the length of time required for clothing to dry if there is 60% humidity.

45. *Grass Growth* The rate of growth of grass in inches per week depends on a number of factors, including rainfall and temperature. For a certain area, this can be approximated by the expression $0.2R^2 + 0.003RT + 0.0001T^2$, where R is the weekly rainfall, in inches, and T is the average weekly temperature, in degrees Fahrenheit. Find the amount of growth of grass for a week in which the rainfall is 2 in. and the average temperature is 70°F.

Challenge Problems/Group Activities

46. Explain why $(-1)^n = 1$ for any even number n.

47. Does $(x + y)^2 = x^2 + y^2$? Complete the table and state your conclusion.

x	y	$(x + y)^2$	$x^2 + y^2$
2	3		
−2	−3		
−2	3		
2	−3		

48. Suppose n represents any natural number. Explain why 1^n equals 1?

Internet/Research Activity

49. When were exponents first used? Write a paper explaining how exponents were first used and when mathematicians began writing them in the present form.

6.2 LINEAR EQUATIONS IN ONE VARIABLE

In Section 6.1, we stated that two algebraic expressions joined by an equal sign form an equation. The solution to some equations, such as $x + 3 = 4$, can be found easily by trial and error. However, solving more complex equations, such as $2x - 3 = 4(x + 3)$, requires understanding the meaning of like terms and learning four basic properties.

The parts that are added or subtracted in an algebraic expression are called *terms*. The expression $4x - 3y - 5$ contains three terms, namely $4x$, $-3y$, and -5. The + and − signs that break the expression into terms are a part of the terms. When listing the terms of an expression, however, it is not necessary to include the + sign at the beginning of the term.

The numerical part of a term is called its ***numerical coefficient*** or, simply, its ***coefficient***. In the term $4x$, the 4 is the numerical coefficient. In the term $-4y$, the -4 is the numerical coefficient.

Like terms are terms that have the same variables with the same exponents on the variables. ***Unlike terms*** have different variables or different exponents on the variables.

Like Terms	Unlike Terms
$2x$, $7x$ (same variable, x)	$2x$, 9 (only first term has a variable)
$-8y$, $3y$ (same variable, y)	$5x$, $6y$ (different variables)
-4, 10 (both constants)	x, 8 (only first term has a variable)
$-5x^2$, $6x^2$ (same variable with same exponent)	$2x^3$, $3x^2$ (different exponents)

To ***simplify an expression*** means to combine like terms by using the commutative, associative, and distributive properties discussed in Chapter 5. For convenience, we list these properties below.

Properties of the Real Numbers

$a(b + c) = ab + ac$	Distributive property
$a + b = b + a$	Commutative property of addition
$ab = ba$	Commutative property of multiplication
$(a + b) + c = a + (b + c)$	Associative property of addition
$(ab)c = a(bc)$	Associative property of multiplication

EXAMPLE 1 *Combining Like Terms*

Combine like terms in each expression.

a) $7x + 3x$ b) $6y - 2y$

c) $x + 12 - 3x + 7$ d) $-2x + 4 - 6y - 11 - 5y + 3x$

SOLUTION:

a) We use the distributive property (in reverse) to combine like terms.

$$\begin{aligned} 7x + 3x &= (7 + 3)x && \text{Distributive property} \\ &= 10x \end{aligned}$$

b) $6y - 2y = (6 - 2)y = 4y$

c)
$$\begin{aligned} x + 12 - 3x + 7 &= x - 3x + 12 + 7 && \text{Rearrange terms, place like terms together.} \\ &= -2x + 19 && \text{Combine like terms.} \end{aligned}$$

d)
$$\begin{aligned} &-2x + 4 - 6y - 11 - 5y + 3x \\ &= -2x + 3x - 6y - 5y + 4 - 11 && \text{Rearrange terms, place like terms together.} \\ &= x - 11y - 7 && \text{Combine like terms.} \end{aligned}$$

DID YOU KNOW

In the Beginning

Greek philosopher Diophantus of Alexandria (A.D. 250), who invented notations for powers of a number and for multiplication and division of simple quantities, is thought to have made the first attempts at algebra. But not until the sixteenth century did French mathematician François Viète (1540–1603) use symbols to represent numbers, the foundation of symbolic algebra. However, the work of René Descartes (1596–1660) is considered to be the starting point of modern-day algebra. In 1707, Sir Isaac Newton (1643–1727) gave symbolic mathematics the name *universal arithmetic.*

We are able to rearrange the terms of an expression, as was done in Example 1(c) and (d) by the commutative and associative properties that were discussed in Section 5.5.

The order of the terms in an expression is not crucial. However, when listing the terms of an expression we generally list the terms in alphabetical order with the constant, the term without a variable, last.

Solving Equations

Recall that to solve an equation means to find the value or values for the variable that make(s) the equation true. In this section, we discuss solving ***linear (or first degree) equations***. A linear equation in one variable is one in which the exponent on the variable is 1. Examples of linear equations are $5x - 1 = 3$ and $2x + 4 = 6x - 5$.

Equivalent equations are equations that have the same solution. The equations $2x - 5 = 1$, $2x = 6$, and $x = 3$ are all equivalent equations since they all have the same solution, 3. When we solve an equation, we write the given equation as a series of simpler equivalent equations until we obtain an equation of the form $x = c$, where c is some real number.

To solve any equation, we have to ***isolate the variable***. That means getting the variable by itself on one side of the equal sign. The four properties of equality that we are about to discuss are used to isolate the variable. The first is the addition property.

Addition Property of Equality

If $a = b$, then $a + c = b + c$ for all real numbers a, b, and c.

The addition property of equality indicates that the same number can be added to both sides of an equation without changing the solution.

EXAMPLE 2 *Using the Addition Property of Equality*

Find the solution to the equation $x - 7 = 10$.

SOLUTION: To isolate the variable, add 7 to both sides of the equation.

$$x - 7 = 10$$
$$x - 7 + 7 = 10 + 7$$
$$x + 0 = 17$$
$$x = 17$$

Check:

$$x - 7 \stackrel{?}{=} 10$$
$$17 - 7 \stackrel{?}{=} 10 \quad \text{Substitute 17 for } x.$$
$$10 = 10 \quad \text{True}$$

In Example 2, we showed the step $x + 0 = 17$. Generally this step is done mentally, and the step is not listed.

Subtraction Property of Equality

If $a = b$, then $a - c = b - c$ for all real numbers a, b, and c.

The subtraction property of equality indicates that the same number can be subtracted from both sides of an equation without changing the solution.

EXAMPLE 3 *Using the Subtraction Property of Equality*

Find the solution to the equation $x + 7 = 15$.

SOLUTION: To isolate the variable, subtract 7 from both sides of the equation.

$$x + 7 = 15$$
$$x + 7 - 7 = 15 - 7$$
$$x = 8$$

Note that we did not subtract 15 from both sides of the equation, since this would not result in getting x on one side of the equal sign by itself. ▲

Multiplication Property of Equality

If $a = b$, then $a \cdot c = b \cdot c$ for all real numbers a, b, and c, where $c \neq 0$.

The multiplication property of equality indicates that both sides of the equation can be multiplied by the same nonzero number without changing the solution.

EXAMPLE 4 *Using the Multiplication Property of Equality*

Find the solution to $\frac{x}{6} = 3$.

SOLUTION: To solve this equation, multiply both sides of the equation by 6.

$$\frac{x}{6} = 3$$
$$6\left(\frac{x}{6}\right) = 6(3)$$
$$\frac{\overset{1}{\cancel{6}}x}{\underset{1}{\cancel{6}}} = 18$$
$$1x = 18$$
$$x = 18$$

▲

In Example 4, we showed the steps $\frac{6x}{6} = 18$ and $1x = 18$. Generally, we will not illustrate these steps.

Division Property of Equality

If $a = b$, then $\frac{a}{c} = \frac{b}{c}$ for all real numbers a, b, and c, $c \neq 0$.

The division property of equality indicates that both sides of an equation can be divided by the same nonzero number without changing the solution. Note that the divisor, c, cannot be 0 because division by 0 is not permitted.

EXAMPLE 5 *Using the Division Property of Equality*

Find the solution to the equation $3x = 15$.

SOLUTION: To solve this equation, divide both sides of the equation by 3.

$$3x = 15$$
$$\frac{3x}{3} = \frac{15}{3}$$
$$x = 5$$

An ***algorithm*** is a general procedure for accomplishing a task. The following general procedure is an algorithm for solving linear (or first-degree) equations. Sometimes the solution to an equation may be found more easily by using a variation of this general procedure. Remember that the primary objective in solving any equation is to isolate the variable.

A General Procedure for Solving Linear Equations

1. If the equation contains fractions, multiply both sides of the equation by the lowest common denominator (or least common multiple). This step will eliminate all fractions from the equation.
2. Use the distributive property to remove parentheses when necessary.
3. Combine like terms on the same side of the equal sign when possible.
4. Use the addition or subtraction property to collect all terms with a variable on one side of the equal sign and all constants on the other side of the equal sign. It may be necessary to use the addition or subtraction property more than once. This process will eventually result in an equation of the form $ax = b$, where a and b are real numbers.
5. Solve for the variable using the division or multiplication property. This will result in an answer in the form $x = c$, where c is a real number.

EXAMPLE 6 *Using the General Procedure*

Solve the equation $2x - 9 = 19$ and check your solution.

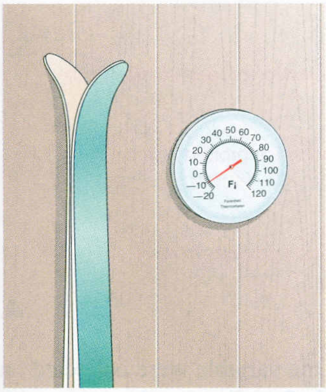
SOLUTION: Our goal is to isolate the variable; therefore, we start by getting the term $2x$ by itself on one side of the equation.

$$2x - 9 = 19$$
$$2x - 9 + 9 = 19 + 9 \quad \text{Add 9 to both sides of the equation (addition property) (step 4).}$$
$$2x = 28$$
$$\frac{2x}{2} = \frac{28}{2} \quad \text{Divide both sides of the equation by 2 (division property) (step 5).}$$
$$x = 14$$

A check will show that 14 is the solution to $2x - 9 = 19$.

DID YOU KNOW

A New Concept



Well into the sixteenth century, mathematicians found it difficult to accept the idea that the solution to a problem (such as Example 7) could be a negative number because negative numbers could not be accepted as physically real. In the early days of algebra, someone working a problem did not isolate a variable by subtracting like terms. Instead, a problem would be put into a form that allowed only positive coefficients and answers. Albert Girard (1595–1637) contributed to the evolution of a correct understanding of negative quantities.

EXAMPLE 7 *Solving a Linear Equation*

Solve the equation $4 = 5 + 2(t + 1)$ for t.

SOLUTION: Our goal is to isolate the variable t. To do so, follow the general procedure for solving equations.

$$
\begin{aligned}
4 &= 5 + 2(t+1) \\
4 &= 5 + 2t + 2 && \text{Distributive property (step 2)} \\
4 &= 2t + 7 && \text{Combine like terms (step 3)} \\
4 - 7 &= 2t + 7 - 7 && \text{Subtraction property (step 4)} \\
-3 &= 2t \\
\frac{-3}{2} &= \frac{2t}{2} && \text{Division property (step 5)} \\
-\frac{3}{2} &= t
\end{aligned}
$$

EXAMPLE 8 *Solving an Equation Containing Fractions*

Solve the equation $\frac{2x}{3} + \frac{1}{3} = \frac{3}{4}$.

SOLUTION: When an equation contains fractions, we generally begin by multiplying each term of the equation by the lowest common denominator, LCD (see Chapter 5). In this example, the LCD is 12, since 12 is the smallest number that is divisible by both 3 and 4.

$$
\begin{aligned}
12\left(\frac{2x}{3} + \frac{1}{3}\right) &= 12\left(\frac{3}{4}\right) && \text{Multiply both sides of the equation by the LCD (step 1).} \\
12\left(\frac{2x}{3}\right) + 12\left(\frac{1}{3}\right) &= 12\left(\frac{3}{4}\right) && \text{Distributive property (step 2)} \\
\overset{4}{\cancel{12}}\left(\frac{2x}{\underset{1}{\cancel{3}}}\right) + \overset{4}{\cancel{12}}\left(\frac{1}{\underset{1}{\cancel{3}}}\right) &= \overset{3}{\cancel{12}}\left(\frac{3}{\underset{1}{\cancel{4}}}\right) && \text{Divide out common factors.} \\
8x + 4 &= 9 \\
8x + 4 - 4 &= 9 - 4 && \text{Subtraction property (step 4)} \\
8x &= 5 \\
\frac{8x}{8} &= \frac{5}{8} && \text{Division property (step 5)} \\
x &= \frac{5}{8}
\end{aligned}
$$

A check will show that $\frac{5}{8}$ is the solution to the equation. You could have worked the problem without first multiplying both sides of the equation by the LCD. Try it!

EXAMPLE 9 *Variables on Both Sides of the Equation*

Solve the equation $6x + 8 = 10x + 12$.

SOLUTION: Note that the equation has an x on both sides of the equal sign. In equations of this type, you might wonder what to do first. It really does not matter as long as you do not forget the goal of isolating the variable x. Let's collect the terms containing a variable on the left-hand side of the equation.

$$
\begin{aligned}
6x + 8 &= 10x + 12 \\
6x + 8 - 8 &= 10x + 12 - 8 && \text{Subtraction property (step 4)} \\
6x &= 10x + 4 \\
6x - 10x &= 10x - 10x + 4 && \text{Subtraction property (step 4)} \\
-4x &= 4 \\
\frac{-4x}{-4} &= \frac{4}{-4} && \text{Division property (step 5)} \\
x &= -1
\end{aligned}
$$

In the solution to Example 9, the terms containing the variable were collected on the left-hand side of the equal sign. Now work Example 9, collecting the terms with the variable on the right-hand side of the equal sign. If you do so correctly, you will get the same result.

EXAMPLE 10 *Solving an Equation Containing Decimals*

Solve the equation $4x - 0.48 = 0.8x + 4$ and check your solution.

SOLUTION: This problem may be solved with the decimals, or you may multiply each term by 100 and eliminate the decimals. We will solve the problem with the decimals.

$$
\begin{aligned}
4x - 0.48 &= 0.8x + 4 \\
4x - 0.48 + 0.48 &= 0.8x + 4 + 0.48 && \text{Addition property} \\
4x &= 0.8x + 4.48 \\
4x - 0.8x &= 0.8x - 0.8x + 4.48 && \text{Subtraction property} \\
3.2x &= 4.48 \\
\frac{3.2x}{3.2} &= \frac{4.48}{3.2} && \text{Division property} \\
x &= 1.4
\end{aligned}
$$

Check:

$$
\begin{aligned}
4x - 0.48 &= 0.8x + 4 \\
4(1.4) - 0.48 &= 0.8(1.4) + 4 && \text{Substitute 1.4 for each } x \text{ in the equation.} \\
5.6 - 0.48 &= 1.12 + 4 \\
5.12 &= 5.12 && \text{True}
\end{aligned}
$$

In Chapter 5, we explained that $a - b$ can be expressed as $a + (-b)$. We use this principle in Example 11.

EXAMPLE 11 *Using the Definition of Subtraction*

Solve $10 = -5 + 3(p - 4)$ for p.

SOLUTION: Our goal is to isolate the variable p. To do so, follow the general procedure for solving equations.

$$\begin{aligned}
10 &= -5 + 3(p - 4) \\
10 &= -5 + 3[p + (-4)] && \text{Definition of subtraction} \\
10 &= -5 + 3(p) + 3(-4) && \text{Distributive property} \\
10 &= -5 + 3p - 12 \\
10 &= 3p - 17 && \text{Combine like terms.} \\
10 + 17 &= 3p - 17 + 17 && \text{Addition property} \\
27 &= 3p && \text{Combine like terms.} \\
\frac{27}{3} &= \frac{3p}{3} && \text{Division property} \\
9 &= p
\end{aligned}$$

TIMELY TIP Remember that the goal in solving an equation is to get the variable alone on one side of the equal sign.

So far, every equation has had exactly one solution. Some equations, however, have no solution and others have more than one solution. Example 12 illustrates an equation that has no solution, and Example 13 illustrates an equation that has an infinite number of solutions.

EXAMPLE 12 *An Equation with No Solution*

Solve $3(x - 4) + x + 2 = 6x - 2(x + 3)$.

SOLUTION:

$$\begin{aligned}
3(x - 4) + x + 2 &= 6x - 2(x + 3) \\
3x - 12 + x + 2 &= 6x - 2x - 6 && \text{Distributive property} \\
4x - 10 &= 4x - 6 && \text{Combine like terms.} \\
4x - 4x - 10 &= 4x - 4x - 6 && \text{Subtraction property} \\
-10 &= -6 && \text{False}
\end{aligned}$$

During the process of solving an equation, if you obtain a false statement like $-10 = -6$, or $-4 = 0$, the equation has ***no solution***. An equation that has no solution is called an ***inconsistent equation***. The equation $3(x - 4) + x + 2 = 6x - 2(x + 3)$ is inconsistent and thus has no solution.

EXAMPLE 13 *An Equation with Infinitely Many Solutions*

Solve $3(x + 2) - 5(x - 3) = -2x + 21$.

SOLUTION:

$$\begin{aligned}
3(x + 2) - 5(x - 3) &= -2x + 21 \\
3x + 6 - 5x + 15 &= -2x + 21 && \text{Distributive property} \\
-2x + 21 &= -2x + 21 && \text{Combine like terms.}
\end{aligned}$$

Note that at this point both sides of the equation are the same. Every real number will satisfy this equation. This equation has an infinite number of solutions. An equation of this type is called an ***identity***. When solving an equation, if you notice that the same expression appears on both sides of the equal sign, the equation is an identity. The solution to any linear equation that is an identity is ***all real numbers***. If you continue to solve an equation that is an identity, you will end up with $0 = 0$, as follows.

$$\begin{aligned} -2x + 21 &= -2x + 21 \\ -2x + 2x + 21 &= -2x + 2x + 21 && \text{Addition property} \\ 21 &= 21 && \text{Combine like terms.} \\ 21 - 21 &= 21 - 21 && \text{Subtraction property} \\ 0 &= 0 && \text{True for any value of } x \end{aligned}$$

Proportions

A ***ratio*** is a quotient of two quantities. An example is the ratio of 2 to 5, which can be written $2:5$ or $\frac{2}{5}$ or $2/5$.

A **proportion** is a statement of equality between two ratios.

An example of a proportion is $\frac{a}{b} = \frac{c}{d}$. Consider the proportion

$$\frac{x+2}{5} = \frac{x+5}{8}$$

We can solve this proportion by first multiplying both sides of the equation by the least common denominator, 40.

$$\begin{aligned} \frac{x+2}{5} &= \frac{x+5}{8} \\ \overset{8}{\cancel{40}}\left(\frac{x+2}{\cancel{5}}\right) &= \overset{5}{\cancel{40}}\left(\frac{x+5}{\cancel{8}}\right) && \text{Multiplication property} \\ 8(x+2) &= 5(x+5) \\ 8x + 16 &= 5x + 25 \\ 3x + 16 &= 25 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

A check will show that 3 is the solution.

Proportions can often be solved more easily by using cross multiplication.

Cross Multiplication

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, $\quad b \neq 0, d \neq 0$.

Let's use cross multiplication to solve the proportion $\frac{x+2}{5} = \frac{x+5}{8}$.

$$\frac{x+2}{5} = \frac{x+5}{8}$$

$$8(x+2) = 5(x+5) \quad \text{Cross multiplication}$$

$$8x + 16 = 5x + 25$$

$$3x + 16 = 25$$

$$3x = 9$$

$$x = 3$$

Many practical application problems can be solved using proportions.

To Solve Application Problems Using Proportions

1. Represent the unknown quantity by a variable.
2. Set up the proportion by listing the given ratio on the left-hand side of the equal sign and the unknown and other given quantity on the right-hand side of the equal sign. When setting up the right-hand side of the proportion, the same respective quantities should occupy the same respective positions on the left and right. For example, an acceptable proportion might be

$$\frac{\text{miles}}{\text{hour}} = \frac{\text{miles}}{\text{hour}}$$

3. Once the proportion is properly written, drop the units and use cross multiplication to solve the equation.
4. Answer the question or questions asked.

EXAMPLE 14 *Water Usage*

The cost for water in Orange County is \$1.42 per 750 gallons (gal) of water used. What is the water bill if 30,000 gallons are used?

SOLUTION: This problem may be solved by setting up a proportion. One proportion that can be used is

$$\frac{\text{cost of 750 gal}}{\text{750 gal}} = \frac{\text{cost of 30,000 gal}}{\text{30,000 gal}}$$

We want to find the cost for 30,000 gallons of water, so we will call this quantity x. The proportion then becomes

$$\text{Given ratio} \quad \frac{1.42}{750} = \frac{x}{30{,}000}$$

Now we solve for x.

$$(1.42)(30{,}000) = 750x$$

$$42{,}600 = 750x$$

$$\frac{42{,}600}{750} = \frac{750x}{750}$$

$$\$56.80 = x$$

The cost of 30,000 gallons of water is \$56.80.

EXAMPLE 15 *Determining the Amount of Insulin*

Insulin comes in 10 cubic centimeter (cc) vials labeled in the number of units of insulin per cubic centimeter of fluid. A vial of insulin marked U40 has 40 units of insulin per cubic centimeter of fluid. If a patient needs 30 units of insulin, how much fluid should be drawn into the syringe from the U40 vial?

SOLUTION: The unknown quantity, x, is the number of cubic centimeters of fluid to be drawn into the syringe. Following is one proportion that can be used to find that quantity.

$$\text{Given ratio}\left\{\frac{40\text{ units}}{1\text{ cc}}\right. = \frac{30\text{ units}}{x\text{ cc}}$$

$$40x = 30(1)$$

$$40x = 30$$

$$x = \frac{30}{40} = 0.75$$

The nurse or doctor putting the insulin in the syringe should draw 0.75 cc of the fluid.

SECTION 6.2 EXERCISES

Concept/Writing Exercises

1. Define and give an example of a *term*.
2. Define and give an example of *like terms*.
3. Define and give an example of a *numerical coefficient*.
4. Define and give an example of a *linear equation*.
5. Explain how to simplify an expression. Give an example.
6. State the addition property of equality. Give an example.
7. State the subtraction property of equality. Give an example.
8. State the multiplication property of equality. Give an example.
9. State the division property of equality. Give an example.
10. Define *algorithm*.
11. Define and give an example of a *ratio*.
12. Define and give an example of a *proportion*.
13. Are $3x$ and $\frac{1}{2}x$ like terms? Explain.
14. Are $4x$ and $4y$ like terms? Explain.

Practice the Skills

In Exercises 15–38, combine like terms.

15. $2x + 9x$
16. $-4x - 7x$
17. $5x - 3x + 12$
18. $-6x + 3x + 21$
19. $7x + 3y - 4x + 8y$
20. $x - 4x + 3$
21. $-3x + 2 - 5x$
22. $-3x + 4x - 2 + 5$
23. $2 - 3x - 2x + 1$
24. $-0.2x + 1.7x - 4$
25. $6.2x - 8.3 + 7.1x$
26. $\frac{2}{3}x + \frac{1}{6}x - 5$
27. $\frac{1}{5}x - \frac{1}{3}x - 4$
28. $7t + 5s + 9 - 3t - 2s - 12$
29. $5x - 4y - 3y + 8x + 3$
30. $3(p + 2) - 4(p + 3)$
31. $2(s + 3) + 6(s - 4) + 1$
32. $6(r - 3) - 2(r + 5) + 10$
33. $0.3(x + 2) + 1.2(x - 4)$
34. $\frac{1}{5}(x + 2) - \frac{1}{10}x$
35. $\frac{2}{3}x + \frac{3}{7} - \frac{1}{4}x$
36. $n - \frac{3}{4} + \frac{5}{9}n - \frac{1}{6}$
37. $0.5(2.6x - 4) + 2.3(1.4x - 5)$
38. $\frac{2}{3}(3x + 9) - \frac{1}{4}(2x + 5)$

In Exercises 39–64, solve the equation.

39. $y + 8 = 13$
40. $2y - 7 = 17$
41. $9 = 12 - 3x$
42. $14 = 3x + 5$
43. $\frac{3}{x} = \frac{7}{8}$
44. $\frac{x - 1}{5} = \frac{x + 5}{15}$
45. $\frac{1}{2}x + \frac{1}{3} = \frac{2}{3}$
46. $\frac{1}{2}y + \frac{1}{3} = \frac{1}{4}$
47. $0.7x - 0.3 = 1.8$
48. $5x + 0.050 = -0.732$

49. $6t - 8 = 4t - 2$

50. $\dfrac{x}{4} + 2x = \dfrac{1}{3}$

51. $\dfrac{x-3}{2} = \dfrac{x+4}{3}$

52. $\dfrac{x-5}{4} = \dfrac{x-9}{3}$

53. $6t - 7 = 8t + 9$

54. $12x - 1.2 = 3x + 1.5$

55. $2(x + 3) - 4 = 2(x - 4)$

56. $3(x + 2) + 2(x - 1) = 5x - 7$

57. $4(x - 4) + 12 = 4(x - 1)$

58. $\dfrac{y}{3} + 4 = \dfrac{2y}{5} - 6$

59. $\frac{1}{4}(x + 4) = \frac{2}{5}(x + 2)$

60. $\frac{2}{3}(x + 5) = \frac{1}{4}(x + 2)$

61. $3x + 2 - 6x = -x - 15 + 8 - 5x$

62. $6x + 8 - 22x = 28 + 14x - 10 + 12x$

63. $2(t - 3) + 2 = 2(2t - 6)$

64. $5.7x - 3.1(x + 5) = 7.3$

Problem Solving

In Exercises 65 and 66, use the DeKalb County water rate of $2.05 per 1000 gallons of water used.

65. *Water Bill* What is the water bill if a resident uses 35,300 gal?

66. *Limiting the Cost* How many gallons of water can the customer use if the water bill is not to exceed $40.68?

67. *Dial Bodywash* A bottle of Dial Bodywash contains 354 milliliters (ml) of soap. If Tony Vaszquez uses 6 ml for each shower, how many times can he shower using one bottle of Dial Bodywash?

68. *Fajitas* A recipe for six servings of beef fajitas requires 16 oz of beef sirloin.

a) If the recipe were to be made for nine servings, how many ounces of beef sirloin would be needed?

b) How many servings of beef fajitas can be made with 32 oz of beef sirloin?

69. *Watching Television* Nielson Media Research determines the number of people who watch a television show. One rating point means that about 1,022,000 households watched the show. The top-rated television show for the week of September 23, 2002, was *Friends*, with a rating of 20.3. About how many households watched *Friends* that week?

70. *Grass Seed Coverage* A 20 lb bag of grass seed will cover an area of 10,000 ft^2.

a) How many pounds are needed to cover an area of 140,000 ft^2?

b) How many bags of grass seed must be purchased to cover an area of 140,000 ft^2?

71. *Speed Limit* When Jacob Abbott crossed over from Niagara Falls, New York, to Niagara Falls, Canada, he saw a sign that said 50 miles per hour (mph) is equal to 80 kilometers per hour (kph).

a) How many kilometers per hour are equal to mph?

b) On a stretch of the Queen Elizabeth Way, the speed limit is 90 kph. What is the speed limit in miles per hour?

72. *The Proper Dosage* A doctor asks a nurse to give a patient 250 milligrams (mg) of the drug Simethicone. The drug is available only in a solution whose concentration is 40 mg Simethicone per 0.6 millimeter (mm) of solution. How many millimeters of solution should the nurse give the patient?

Amount of Insulin In Exercises 73 and 74, how much insulin (in cc) would be given for the following doses? (Refer to Example 15 on page 302.)

73. 12 units of insulin from a vial marked U40

74. 35 units of insulin from a vial marked U40

75. a) In your own words, summarize the procedure to use to solve an equation.

b) Solve the equation $2(x + 3) = 4x + 3 - 5x$ with the procedure you outlined in part (a).

76. a) What is an identity?

b) When solving an equation, how will you know if the equation is an identity?

77. a) What is an inconsistent equation?

b) When solving an equation, how will you know if the equation is inconsistent?

Challenge Problems/Group Activities

78. *Depth of a Submarine* The pressure, P, in pounds per square inch (psi), exerted on an object x ft below the sea is given by the formula $P = 14.70 + 0.43x$. The 14.70 represents the weight in pounds of the column of air (from sea level to the top of the atmosphere) standing over a 1 in. by 1 in. square of seawater. The $0.43x$ represents the weight in pounds of a column of water 1 in. by 1 in. by x ft (see Fig. 6.1).

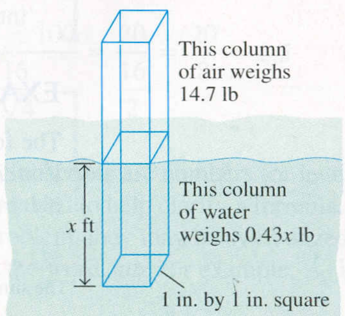


Figure 6.1

a) A submarine is built to withstand a pressure of 148 psi. How deep can that submarine go?

b) If the pressure gauge in the submarine registers a pressure of 128.65 psi, how deep is the submarine?

79. a) *Gender Ratios* If the ratio of males to females in a class is 2 : 3, what is the ratio of males to all the students in the class? Explain your answer.

b) If the ratio of males to females in a class is $m : n$, what is the ratio of males to all the students in the class?

Internet/Research Activities

80. Ratio and proportion are used in many different ways in everyday life. Submit two articles from newspapers, magazines, or the Internet in which ratios and/or proportions are used. Write a brief summary of each article explaining how ratio and/or proportion were used.

81. Write a report explaining how the ancient Egyptians used equations. Include in your discussion the forms of the equations used.

6.3 FORMULAS

A ***formula*** is an equation that typically has a real-life application. To ***evaluate a formula***, substitute the given values for their respective variables and then evaluate using the order of operations given in Section 6.1. Many of the formulas given in this section are discussed in greater detail in other parts of the book.

EXAMPLE 1 *Simple Interest*

The simple interest formula*, interest = principal × rate × time, or $i = prt$, is used to find the interest you must pay on a simple interest loan when you borrow principal, p, at simple interest rate, r, in decimal form, for time, t. Chris Campbell borrows \$3000 at a simple interest rate of 9% for 3 years.

a) How much will Chris Campbell pay in interest at the end of 3 years?

b) What is the total amount he will repay the bank at the end of 3 years?

SOLUTION:

a) Substitute the values of p, r, and t into the formula; then evaluate.

$$\begin{aligned} i &= prt \\ &= 3000(0.09)(3) \\ &= 810 \end{aligned}$$

Thus, Chris must pay \$810 interest.

b) The total he must pay at the end of 3 years is the principal, \$3000, plus the \$810 interest, for a total of \$3810.

EXAMPLE 2 *Volume of a Cereal Box*

The formula for the volume of a box* is volume = length × width × height or $V = lwh$. Use the formula $V = lwh$ to find the width of a Sweet Treats cereal box if $l = 7.5$ in., $h = 10.5$ in., and $V = 196.875$ in.^{3}.

*The simple interest formula is discussed in Section 10.2. The volume formula is discussed in Section 9.4.

SOLUTION: We substitute the appropriate values into the volume formula and solve for the desired quantity, w.

$$
\begin{aligned}
V &= lwh \\
196.875 &= (7.5)w(10.5) \\
196.875 &= 78.75w \\
\frac{196.875}{78.75} &= w \\
2.5 &= w
\end{aligned}
$$

Therefore, the width of a Sweet Treats cereal box is 2.5 in.

In Example 1, we used the formula $i = prt$. In Example 2, we used the formula $V = lwh$. In these examples, we used a mathematical equation to represent real phenomena. When we represent real phenomena, such as finding simple interest, mathematically we say we have created a ***mathematical model*** or simply a ***model*** to represent the situation. A model may be a single formula, or equation, or a system of many equations. By using models we gain insight into real-life situations, such as how much interest you will accumulate in your savings account. We will use mathematical models throughout this chapter and elsewhere in the text. In some exercises in this and the next chapter, when you are asked to determine an equation to represent a real-life situation, we will sometimes write the word *model* in the instructions.

Many formulas contain Greek letters, such as μ (mu), σ (sigma), Σ (capital sigma), δ (delta), ϵ (epsilon), π (pi), θ (theta), and λ (lambda). Example 3 makes use of Greek letters.

EXAMPLE 3 *A Statistics Formula*

A formula used in the study of statistics to find the standard score (or z-score) is

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Find the value of z when $\bar{x}$ (read "x bar") $= 120$, $\mu = 100$, $\sigma = 16$, and $n = 4$.

SOLUTION:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{120 - 100}{\frac{16}{\sqrt{4}}} = \frac{20}{\frac{16}{2}} = \frac{20}{8} = 2.5$$

Some formulas contain ***subscripts***. Subscripts are numbers (or letters) placed below and to the right of variables. They are used to help clarify a formula. For example, if two different amounts are used in a problem, they may be symbolized as A and A_0, or A_1 and A_2. Subscripts are read using the word *sub*; for example, A_0 is read "A sub zero" and A_1 is read "A sub one."

Exponential Equations

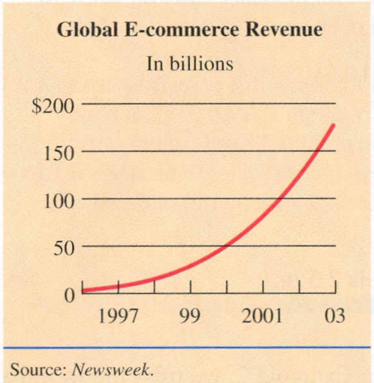


Figure 6.2

Many real-life problems, including population growth, growth of bacteria, and decay of radioactive substances, increase or decrease at a very rapid rate. For example, in Fig. 6.2, which shows global electronic business revenue, in billions of dollars, from 1996 through 2003, the graph is increasing rapidly. This is an example where the graph is increasing ***exponentially***. The equation of a graph that increases or decreases exponentially is called an ***exponential equation*** (or ***exponential formula***). An exponential equation is of the form $y = a^x$, $a > 0$, $a \neq 1$. We often use exponential equations to model real-life problems. In Section 6.10, we will discuss exponential equations (and exponential formulas) in more detail.

In an exponential formula, letters other than x and y may be used to represent the variables. The following equations are examples of exponential formulas: $y = 2^x$, $A = (\frac{1}{2})^x$ and $P = 2.3^t$. Note in the exponential formula that the variable is the exponent of some positive constant that is not equal to 1. In many real-life applications, the variable t will be used to represent time. Problems involving exponential formulas can be evaluated much more easily if you use a calculator containing a $\boxed{y^x}$, $\boxed{x^y}$, or $\boxed{\wedge}$ key.

The following formula, referred to as the ***exponential growth*** or ***decay formula***, is used to solve many real-life problems.

$$P = P_0 a^{kt}, \quad a > 0, \quad a \neq 1$$

In the formula, P_0 represents the original amount present, P represents the amount present after t years, and a and k are constants.

When $k > 0$, P increases as t increases and we have exponential growth. When $k < 0$, P decreases as t increases and we have exponential decay.

EXAMPLE 4 *Using an Exponential Decay Formula*

Carbon dating is used by scientists to find the age of fossils, bones, and other items. The formula used in carbon dating is

$$P = P_0 2^{-t/5600}$$

where P_0 represents the original amount of carbon 14 (C_{14}) present and P represents the amount of C_{14} present after t years. If 10 mg of C_{14} is present in an animal bone recently excavated, how many milligrams will be present in 3000 years?

SOLUTION: Substituting the values in the formula gives

$$P = P_0 2^{-t/5600}$$
$$P = 10(2)^{-3000/5600}$$
$$P \approx 10(2)^{-0.54} \quad \text{(Recall that } \approx \text{ means "is approximately equal to")}$$
$$P \approx 10(0.69)$$
$$P \approx 6.9 \text{ mg}$$

Thus, in 3000 years, approximately 6.9 mg of the original 10 mg of C_{14} will remain. ▲

In Example 4, we used a calculator to evaluate $(2)^{-3000/5600}$. The steps used to find this quantity on a calculator with a $\boxed{y^x}$ key are

$$2\ \boxed{y^x}\ \boxed{(}\ 3000\ \boxed{+/-}\ \boxed{\div}\ 5600\ \boxed{)}\ \boxed{=}\ .6898170602$$

After the $\boxed{=}$ key is pressed, the calculator displays the answer 0.689817. To evaluate $10(2)^{-3000/5600}$ on a scientific calculator, we can press the following keys.

$$10\ \boxed{\times}\ 2\ \boxed{y^x}\ \boxed{(}\ 3000\ \boxed{+/-}\ \boxed{\div}\ 5600\ \boxed{)}\ \boxed{=}\ 6.898170602$$

Notice that the answer obtained using the calculator steps shown above is a little more accurate than the answer we gave when we rounded the values before the final answer in Example 4.

When the a in the formula $P = P_0a^{kt}$ is replaced with the very special letter e, we get the ***natural exponential formula***

$$P = P_0e^{kt}$$

The letter e represents an irrational number whose value is approximately 2.7183. The number e plays an important role in mathematics and is used in finding the solution to many application problems.

To evaluate $e^{(0.04)5}$ on a calculator, as will be needed in Example 5, press*

$$\boxed{(}\ .04\ \boxed{\times}\ 5\ \boxed{)}\ \underbrace{\boxed{\text{inv}}}_{\text{Inverse key}}\ \underbrace{\boxed{\ln}}_{\text{Natural logarithm key}}\ 1.221402758$$

After the $\boxed{\ln}$ key is pressed, the calculator displays the answer 1.221402758.

To evaluate $10{,}000e^{(0.04)5}$ on a calculator, press

$$10000\ \boxed{\times}\ \boxed{(}\ .04\ \boxed{\times}\ 5\ \boxed{)}\ \boxed{\text{inv}}\ \boxed{\ln}\ \boxed{=}\ 12214.02758$$

In this calculation, after the $\boxed{=}$ key is pressed, the calculator displays the answer 12214.02758.

EXAMPLE 5 *Using an Exponential Growth Formula*

Banks often credit compound interest continuously. When that is done, the principal amount in the account, P, at any time t can be calculated by the natural exponential formula $P = P_0e^{kt}$, where P_0 is the initial principal invested, k is the interest rate in decimal form, and t is the time.

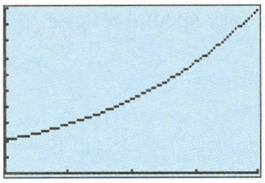
*Keys to press may vary on some calculators.

Suppose \$10,000 is invested in a savings account at a 4% interest rate compounded continuously. What will be the balance (or principal) in the account in 5 years?

SOLUTION:

$$\begin{aligned} P &= P_0e^{kt} \\ &= 10{,}000e^{(0.04)5} \\ &= 10{,}000e^{(0.20)} \\ &\approx 10{,}000(1.221402758) \\ &\approx 12{,}214 \end{aligned}$$

Thus, after 5 years, the account's value will have grown from \$10,000 to about \$12,214, an increase of about \$2214. ▲



$P = 10{,}000e^{0.04t}$

Figure 6.3

Graphing calculators are a tool that can be used to graph equations. Figure 6.3 shows the graph of $P = 10{,}000e^{0.04t}$ as it appears on the screen (or window) of a Texas Instrument TI-83 Plus graphing calculator. To obtain this screen, the domain (or the x-values) and range (or the y-values) of the window need to be set to selected values. We will speak a little more about graphing calculators shortly. In Section 6.10, we will explain how to graph exponential equations by plotting points.

EXAMPLE 6 *Population of Nevada*

The population of Nevada, which was the fastest growing state in every decade of the twentieth century except for the 1950s, is continuing to grow exponentially at the rate of about 5.10% per year. In 2000, the population of Nevada was 1,998,257. Nevada's expected population, t years after 2000 is given by the formula $P = 1{,}998{,}257e^{0.0510t}$. Find the expected population of Nevada in the year 2010.

SOLUTION: Since 2010 is 10 years after 2000, $t = 10$ years.

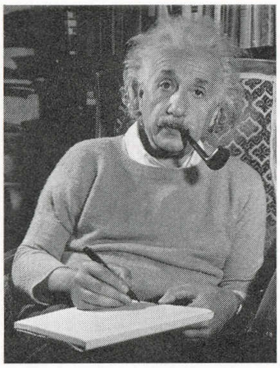
$$\begin{aligned} P &= 1{,}998{,}257e^{0.0510t} \\ &= 1{,}998{,}257e^{0.0510(10)} && \text{Substitute 10 for } t. \\ &= 1{,}998{,}257e^{0.510} \\ &\approx 1{,}998{,}257(1.665291195) \\ &\approx 3{,}327{,}679.787 \end{aligned}$$

Thus, in the year 2010, the population of Nevada is expected to be about 3,327,697 people. ▲

TIMELY TIP When doing calculations on the calculator, do not round any value before obtaining the final answer. By not rounding, you will obtain a more accurate answer. For example, if we work Example 6 on a calculator, and rounded $e^{0.510}$ to 1.67, we would determine that the population of Nevada in 2010 is expected to be about 3,337,089 which is a less accurate answer.

DID YOU KNOW

$\cancel{L} = mc^2$
$E = mc^2$



Many of you recognize the formula $E = mc^2$ used by Albert Einstein in 1912 to describe his groundbreaking theory of relativity. In the formula, E is energy, m is mass, and c is the speed of light. In his theory, Einstein hypothesized that time was not absolute and that mass and energy were related.

Einstein's 72-page handwritten manuscript, which went on display in 1999 at Jerusalem's Israel Museum, shows that Einstein toyed with using an L rather than an E to represent energy. He wrote $L = mc^2$, and then scratched out the L and replaced it with an E to produce the equation $E = mc^2$. Yisrail Galili at Jerusalem's Hebrew University said the L might have been a lambda, a Greek letter sometimes used in functions related to energy. He said it might also be related to Joseph Lagrange, an eighteenth-century Italian mathematician.

Solving for a Variable in a Formula or Equation

Often in mathematics and science courses, you are given a formula or an equation expressed in terms of one variable and asked to express it in terms of a different variable. For example, you may be given the formula $P = i^2r$ and asked to solve the formula for r. To do so, treat each of the variables, except the one you are solving for, as if it were a constant. Then solve for the variable desired, using the properties previously discussed. Examples 7 through 9 show how to do this task.

When graphing equations in Section 6.7, you will sometimes have to solve the equation for the variable y as is done in Example 7.

EXAMPLE 7 *Solving an Equation Containing More Than One Variable*

Solve the equation $2x + 5y - 10 = 0$ for y.

SOLUTION: We need to isolate the term containing the variable y. Begin by moving the constant, -10, and the term $2x$ to the right-hand side of the equation.

$$2x + 5y - 10 = 0$$

$$2x + 5y - 10 + 10 = 0 + 10 \qquad \text{Addition property}$$

$$2x + 5y = 10$$

$$-2x + 2x + 5y = -2x + 10 \qquad \text{Subtraction property}$$

$$5y = -2x + 10$$

$$\frac{5y}{5} = \frac{-2x + 10}{5} \qquad \text{Division property}$$

$$y = \frac{-2x + 10}{5}$$

$$y = -\frac{2x}{5} + \frac{10}{5}$$

$$y = -\frac{2}{5}x + 2$$

Note that once you have found $y = \frac{-2x + 10}{5}$, you have solved the equation for y. The solution can also be expressed in the form $y = -\frac{2}{5}x + 2$. This form of the equation is convenient for graphing equations, as will be explained in Section 6.7. Example 7 can also be solved by moving the y term to the right-hand side of the equal sign. Do so now and note that you obtain the same answer.

EXAMPLE 8 *Solving for a Variable in a Formula*

An important formula used in statistics is

$$z = \frac{x - \mu}{\sigma}$$

Solve this formula for x.

SOLUTION: To isolate the term x, use the general procedure for solving linear equations given in Section 6.2. Treat each letter, except x, as if it were a constant.

$$z = \frac{x - \mu}{\sigma}$$

$$z \cdot \sigma = \frac{x - \mu}{\cancel{\sigma}} \cdot \cancel{\sigma} \qquad \text{Multiplication property}$$

$$z\sigma = x - \mu$$

$$z\sigma + \mu = x - \mu + \mu \qquad \text{Add } \mu \text{ to both sides of the equation.}$$

$$z\sigma + \mu = x$$

$$\text{or} \quad x = z\sigma + \mu$$

EXAMPLE 9 *The Tax Free Yield Formula*

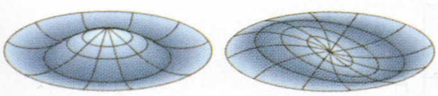
A formula that may be important to you now or sometime in the future is the tax-free yield formula, $T_f = T_a(1 - F)$. This formula can be used to convert a taxable yield, T_a, into its equivalent tax-free yield, T_f, where F is the federal income tax bracket of the individual. A taxable yield is an interest rate for which income tax is paid on the interest made. A tax-free yield is an interest rate for which income tax does not have to be paid on the interest made.

PROFILE IN MATHEMATICS

SOPHIE GERMAIN

Because she was a woman, Sophie Germain (1776–1831) was denied admission to the École Polytechnic, the French academy of mathematics and science. Not to be stopped, she obtained lecture notes from courses in which she had an interest, including one taught by Joseph-Louis Lagrange. Under the pen name M. LeBlanc, she submitted a paper on analysis to Lagrange, who was so impressed with the report that he wanted to meet the author and personally congratulate "him." When he found out that the author was a woman, he became a great help and encouragement to her. Lagrange introduced Germain to many of the French scientists of the time.

In 1801, Germain wrote the great German mathematician Carl Friedrich Gauss to discuss Fermat's equation, $x^n + y^n = z^n$. He commended her for showing "the noblest courage, quite extraordinary talents and a superior genius." Germain's interests included work in number theory and mathematical physics. She would have received an honorary doctorate from the University of Göttingen, based on Gauss's recommendation, but died before the honorary doctorate could be awarded.



Germain was the first person to devise a formula describing elastic motion. The study of the equations for the elasticity of different materials aided the development of acoustical diaphragms in loudspeakers and telephones.

a) For someone in a 25% tax bracket, find the equivalent tax-free yield of a 4% taxable investment.

b) Solve this formula for T_a. That is, write a formula for taxable yield in terms of tax-free yield.

SOLUTION:

a) $T_f = T_a(1 - F)$
$= 0.04(1 - 0.25) = 0.04(0.75) = 0.03$, or 3%

Thus, a taxable investment of 4% is equivalent to a tax-free investment of 3% for a person in a 25% income tax bracket.

b) $T_f = T_a(1 - F)$

$$\frac{T_f}{1-F} = \frac{T_a\cancel{(1-F)}}{\cancel{1-F}}$$ Divide both sides of the equation by $1 - F$.

$$\frac{T_f}{1-F} = T_a, \quad \text{or} \quad T_a = \frac{T_f}{1-F}$$

SECTION 6.3 EXERCISES

Concept/Writing Exercises

1. What is a formula?
2. Explain how to evaluate a formula.
3. What are subscripts?
4. What is the simple interest formula?
5. What is an exponential equation?
6. a) In an exponential equation of the form $y = a^x$, what are the restrictions on a?
   b) In an exponential equation of the form $y = P_0a^{kt}$, what does P_0 represent?

Practice the Skills

In Exercises 7–38, use the formula to find the value of the indicated variable for the values given. Use a calculator when one is needed. When necessary, round answers to the nearest hundredth.

7. $P = 4s$; find P when $s = 5$ (geometry).
8. $P = a + b + c$; find P when $a = 25$, $b = 53$ and $c = 32$ (geometry).
9. $P = 2l + 2w$; find P when $l = 12$ and $w = 16$ (geometry).
10. $F = ma$; find m when $F = 40$ and $a = 5$ (physics).
11. $E = mc^2$; find m when $E = 400$ and $c = 4$ (physics).
12. $p = i^2r$; find r when $p = 62{,}500$ and $i = 5$ (electronics).
13. $A = \pi(R^2 - r^2)$; find A when $R = 6$, $\pi = 3.14$, and $r = 4$ (geometry).
14. $B = \dfrac{703w}{h^2}$; find B when $w = 130$ and $h = 67$ (for finding body mass index).
15. $z = \dfrac{x - \mu}{\sigma}$; find μ when $z = 2.5$, $x = 42.1$, and $\sigma = 2$ (statistics).
16. $S = B + \frac{1}{2}Ps$; find P when $s = 10$, $S = 300$, and $B = 100$ (geometry).
17. $T = \dfrac{PV}{k}$; find P when $T = 80$, $V = 20$, and $k = 0.5$ (physics).
18. $m = \dfrac{a + b + c}{3}$; find a when $m = 70$, $b = 60$, and $c = 90$ (statistics).
19. $A = P(1 + rt)$; find P when $A = 3600$, $r = 0.04$, and $t = 5$ (economics).
20. $m = \dfrac{a + b}{2}$; find a when $m = 70$ and $b = 77$ (statistics).
21. $v = \frac{1}{3}at^2$; find a when $v = 576$ and $t = 12$ (physics).
22. $F = \frac{9}{5}C + 32$; find F when C = 7 (temperature conversion).
23. $C = \frac{5}{9}(F - 32)$; find C when F = 77 (temperature conversion).

24. $K = \dfrac{F - 32}{1.8} + 273.1$; find K when F = 100 (chemistry).

25. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$; find m when $y_2 = 8$, $y_1 = -4$, $x_2 = -3$, and $x_1 = -5$ (mathematics).

26. $z = \dfrac{\bar{x} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$; find z when $\bar{x} = 66$, $\mu = 60$, $\sigma = 15$, and $n = 25$ (statistics).

27. $S = R - rR$; find R when $S = 186$ and $r = 0.07$ (for determining sale price when an item is discounted).

28. $S = C + rC$; find C when $S = 115$ and $r = 0.15$ (for determining selling price when an item is marked up).

29. $E = a_1p_1 + a_2p_2 + a_3p_3$; find E when $a_1 = 5$, $p_1 = 0.2$, $a_2 = 7$, $p_2 = 0.6$, $a_3 = 10$, and $p_3 = 0.2$ (probability).

30. $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$; find x when $a = 2$, $b = -5$, and $c = -12$ (mathematics).

31. $s = -16t^2 + v_0t + s_0$; find s when $t = 4$, $v_0 = 30$, and $s_0 = 150$ (physics).

32. $R = O + (V - D)r$; find O when $R = 670$, $V = 100$, $D = 10$, and $r = 4$ (economics).

33. $P = \dfrac{f}{1 + i}$; find f when $i = 0.08$ and $P = 3000$ (investment banking).

34. $c = \sqrt{a^2 + b^2}$; find c when $a = 5$ and $b = 12$ (geometry).

35. $F = \dfrac{Gm_1m_2}{r^2}$; find G when $F = 625$, $m_1 = 100$, $m_2 = 200$, and $r = 4$ (physics).

36. $P = \dfrac{nRT}{V}$; find V if $P = 12$, $n = 10$, $R = 60$, and $T = 8$ (chemistry).

37. $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$; find S_n when $a_1 = 8$, $r = \frac{1}{2}$, and $n = 3$ (mathematics).

38. $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$; find A when $P = 100$, $r = 6\%$, $n = 1$, and $t = 3$ (banking).

In Exercises 39–48, solve the equation for y.

39. $10x - 4y = 13$

40. $8x - 6y = 21$

41. $4x + 7y = 14$

42. $-2x + 4y = 9$

43. $2x - 3y + 6 = 0$

44. $3x + 4y = 0$

45. $-2x + 3y + z = 15$

46. $5x + 3y - 2z = 22$

47. $9x + 4z = 7 + 8y$

48. $2x - 3y + 5z = 0$

In Exercises 49–68, solve for the variable indicated.

49. $E = IR$ for R

50. $p = irt$ for t

51. $p = a + b + c$ for a

52. $p = a + b + s_1 + s_2$ for s_1

53. $V = \frac{1}{3}Bh$ for B

54. $V = \pi r^2h$ for h

55. $C = 2\pi r$ for r

56. $r = \dfrac{2gm}{c^2}$ for m

57. $y = mx + b$ for b

58. $y = mx + b$ for m

59. $P = 2l + 2w$ for w

60. $A = \dfrac{d_1d_2}{2}$ for d_2

61. $A = \dfrac{a + b + c}{3}$ for c

62. $A = \frac{1}{2}bh$ for b

63. $P = \dfrac{KT}{V}$ for T

64. $\dfrac{P_1V_1}{T_1} = \dfrac{P_2V_2}{T_2}$ for V_2

65. $F = \frac{9}{5}C + 32$ for C

66. $C = \frac{5}{9}(F - 32)$ for F

67. $S = \pi r^2 + \pi rs$ for s

68. $A = \dfrac{1}{2}h(b_1 + b_2)$ for b_2

Problem Solving

69. ***Refund Check*** Joel and Patti Karpel received a \$600 income tax refund check from the federal government and decided to deposit the check in a money market account that paid 2% simple interest per year. Determine

a) how much interest was added to their account at the end of 1 year.

b) the balance in their account at the end of 1 year.

70. ***Interest on a Loan*** Jeff Hubbard borrowed \$800 from his brother for 2 years. At the end of 2 years, he repaid the \$800 plus \$128 in interest. What simple interest rate did he pay?

71. ***Volume in a Soup Can*** Determine the volume of a cylindrical soup can if its diameter is 2.5 in. and its height is 3.75 in. (The formula for the volume of a cylinder is $V = \pi r^2h$. Use your pi key, $\boxed{\pi}$, on your calculator, or 3.14 for π if your calculator does not have a $\boxed{\pi}$ key.) Round your answer to the nearest tenth.

72. ***Body Mass Index*** A person's body mass index (BMI) is found by the formula $B = \dfrac{703w}{h^2}$, where w is the person's weight, in pounds, and h is the person's height, in inches. Lance Bass is 6 ft tall and weighs 200 lb.

a) Determine his BMI.

b) If Lance would like to have a BMI of 26, how much weight would he need to gain or lose?

73. *Bacteria* The number of a certain type of bacteria, y, present in a culture is determined by the equation $y = 2000(3)^x$, where x is the number of days the culture has been growing. Find the number of bacteria present after 5 days.

74. *Adjusting for Inflation* If P is the price of an item today, the price of the same item n years from today, P_n, is $P_n = P(1 + r)^n$, where r is the constant rate of inflation. Determine the price of a movie ticket 10 years from today if the price today is \$8.00 and the annual rate of inflation is constant at 3%.

75. *Value of New York City* Assume the value of the island of Manhattan has grown at an exponential rate of 8% per year since 1626 when Peter Minuit of the Dutch West India Company purchased the island for \$24. The value of the island, V, at any time, t, in years after 1626, can be found by the formula $V = 24e^{0.08t}$. What is the value of the island in 2003, 377 years after Minuit purchased it?

76. *Radioactive Decay* Strontium 90 is a radioactive isotope that decays exponentially at a rate of 2.8% per year. The amount of strontium 90, S, remaining after t years can be found by the formula $S = S_0e^{-0.028t}$, where S_0 is the original amount present. If there are originally 1000 g of strontium 90, find the amount of strontium 90 remaining after 30 years.

Challenge Problem/Group Activity

77. Determine the volume of the block shown in Fig. 6.4, excluding the hole.

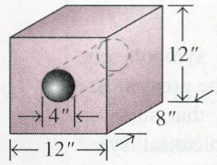


Figure 6.4

Recreational Mathematics

78. *Triangle Seek* In this word seek, you are looking for six-letter words that form triangles. The first letter of the words can be in any position in the triangle; the remaining letters of the words are in order moving clockwise or counterclockwise. Triangles may overlap other triangles and the triangles can point up or down. For example, the word LINEAR is indicated below. The word list is below on the right.

S F A R R L I D G R T P
C C J O A N Y E B O L F
R I E D N E N D O M S O
T E E E W T O M E Y I H
F S E R G H T L Y Z M T
A P J N R W C I N C U D
M D P P R O L R N A T A
G R O I V E U U F R O U

Word List
LINEAR
SYMBOL
DEGREE
FACTOR
GROWTH
NEWTON

6.4 APPLICATIONS OF LINEAR EQUATIONS IN ONE VARIABLE

One of the main reasons for studying algebra is that it can be used to solve everyday problems. In this section, we will do two things: (1) show how to translate a written problem into a mathematical equation and (2) show how linear equations can be used in solving everyday problems. We begin by illustrating how English phrases can be written as mathematical expressions. When writing a mathematical expression, we may use any letter to represent the variable. In the following illustrations, we use the letter x.

Phrase	Mathematical expression
Six more than a number	$x + 6$
A number increased by 3	$x + 3$
Four less than a number	$x - 4$
A number decreased by 9	$x - 9$
Twice a number	$2x$
Four times a number	$4x$
3 decreased by a number	$3 - x$
The difference between a number and 5	$x - 5$

DID YOU KNOW

Survival of the Fittest

Symbols come and symbols go; the ones that find the greatest acceptance are the ones that survive. The Egyptians used pictorial symbols: a pair of legs walking forward for addition or backward for subtraction. Robert Recorde (1510–1558) used two parallel lines to represent "equals" because "no 2 thynges can be moore equalle." Some symbols evolved from abbreviations, such as the "+" sign, which comes from the Latin *et* meaning "and." The evolution of others is less clear. The invention of the printing press in the fifteenth century led to a greater standardization of symbols already in use.

Sometimes the phrase that must be converted to a mathematical expression involves more than one operation.

Phrase	Mathematical expression
Four less than 3 times a number	$3x - 4$
Ten more than twice a number	$2x + 10$
The sum of 5 times a number and 3	$5x + 3$
Eight times a number decreased by 7	$8x - 7$

The word *is* often represents the equal sign.

Phrase	Mathematical equation
Six more than a number is 10.	$x + 6 = 10$
Five less than a number is 20.	$x - 5 = 20$
Twice a number decreased by 6 is 12.	$2x - 6 = 12$
A number decreased by 13 is 6 times the number.	$x - 13 = 6x$

The following is a general procedure for solving word problems.

To Solve a Word Problem

1. Read the problem carefully at least twice to be sure that you understand it.
2. If possible, draw a sketch to help visualize the problem.
3. Determine which quantity you are being asked to find. Choose a letter to represent this unknown quantity. Write down exactly what this letter represents.
4. Write the word problem as an equation.
5. Solve the equation for the unknown quantity.
6. Answer the question or questions asked.
7. Check the solution.

This general procedure for solving word problems is illustrated in Examples 1 through 4. In these examples, the equations we obtain are mathematical models of the given situations.

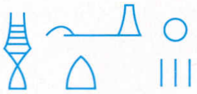
EXAMPLE 1 *How Much Can You Purchase?*

Roberto Raynor spent \$339.95 on textbooks at the bookstore. In addition to textbooks, he wanted to purchase as many notebooks as possible, but he only has a total of \$350 to spend. If a notebook, including tax, costs \$0.99, how many notebooks can he purchase?

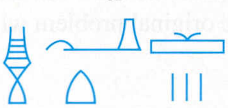
SOLUTION: In this problem, the unknown quantity is the number of notebooks Roberto can purchase. Let's select n to represent the number of notebooks he can purchase. Then we construct an equation using the given information that will allow us to solve for n.

DID YOU KNOW

Aha!

Aha: 

or

Heap: 

The Egyptians as far back as 1650 B.C. had a knowledge of linear equations. They used the words *aha* or *heap* in place of the variable. Problems involving linear equations can be found in the Rhind Papyrus (See Chapter 4, p. 168).

Let

$$n = \text{number of notebooks Roberto can purchase}$$

Then

$$\$0.99n = \text{cost for } n \text{ notebooks at } \$0.99 \text{ per notebook}$$

$$\begin{array}{ccccc} \text{Cost of textbooks} & + & \text{cost of notebooks} & = & \text{total amount to spend} \\ \$339.95 & + & \$0.99n & = & \$350 \end{array}$$

Now solve the equation.

$$\begin{aligned} 339.95 + 0.99n &= 350 \\ 339.95 - 339.95 + 0.99n &= 350 - 339.95 \\ 0.99n &= 10.05 \\ \frac{0.99n}{0.99} &= \frac{10.05}{0.99} \\ n &\approx 10.15 \end{aligned}$$

Therefore, Roberto can purchase 10 notebooks. When we solve the equation, we obtain $n = 10.15$ (to the nearest hundredth). Since he cannot purchase a part of a notebook, only 10 notebooks can be purchased.

Check: The check is made with the information given in the original problem.

$$\begin{aligned} \text{Total amount to spend} &= \text{cost of textbooks} + \text{cost of notebooks} \\ &= 339.95 + 0.99(10) \\ &= 339.95 + 9.90 \\ &= 349.85 \end{aligned}$$

This result would leave 15 cents change from the \$350 he has to spend, which is not enough to purchase another notebook. Therefore, this answer checks.

EXAMPLE 2

Forty hours of overtime must be split among three workers. One worker will be assigned twice the number of hours as each of the other two. How many hours of overtime will be assigned to each worker?

SOLUTION: Two workers receive the same amount of overtime, and the third worker receives twice that amount.

Let

$$\begin{aligned} x &= \text{number of hours of overtime for the first worker} \\ x &= \text{number of hours of overtime for the second worker} \\ 2x &= \text{number of hours of overtime for the third worker} \end{aligned}$$

Then

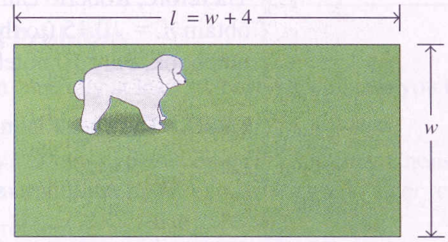
$$x + x + 2x = \text{total amount of overtime}$$
$$x + x + 2x = 40$$
$$4x = 40$$
$$x = 10$$

Thus, two workers are assigned 10 hours of overtime, and the third worker is assigned 2(10), or 20, hours of overtime. A check in the original problem will verify that this answer is correct.

EXAMPLE 3 *Dimensions of a Fenced in Area*

Robert Koch wants to fence in a rectangular region in his backyard for his poodle. He only has 56 ft of fencing to use for the perimeter of the region. What should the dimensions of the region be if he wants the length to be 4 ft greater than the width?

SOLUTION: The formula for finding the perimeter of a rectangle is $P = 2l + 2w$, where P is the perimeter, l is the length, and w is the width. A diagram, such as the one shown below, is often helpful in solving problems of this type.



Let w equal the width of the region. The length is 4 ft more than the width, so $l = w + 4$. The total distance around the region P, is 56 ft.

Substitute the known quantities in the formula.

$$P = 2w + 2l$$
$$56 = 2w + 2(w + 4)$$
$$56 = 2w + 2w + 8$$
$$56 = 4w + 8$$
$$48 = 4w$$
$$12 = w$$

The width of the region is 12 ft, and the length of the region is $12 + 4$ or 16 ft. ▲

In shopping and other daily activities, we are occasionally asked to solve problems using percents. The word *percent* means "per hundred." Thus, for example, 7% means 7 per hundred, or $\frac{7}{100}$. When $\frac{7}{100}$ is converted to a decimal number, we obtain 0.07. Thus, $7\% = 0.07$.

Let's look at one example involving percent. (See Section 11.1 for a more detailed discussion of percent.)

EXAMPLE 4 *Art Show*

Peggy McMahon is planning to sell her original paintings at an art show. Determine the cost of a painting before tax if the total cost of a painting, including an 8% sales tax, is to be \$145.80.

SOLUTION: We are asked to find the cost of a painting before sales tax.

Let

$$x = \text{cost of a painting before sales tax.}$$

Then

$$0.08x = 8\% \text{ of the cost of the painting (the sales tax)}$$

$$\begin{aligned}
\text{Cost of a painting before tax} + \text{tax on a painting} &= 145.80 \\
x + 0.08x &= 145.80 \\
1.08x &= 145.80 \\
\frac{1.08x}{1.08} &= \frac{145.80}{1.08} \\
x &= \frac{145.80}{1.08} \\
x &= 135
\end{aligned}$$

Thus, the cost of a painting before tax is \$135.

SECTION 6.4 EXERCISES

Concept/Writing Exercises

1. What is the difference between a mathematical expression and an equation?
2. Give an example of a mathematical expression and an example of a mathematical equation.

Practice the Skills

In Exercises 3–14, write the phrase as a mathematical expression.

3. 4 increased by 3 times x
4. 6 times x decreased by 2
5. 5 more than 6 times r
6. 10 times s decreased by 13
7. 15 decreased by twice r
8. 6 more than x
9. 2 times m increased by 9
10. 8 increased by 5 times x
11. 18 decreased by s, divided by 4
12. The sum of 8 and t, divided by 2
13. 6 less than the product of 5 times y, increased by 3
14. The quotient of 8 and y, decreased by 3 times x

In Exercises 15–26, write an equation and solve.

15. A number decreased by 6 is 5.
16. The sum of a number and 7 is 15.
17. The difference between a number and 4 is 20.
18. A number multiplied by 7 is 42.
19. Twelve increased by 5 times a number is 47.
20. Four times a number decreased by 10 is 42.
21. Sixteen more than 8 times a number is 88.
22. Six more than five times a number is 7 times the number decreased by 18.
23. A number increased by 11 is 1 more than 3 times the number.
24. A number divided by 3 is 4 less than the number.

25. A number increased by 10 is 2 times the sum of the number and 3.

26. The product of 2 and a number decreased by 3 is 4 more than the number.

Problem Solving

In Exercises 27–46, set up an equation that can be used to solve the problem. Solve the equation and find the desired value(s).

27. **MODELING -** *Ticket Sales* New Hyde Park High School sold 600 tickets to the play *The Wiz*. The number of tickets sold to students was three times the number of tickets sold to nonstudents. How many tickets were sold to students and to nonstudents?

28. **MODELING -** *New Clothing* Miguel Garcia purchases two new pairs of pants at The Gap for \$60. If one pair was \$10 more than the other, how much was the more expensive pair?

29. **MODELING -** *Income Tax* From 1999 to 2000, there was an 11.6% increase in the number of taxpayers filing their taxes electronically. If 34.20 million taxpayers filed their taxes electronically in 2000, how many million taxpayers filed their taxes electronically in 1999?

30. **MODELING -** *Sales Commission* Vinny Raineri receives a weekly salary of \$400 at Abbott's Appliances. He also receives a 6% commission on the total dollar amount of all sales he makes. What must his total sales be in a week if he is to make a total of \$790?

31. **MODELING -** *Pet Supplies* PetSmart has a sale offering 10% off of all pet supplies. If Amanda Miller spent \$15.72 on pet supplies before tax, what was the price of the pet supplies he purchased before the discount?

32. **MODELING -** *Copying* Ronnie McNeil pays 8¢ to make a copy of a page at a copy shop. She is considering purchasing a photocopy machine that is on sale for \$250, including tax. How many copies would Ronnie have to make in the copy shop for her cost to equal the purchase price of the photocopy machine she is considering buying?

33. **MODELING -** *Number of CD's* Samantha Silverstone and Josie Appleton receive 12 free compact discs by joining a compact disc club. How many CD's will each receive if Josie is to have three times as many as Samantha?

34. **MODELING -** *Scholarship Donation* Each year, Andrea Choi donates a total of \$1000 for scholarships at Mercer County Community College. This year she wants the amount she donates for scholarships for liberal arts to be three times the amount she donates for scholarships for business. Determine the amount she will donate for each type of scholarship.

35. **MODELING -** *Homeowners Association* Cross Creek Townhouses Homeowners Association needs to charge each homeowner a supplemental assessment to help pay for some unexpected repairs to the townhouses. The association has \$2000 in its reserve fund that it will use to help pay for the repairs. How much must the association charge each of the 50 homeowners if the total cost for the repairs is \$13,350?

36. **MODELING -** *Dimensions of a Deck* Jim Yuhas is building a rectangular deck and wants the length to be 3 ft greater than the width. What will be the dimensions of the deck if the perimeter is to be 54 ft?

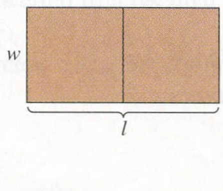
37. **MODELING -** *Floor Area* The total floor space in three barns is 45,000 ft^2. The two smaller ones have the same area, and the largest one is three times the area of the smaller ones.
    a) Determine the floor space for each barn.
    b) Can merchandise that takes up 8500 ft^2 of floor space fit into either of the smaller barns?

38. **MODELING -** *Average Salary* According to the Bureau of Economic Analysis, the average per capita income by state in 2000 was highest in Connecticut and lowest in Mississippi. The average per capita income in Connecticut was \$1346 less than two times the per capita income in Mississippi. If the sum of the average per capita income in Connecticut and Mississippi is \$61,663, what is the average per capita income in each state?

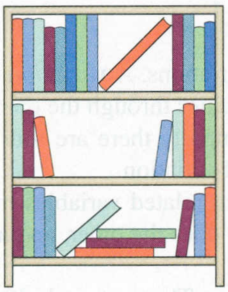
39. **MODELING -** *Vacation Days* According to the World Tourism Organization, the average number of vacation days for employees in Italy is 3 more than 3 times the average number of vacation days for employees in the United States. The sum of the average number of vacation days in Italy and in the United States is 55. Determine the average number of vacation days in each country.

40. **MODELING -** *Car Purchase* The Gilberts purchased a car. If the total cost, including a 5% sales tax, was \$14,512, find the cost of the car before tax.

41. **MODELING -** *Enclosing Two Pens* Chuck Salvador has 140 ft of fencing in which he wants to build two connecting, adjacent square pens (see the figure). What will be the dimensions if the length of the entire enclosed region is to be twice the width?



42. **MODELING - *Dimensions of a Bookcase*** A bookcase with three shelves is to be built by a woodworking student. If the height of the bookcase is to be 2 ft longer than the length of a shelf and the total amount of wood to be used is 32 ft, find the dimensions of the bookcase.



43. **MODELING - *Laundry Cost*** The cost of doing the family laundry for a month at a local laundromat is \$70. A new washer and dryer cost a total of \$760. How many months would it take for the cost of doing the laundry at the laundromat to equal the cost of a new washer and dryer?

44. **MODELING - *Health Club Cost*** A health club is offering two new membership plans. Plan A costs \$56 per month for unlimited use. Plan B costs \$20 per month plus \$3 for every visit. How many visits to the health club must Doug Jones make per month for Plan A to result in the same cost as Plan B?

45. **MODELING - *Airfare*** Rachel James has been told that with her half-off airfare coupon, her airfare from New York to San Diego will be \$257.00. The \$257.00 includes a 7% tax *on the regular fare*. On the way to the airport, Rachel realizes that she has lost her coupon. What will her regular fare be before tax?

Hotel del Coronado, San Diego, CA

46. **MODELING - *Truck Rentals*** The cost of renting a small truck at the U-Haul rental agency is \$35 per day plus 20¢ a mile. The cost of renting the same truck at the Ryder rental agency is \$25 per day plus 32¢ a mile. How far would you have to drive in one day for the cost of renting from U-Haul to equal the cost of renting from Ryder?

Challenge Problems/Group Activities

47. ***Income Tax*** Some states allow a husband and wife to file individual tax returns (on a single form) even though they have filed a joint federal tax return. It is usually to the taxpayers' advantage to do so when both husband and wife work. The smallest amount of tax owed (or the largest refund) will occur when the husband's and wife's taxable incomes are the same.

    Mr. McAdams's 2003 taxable income was \$24,200, and Mrs. McAdams's taxable income for that year was \$26,400. The McAdams's total tax deduction for the year was \$3640. This deduction can be divided between Mr. and Mrs. McAdams any way they wish. How should the \$3640 be divided between them to result in each individual's having the same taxable income and therefore the greatest tax refund?

48. Write each equation as a sentence. There are many correct answers.
    a) $x + 3 = 13$ b) $3x + 5 = 8$
    c) $3x - 8 = 7$

49. Show that the sum of any three consecutive integers is 3 less than 3 times the largest.

50. ***Auto Insurance*** A driver education course at the East Lake School of Driving costs \$45 but saves those under 25 years of age 10% of their annual insurance premiums until they are 25. Dan has just turned 18, and his insurance costs \$600.00 per year.
    a) When will the amount saved from insurance equal the price of the course?
    b) Including the cost of the course, when Dan turns 25, how much will he have saved?

Recreational Mathematics

51. The relationship between Fahrenheit temperature (F) and Celsius temperature (C) is shown by the formula $F = \frac{9}{5}C + 32$. At what temperature will a Fahrenheit thermometer read the same as a Celsius thermometer?

6.5 VARIATION

In Sections 6.3 and 6.4, we presented many applications of algebra. In this section, we introduce variation, which is an important tool in solving applied problems.

Direct Variation

Many scientific formulas are expressed as variations. A ***variation*** is an equation that relates one variable to one or more other variables through the operations of multiplication or division (or both operations). Essentially there are four types of variation problems: direct, inverse, joint, and combined variation.

In ***direct variation***, the values of the two related variables increase together or decrease together; that is, as one increases so does the other, and as one decreases so does the other.

Consider a car traveling at 40 miles an hour. The car travels 40 miles in 1 hour, 80 miles in 2 hours, and 120 miles in 3 hours. Note that, as the time increases, the distance traveled increases, and, as the time decreases, the distance traveled decreases. The formula used to calculate distance traveled is

$$\text{Distance} = \text{rate} \cdot \text{time}$$

Since the rate is a constant 40 miles per hour, the formula can be written

$$d = 40t$$

We say that distance *varies directly* as time or that distance is *directly proportional* to time.

The preceding equation is an example of direct variation.

Direct Variation

If a variable y varies directly with a variable x, then

$$y = kx$$

where k is the **constant of proportionality** (or the variation constant).

Examples 1 through 4 illustrate direct variation.

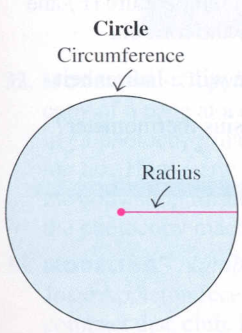


Figure 6.5

EXAMPLE 1 *Direct Variation in Geometry*

The circumference of a circle, C, is directly proportional to (or varies directly as) its radius, r; see Fig. 6.5. Write the equation for the circumference of a circle if the constant of proportionality, k, is 2π.

SOLUTION:

$C = kr$ C varies directly as r.

$C = 2\pi r$ Constant of proportionality is 2π.

EXAMPLE 2 *Direct Variation in Medicine*

The recommended dosage, d, of the antibiotic drug vancomycin is directly proportional to a person's weight, w.

a) Write this variation as an equation.

b) Find the recommended dosage, in milligrams, for Doug Kulzer, who weighs 192 lb. Assume the constant of proportionality for the dosage is 18.

SOLUTION:

a) $d = kw$

b) $d = 18(192) = 3456$

The recommended dosage for Doug Kulzer is 3456 mg.

In certain variation problems, the constant of proportionality, k, may not be known. In such cases, we can often find it by substituting the given values in the variation formula and solving for k.

EXAMPLE 3 *Finding the Constant of Proportionality*

Suppose w varies directly as the square of y. If w is 60 when y is 20, find the constant of proportionality.

SOLUTION: Since w varies directly as the *square of* y, we begin with the formula $w = ky^2$. Since the constant of proportionality is not given, we must find k using the given information. Substitute 60 for w and 20 for y.

$$w = ky^2$$
$$60 = k(20)^2$$
$$60 = 400k$$
$$\frac{60}{400} = \frac{400k}{400}$$
$$0.15 = k$$

Thus, the constant of proportionality is 0.15.

EXAMPLE 4 *Using the Constant of Proportionality*

The length that a spring will stretch, S, varies directly with the force (or weight), F, attached to the spring. If a spring stretches 4.2 in. when a 60-lb weight is attached, how far will it stretch when a 30-lb weight is attached?

SOLUTION: We begin with the formula $S = kF$. Since the constant of proportionality is not given, we must find k using the given information.

$$S = kF$$
$$4.2 = k(60)$$
$$\frac{4.2}{60} = k$$
$$0.07 = k$$

We now use $k = 0.07$ to find S when $F = 30$.

$$S = kF$$
$$S = 0.07F$$
$$S = 0.07(30)$$
$$S = 2.1 \text{ in.}$$

Thus, a spring will stretch 2.1 in. when a force of 30 lb is attached. ▲

Inverse Variation

A second type of variation is ***inverse variation***. When two quantities vary inversely, as one quantity increases, the other quantity decreases, and vice versa.

To explain inverse variation, we use the formula, distance = rate · time. If we solve for time, we get time = distance/rate. Assume the distance is fixed at 100 miles; then

$$\text{Time} = \frac{100}{\text{rate}}$$

At 100 miles per hour it would take 1 hour to cover this distance. At 50 miles an hour, it would take 2 hours. At 25 miles an hour, it would take 4 hours. Note that as the rate (or speed) decreases, the time increases and vice versa.

The preceding equation can be written

$$t = \frac{100}{r}$$

This equation is an example of an inverse variation. The time and rate are inversely proportional. The constant of proportionality in this case is 100.

Inverse Variation
If a variable y varies inversely with a variable x, then

$$y = \frac{k}{x}$$

where k is the constant of proportionality.

Two quantities *vary inversely*, or are *inversely proportional*, when as one quantity increases the other quantity decreases and vice versa. Examples 5 and 6 illustrate inverse variation.

EXAMPLE 5 *Inverse Variation in Astronomy*

The velocity, v, of a meteorite approaching Earth varies inversely as the square root of its distance from the center of Earth. Assuming the velocity is 2 miles per second at a distance of 6400 miles from the center of Earth, find the equation that expresses the relationship between the velocity of a meteorite and its distance from the center of Earth.

SOLUTION: Since the velocity of the meteorite varies inversely as the *square root* of its distance from the center of Earth, the general form of the equation is

$$v = \frac{k}{\sqrt{d}}$$

To find k, we substitute the given values for v and d.

$$2 = \frac{k}{\sqrt{6400}}$$

$$2 = \frac{k}{80}$$

$$(2)(80) = k$$

$$160 = k$$

Thus, the formula is $v = \frac{160}{\sqrt{d}}$.

EXAMPLE 6 *Using the Constant of Proportionality*

Suppose y varies inversely as x. If $y = 8$ when $x = 15$, find y when $x = 18$.

SOLUTION: First write the inverse variation, then solve for k.

$$y = \frac{k}{x}$$

$$8 = \frac{k}{15}$$

$$120 = k$$

Now substitute 120 for k in $y = \frac{k}{x}$ and find y when $x = 18$.

$$y = \frac{120}{x} = \frac{120}{18} = 6.7 \quad \text{(to the nearest tenth)}$$

Joint Variation

One quantity may vary directly as a product of two or more other quantities. This type of variation is called *joint variation*.

Joint Variation

The general form of a joint variation, where y varies directly as x and z, is

$$y = kxz$$

where k is the constant of proportionality.

EXAMPLE 7 *Joint Variation in Geometry*

The area, A, of a triangle varies jointly as its base, b, and height, h. If the area of a triangle is 48 in.^{2} when its base is 12 in. and its height is 8 in., find the area of a triangle whose base is 15 in. and whose height is 20 in.

SOLUTION: First write the joint variation, then substitute the known values and solve for k.

$$\begin{aligned} A &= kbh \\ 48 &= k(12)(8) \\ 48 &= k(96) \\ \frac{48}{96} &= k \\ \tfrac{1}{2} &= k \end{aligned}$$

Now solve for the area of the given triangle.

$$\begin{aligned} A &= kbh \\ &= \tfrac{1}{2}(15)(20) \\ &= 150 \text{ in.}^2 \end{aligned}$$

Summary of Variations

Direct	Inverse	Joint
$y = kx$	$y = \frac{k}{x}$	$y = kxz$

Combined Variation

Often in real-life situations, one variable varies as a combination of variables. The following examples illustrate the use of ***combined variations***.

EXAMPLE 8 *Combined Variation in Engineering*

The load, L, that a horizontal beam can safely support varies jointly as the width, w, and the square of the depth, d, and inversely as the length, l. Express L in terms of w, d, l, and the constant of proportionality, k.

SOLUTION:

$$L = \frac{kwd^2}{l}$$

EXAMPLE 9 *Pretzel Price, Combined Variation*

The owners of the Colonel Mustard Pretzel Shop find that their weekly sales of pretzels, S, vary directly with their advertising budget, A, and inversely with their pretzel price, P. When their advertising budget is \$600 and the price is \$1.20, they sell 6500 pretzels.

a) Write an equation of variation expressing S in terms of A and P. Include the value of the proportionality constant.

b) Find the expected sales if the advertising budget is \$900 and the pretzel price is \$1.50.

SOLUTION:

a) Since S varies directly as A and inversely as P, we begin with the equation

$$S = \frac{kA}{P}$$

We now find k using the known values.

$$6500 = \frac{k(600)}{1.20}$$

$$6500 = 500k$$

$$13 = k$$

Therefore, the equation for the sales of pretzels is $S = \dfrac{13A}{P}$.

b) $S = \dfrac{13A}{P}$

$$= \frac{13(900)}{1.50} = 7800$$

They can expect to sell 7800 pretzels.

EXAMPLE 10 *Combined Variation*

A varies jointly as B and C and inversely as the square of D. If $A = 1$ when $B = 9$, $C = 4$, and $D = 6$, find A when $B = 8$, $C = 12$, and $D = 5$.

SOLUTION: We begin with the equation

$$A = \frac{kBC}{D^2}$$

We must first find the constant of proportionality, k, by substituting the known values for A, B, C, and D and solving for k.

$$1 = \frac{k(9)(4)}{6^2}$$

$$1 = \frac{36k}{36}$$

$$1 = k$$

Thus, the constant of proportionality equals 1. Now we find A for the corresponding values of B, C, and D.

$$A = \frac{kBC}{D^2}$$

$$A = \frac{(1)(8)(12)}{5^2} = \frac{96}{25} = 3.84$$

SECTION 6.5 EXERCISES

Concept/Writing Exercises

In Exercises 1–4, use complete sentences to answer the question.

1. Describe inverse variation.
2. Describe direct variation.
3. Describe joint variation.
4. Describe combined variation.

In Exercises 5–20, use your intuition to determine whether the variation between the indicated quantities is direct or inverse.

5. The distance between two cities on a map and the actual distance between the two cities
6. The time required to fill a pool with a hose and the volume of water coming from the hose

Allen R. Angel, author

7. The time required to boil water on a burner and the temperature of the burner
8. The time a person spends walking on a treadmill and the number of calories the person burns
9. The interest earned on an investment and the interest rate
10. The volume of a balloon and its radius
11. A person's speed and the time needed for the person to complete the race
12. The time required to cool a room and the temperature of the room
13. The number of painters hired to paint a house and the time required to paint the house
14. The number of calories eaten and the amount of exercise required to burn off those calories
15. The time required to defrost frozen hamburger in a room and the temperature of the room
16. On Earth, the weight and mass of an object
17. The number of people in the cashier line at the bookstore and the time required to stand in line
18. The number of books that can be placed upright on a shelf 3 ft long and the width of the books
19. The displacement, in liters, and the horsepower of an engine
20. The speed of a rider lawn mower and the time it takes to cut a lawn

In Exercises 21 and 22, use Exercises 5–20 as a guide.

21. Name two items that have not been mentioned in this section that have a direct variation.
22. Name two items that have not been mentioned in this section that have an inverse variation.

Practice the Skills

In Exercises 23–40, (a) write the variation and (b) find the quantity indicated.

23. y varies directly as x. Find y when $x = 5$ and $k = 3$.
24. x varies inversely as y. Find x when $y = 12$ and $k = 15$.

25. m varies inversely as the square of n. Find m when $n = 8$ and $k = 16$.

26. r varies directly as the square of s. Find r when $s = 2$ and $k = 13$.

27. R varies inversely as W. Find R when $W = 160$ and $k = 8$.

28. D varies directly as J and inversely as C. Find D when $J = 10$, $C = 25$, and $k = 5$.

29. F varies jointly as D and E. Find F when $D = 3$, $E = 10$, and $k = 7$.

30. A varies jointly as R_1 and R_2 and inversely as the square of L. Find A when $R_1 = 120$, $R_2 = 8$, $L = 5$, and $k = \frac{3}{2}$.

31. t varies directly as the square of d and inversely as f. If $t = 192$ when $d = 8$ and $f = 4$, find t when $d = 10$ and $f = 6$.

32. y varies directly as the square root of t and inversely as s. If $y = 12$ when $t = 36$ and $s = 2$, find y when $t = 81$ and $s = 4$.

33. Z varies jointly as W and Y. If $Z = 12$ when $W = 9$ and $Y = 4$, find Z when $W = 50$ and $Y = 6$.

34. y varies directly as the square of R. If $y = 4$ when $R = 4$, find y when $R = 8$.

35. H varies directly as L. If $H = 15$ when $L = 50$, find H when $L = 10$.

36. C varies inversely as J. If $C = 7$ when $J = 0.7$, find C when $J = 12$.

37. A varies directly as the square of B. If $A = 245$ when $B = 7$, find A when $B = 12$.

38. F varies jointly as M_1 and M_2 and inversely as the square of d. If $F = 20$ when $M_1 = 5$, $M_2 = 10$, and $d = 0.2$, find F when $M_1 = 10$, $M_2 = 20$, and $d = 0.4$.

39. F varies jointly as q_1 and q_2 and inversely as the square of d. If $F = 8$ when $q_1 = 2$, $q_2 = 8$, and $d = 4$, find F when $q_1 = 28$, $q_2 = 12$, and $d = 2$.

40. S varies jointly as I and the square of T. If $S = 8$ when $I = 20$ and $T = 4$, find S when $I = 2$ and $T = 2$.

Problem Solving

In Exercises 41–49, (a) write the variation and (b) find the quantity indicated.

41. **Resistance** The resistance, R, of a wire varies directly as its length, L. If the resistance of a 30 ft length of wire is 0.24 ohm, find the resistance of a 40 ft length of wire.

42. **Finding Interest** The amount of interest earned on an investment, I, varies directly as the interest rate, r. If the interest earned is \$40 when the interest rate is 4%, find the amount of interest earned when the interest rate is 6%.

43. **Speaker Loudness** The loudness of a stereo speaker, l, measured in decibels (dB), is inversely proportional to the square of the distance, d, of the listener from the speaker. If the loudness is 20 dB when the listener is 6 ft from the speaker, find the loudness when the listener is 3 ft from the speaker.

44. **Building a Deck** The time, t, it takes to build a deck for a specific house is inversely proportional to the number, n, of workers building the deck. If it takes two workers 16 hours to build the deck, how many hours will it take for four workers to build the deck?

45. **Video Rentals** The weekly videotape rentals, R, at Busterblock Video vary directly with their advertising budget, A, and inversely with the daily rental price, P. When the video store's advertising budget is \$600 and the rental price is \$3 per day, it rents 4800 tapes per week. How many tapes would it rent per week if the store increased their advertising budget to \$700 and raised its rental price to \$3.50?

46. **Area and Projection** The area, a, of a projected picture on a movie screen varies directly as the square of the distance, d, from the projector to the screen. If a projector at a distance of 25 feet projects a picture with an area of 100 square feet, what is the area of the projected picture when the projector is at a distance of 40 feet?

47. **Strength of a Beam** The strength, s, of a rectangular beam varies jointly as its width, w, and the square of its depth, d. If the strength of a beam 2 inches wide and 10 inches deep is 2250 pounds per square inch, find the strength of a beam 4 inches wide and 12 inches deep.

48. **Electric Resistance** The electrical resistance of a wire, R, varies directly as its length, L, and inversely as its cross-sectional area, A. If the resistance of a wire is 0.2 ohm when the length 200 ft and its cross-sectional area is 0.05 in.^{2}, find the resistance of a wire whose length is 5000 ft with a cross-sectional area of 0.01 in^2.

49. **Phone Calls** The number of phone calls between two cities during a given time period, N, varies directly as the populations p_1 and p_2 of the two cities and inversely to the distance, d, between them. If 100,000 calls are made between two cities 300 mi apart and the populations of the cities are 60,000 and 200,000, how many calls are made between two cities with populations of 125,000 and 175,000 that are 450 mi apart?

50. **a)** If y varies directly as x and the constant of proportionality is 2, does x vary directly or inversely as y? Explain.

b) Give the new constant of proportionality for x as a variation of y.

51. **a)** If y varies inversely as x and the constant of proportionality is 0.3, does x vary directly or inversely as y? Explain.

b) Give the new constant of proportionality for x as a variation of y.

Challenge Problems/Group Activities

52. ***Photography*** An article in the magazine *Outdoor and Travel Photography* states, "If a surface is illuminated by a point-source of light, the intensity of illumination produced is inversely proportional to the square of the distance separating them. In practical terms, this means that foreground objects will be grossly overexposed if your background subject is properly exposed with a flash. Thus direct flash will not offer pleasing results if there are any intervening objects between the foreground and the subject."

If the subject you are photographing is 4 ft from the flash and the illumination on this subject is $\frac{1}{16}$ of the light of the flash, what is the intensity of illumination on an intervening object that is 3 ft from the flash?

53. ***Water Cost*** In a specific region of the country, the amount of a customer's water bill, W, is directly proportional to the average daily temperature for the month, T, the lawn area, A, and the square root of F, where F is the family size, and inversely proportional to the number of inches of rain, R.

In one month, the average daily temperature is 78°F and the number of inches of rain is 5.6. If the average family of four who has a thousand square feet of lawn pays \$72.00 for water for that month, estimate the water bill in the same month for the average family of six who has 1500 ft^2 of lawn.

6.6 LINEAR INEQUALITIES

The first four sections of this chapter have dealt with equations. However, we often encounter statements of inequality. The symbols of inequality are as follows.

Symbols of Inequality

$a < b$ means that a is less than b.

$a \leq b$ means that a is less than or equal to b.

$a > b$ means that a is greater than b.

$a \geq b$ means that a is greater than or equal to b.

An ***inequality*** consists of two (or more) expressions joined by an inequality sign.

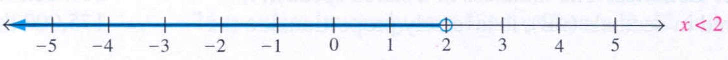
Examples of inequalities

$$3 < 5, \qquad x < 2, \qquad 3x - 2 \geq 5$$

A statement of inequality can be used to indicate a set of real numbers. For example, $x < 2$ represents the set of all real numbers less than 2. Listing all these numbers is impossible, but some are $-2, -1.234, -1, -\frac{1}{2}, 0, \frac{97}{163}, 5, 9$.

A method of picturing all real numbers less than 2 is to graph the solution on the number line. The number line was discussed in Chapter 5.

To indicate the solution set of $x < 2$ on the number line, we draw an open circle at 2 and a line to the left of 2 with an arrow at its end. This technique indicates that all points to the left of 2 are part of the solution set. The open circle indicates that the solution set does not include the number 2.

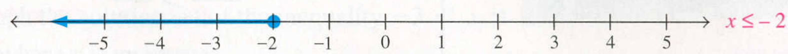


To indicate the solution set of $x \leq 2$ on the number line, we draw a closed (or darkened) circle at 2 and a line to the left of 2 with an arrow at its end. The closed circle indicates that the 2 is part of the solution.

EXAMPLE 1 *Graphing a Less Than or Equal to Inequality*

Graph the solution set of $x \leq -2$, where x is a real number, on the number line.

SOLUTION: The numbers less than or equal to -2 are all the points on the number line to the left of -2 and -2 itself. The closed circle at -2 shows that -2 is included in the solution set.

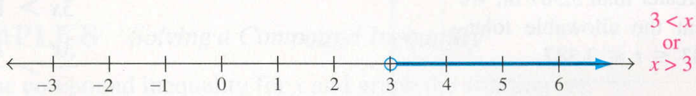


The inequality statements $x < 2$ and $2 > x$ have the same meaning. Note that the inequality symbol points to the x in both cases. Thus, one inequality may be written in place of the other. Likewise, $x > 2$ and $2 < x$ have the same meaning. Note that the inequality symbol points to the 2 in both cases. We make use of this fact in Example 2.

EXAMPLE 2 *Graphing a Less Than Inequality*

Graph the solution set of $3 < x$, where x is a real number, on the number line.

SOLUTION: We can restate $3 < x$ as $x > 3$. Both statements have identical solutions. Any number that is greater than 3 satisfies the inequality $x > 3$. The graph includes all the points to the right of 3 on the number line. To indicate that 3 is not part of the solution set, we place an open circle at 3.



We can find the solution to an inequality by adding, subtracting, multiplying, or dividing both sides of the inequality by the same number or expression. We use the procedure discussed in Section 6.2 to isolate the variable, with one important exception: *When both sides of an inequality are multiplied or divided by a negative number, the direction of the inequality symbol is reversed.*

EXAMPLE 3 *Multiplying by a Negative Number*

Solve the inequality $-x > 3$ and graph the solution set on the number line.

SOLUTION: To solve this inequality, we must eliminate the negative sign in front of the x. To do so, we multiply both sides of the inequality by -1 and change the direction of the inequality symbol.

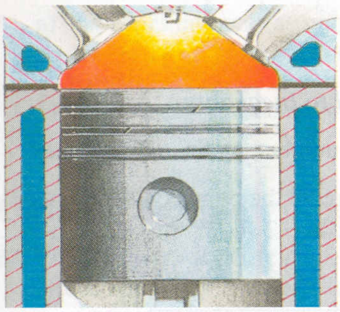
$$-x > 3$$

$$-1(-x) < -1(3) \quad \text{Multiply both sides of the inequality by } -1 \text{ and change the direction of the inequality symbol.}$$

$$x < -3$$

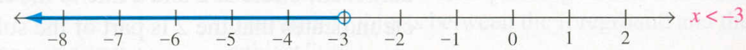
DID YOU KNOW

What a Bore



In the production of machine parts, engineers must allow a certain tolerance in the way parts fit together. For example, the boring machines that grind cylindrical openings in an automobile's engine block must create a cylinder that allows the piston to move freely up and down, but still fit tightly enough to ensure that compression and combustion are complete. The allowable tolerance between parts can be expressed as an inequality. For example, the diameter of a cylinder may need to be no less than 3.383 in. and no greater than 3.387 in. We can represent the allowable tolerance as $3.383 \le t \le 3.387$.

The solution set is graphed on the number line as follows.



EXAMPLE 4 *Dividing by a Negative Number*

Solve the inequality $-4x < 16$ and graph the solution set on the number line.

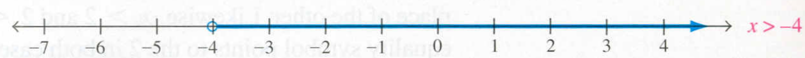
SOLUTION: Solving the inequality requires making the coefficient of the x term 1. To do so, divide both sides of the inequality by -4 and change the direction of the inequality symbol.

$$-4x < 16$$

$$\frac{-4x}{-4} > \frac{16}{-4} \qquad \text{Divide both sides of the inequality by } -4 \text{ and change the direction of the inequality symbol.}$$

$$x > -4$$

The solution set is graphed on the number line as follows.



EXAMPLE 5 *Solving an Inequality*

Solve the inequality $3x - 5 > 13$ and graph the solution set on the number line.

SOLUTION: To find the solution set, isolate x on one side of the inequality symbol.

$$3x - 5 > 13$$

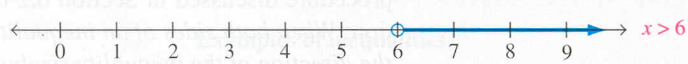
$$3x - 5 + 5 > 13 + 5 \qquad \text{Add 5 to both sides of the inequality.}$$

$$3x > 18$$

$$\frac{3x}{3} > \frac{18}{3} \qquad \text{Divide both sides of the inequality by 3.}$$

$$x > 6$$

Thus, the solution set to $3x - 5 > 13$ is all real numbers greater than 6.



Note that in Example 5, the *direction of the inequality symbol* did not change when both sides of the inequality were divided by the positive number 3.

EXAMPLE 6 *A Solution of Only Integers*

Solve the inequality $x + 4 < 7$, where x is an integer, and graph the solution set on the number line.

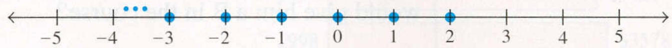
SOLUTION:

$$x + 4 < 7$$

$$x + 4 - 4 < 7 - 4$$

$$x < 3$$

Since x is an integer and is less than 3, the solution set is the set of integers less than 3, or $\{\ldots -3, -2, -1, 0, 1, 2\}$. To graph the solution set, we make solid dots at the corresponding points on the number line. The three smaller dots to the left of -3 indicate that all the integers to the left of -3 are included.



$x < 3$,
x an integer

An inequality of the form $a < x < b$ is called a *compound inequality*. Consider the compound inequality $-3 < x \leq 2$, which means that $-3 < x$ *and* $x \leq 2$.

EXAMPLE 7 *A Compound Inequality*

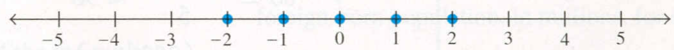
Graph the solution set of the inequality $-3 < x \leq 2$

a) where x is an integer.

b) where x is a real number.

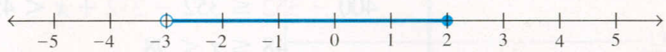
SOLUTION:

a) The solution set is all the integers between -3 and 2, including the 2 but not including the -3, or $\{-2, -1, 0, 1, 2\}$.



$-3 < x \leq 2$,
x an integer

b) The solution set consists of all the real numbers between -3 and 2, including the 2 but not including the -3.



$-3 < x \leq 2$

EXAMPLE 8 *Solving a Compound Inequality*

Solve the compound inequality for x and graph the solution set.

$$-4 < \frac{x+3}{2} \leq 5$$

SOLUTION: To solve a compound inequality, we must isolate the x as the middle term. To do so, we use the same principles used to solve inequalities.

$$-4 < \frac{x+3}{2} \leq 5$$

$$2(-4) < \cancel{2}\left(\frac{x+3}{\cancel{2}}\right) \leq 2(5) \quad \text{Multiply each part of the inequality by 2.}$$

$$-8 < x + 3 \leq 10$$

$$-8 - 3 < x + 3 - 3 \leq 10 - 3 \quad \text{Subtract 3 from each part of the inequality.}$$

$$-11 < x \leq 7$$

The solution set is graphed on the number line as follows.

$-11 < x \leq 7$

"In mathematics the art of posing problems is easier than that of solving them."
Georg Cantor

EXAMPLE 9 *Average Grade*

A student must have an average (the mean) on five tests that is greater than or equal to 80% but less than 90% to receive a final grade of B. Devon's grades on the first four tests were 98%, 76%, 86%, and 92%. What range of grades on the fifth test would give him a B in the course?

SOLUTION: The unknown quantity is the range of grades on the fifth test. First construct an inequality that can be used to find the range of grades on the fifth exam. The average (mean) is found by adding the grades and dividing the sum by the number of exams.

Let x = the fifth grade. Then

$$\text{Average} = \frac{98 + 76 + 86 + 92 + x}{5}$$

For Devon to obtain a B, his average must be greater than or equal to 80 but less than 90.

$$80 \le \frac{98 + 76 + 86 + 92 + x}{5} < 90$$

$$80 \le \frac{352 + x}{5} < 90$$

$$5(80) \le \cancel{5}\left(\frac{352 + x}{\cancel{5}}\right) < 5(90) \qquad \text{Multiply the three terms of the inequality by 5.}$$

$$400 \le 352 + x < 450$$

$$400 - 352 \le 352 - 352 + x < 450 - 352 \qquad \text{Subtract 352 from all three terms.}$$

$$48 \le x < 98$$

Thus, a grade of 48% up to but not including a grade of 98% on the fifth test will result in a grade of B.

TIMELY TIP Remember to change the direction of the inequality symbol when multiplying or dividing both sides of an inequality by a negative number.

SECTION 6.6 EXERCISES

Concept/Writing Exercises

1. Give the four inequality symbols we use in this section and indicate how each is read.
2. a) What is an inequality?
   b) Give an example of three inequalities.
3. When solving an inequality, under what conditions do you need to change the direction of the inequality symbol?
4. Does $x < 2$ have the same meaning as $2 > x$? Explain.
5. Does $x > -3$ have the same meaning as $-3 < x$? Explain.
6. When graphing the solution set to an inequality on the number line, when should you use an open circle and when should you use a closed circle?

Practice the Skills

In Exercises 7–24, graph the solution set of the inequality, where x is a real number, on the number line.

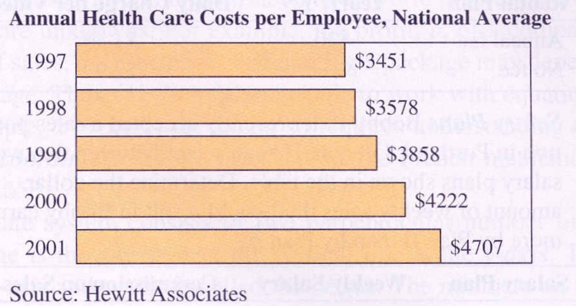
7. $x > 6$
8. $x \leq 9$
9. $x + 4 \geq 7$
10. $3x > 9$
11. $-3x \leq 18$
12. $-4x < 12$
13. $\frac{x}{6} < -2$
14. $\frac{x}{2} > 4$
15. $\frac{-x}{3} \geq 3$
16. $\frac{x}{2} \geq -4$
17. $2x + 6 \geq 14$
18. $3x + 12 < 5x + 14$
19. $4(x - 1) < 6$
20. $-5(x + 1) + 2x > -3x + 6$
21. $3(x + 4) - 2 < 3x + 10$
22. $-2 \leq x \leq 1$
23. $3 < x - 7 \leq 6$
24. $\frac{1}{2} < \frac{x + 4}{2} \leq 4$

In Exercises 25–44, graph the solution set of the inequality, where x is an integer, on the number line.

25. $x \geq 2$
26. $-3 < x$
27. $-3x \leq 27$
28. $3x \geq 27$
29. $x - 2 < 4$
30. $-5x \leq 15$
31. $\frac{x}{3} \leq -2$
32. $\frac{x}{4} \geq -3$
33. $\frac{x}{6} \geq 3$
34. $\frac{2x}{3} \leq 4$
35. $-11 < -5x + 4$
36. $2x + 5 < -3 + 6x$
37. $3(x + 4) \geq 4x + 13$
38. $2(x - 1) < 3(x - 4) + 5$
39. $5(x + 4) - 6 \leq 2x + 8$
40. $-3 \leq x < 5$
41. $1 > -x > -5$
42. $2 < 2x + 3 \leq 6$
43. $0.2 \leq \frac{x - 4}{10} \leq 0.4$
44. $-\frac{1}{3} < \frac{x - 2}{12} \leq \frac{1}{4}$

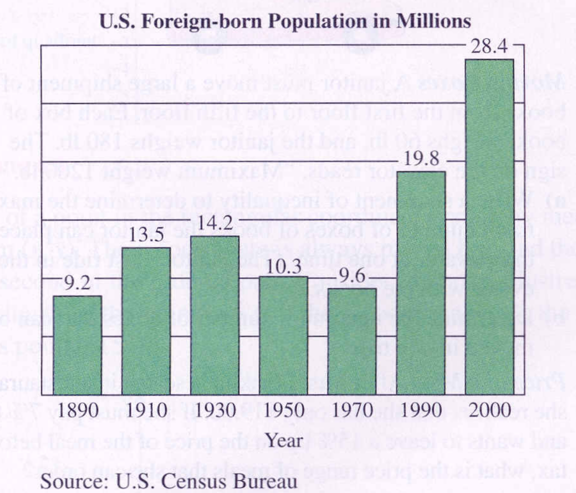
Problem Solving

45. ***Health Care Costs*** The following chart shows the national average for employers for annual health care costs per employee for the years 1997 through 2000 and projected for 2001.



a) In which years was the national average for annual health care costs per employee $> \$4000$?
b) In which years was the national average for annual health care costs per employee $< \$3858$?
c) In which years was the national average for annual health care costs per employee $\leq \$4222$?
d) In which years was the national average for annual health care costs per employee $\geq \$3578$?

46. ***U.S. Population*** The following bar graph shows the U.S. foreign-born population, in millions, for selected years.



a) In which of the years listed on the bar graph was the U.S. foreign-born population > 14.2 million?
b) In which of the years listed on the bar graph was the U.S. foreign-born population ≤ 13.5 million?
c) In which of the years listed on the bar graph was the U.S. foreign-born population ≥ 28.4 million?
d) In which of the years listed on the bar graph was the U.S. foreign-born population > 19.8 million?

47. ***Video Rental*** Movie Mania offers two rental plans. One has an annual fee and the other has no annual fee. The annual membership fee and the daily charge per video for each plan are shown in the table on page 334. Determine

the maximum number of videos that can be rented for the no fee plan to cost less than the annual fee plan.

Rental Plan	Yearly Fee	Daily Charge per Video
Annual fee	$30	$1.49
No fee	None	$2.99

48. *Salary Plans* Bobby Exler recently accepted a sales position in Portland, Oregon. He can select between the two salary plans shown in the table. Determine the dollar amount of weekly sales that would result in Bobby earning more by Plan B than by Plan A.

Salary Plan	Weekly Salary	Commission on Sales
Plan A	$500	6%
Plan B	$400	8%

49. *Van Rental* The Berrys need to rent a van for their family vacation. They can rent a van from Jason's Auto Rentals for $200 per week with no charge for mileage or from Fred's Fine Autos for $110 per week plus $0.25 per mile. Determine the distances the Berrys can drive in the van if the cost of renting from Fred's is to be less than the cost of renting from Jason's.

50. *Moving Boxes* A janitor must move a large shipment of books from the first floor to the fifth floor. Each box of books weighs 60 lb, and the janitor weighs 180 lb. The sign on the elevator reads, "Maximum weight 1200 lb."

a) Write a statement of inequality to determine the maximum number of boxes of books the janitor can place on the elevator at one time. (The janitor must ride in the elevator with the books.)

b) Determine the maximum number of boxes that can be moved in one trip.

51. *Price of a Meal* After Mrs. Franklin is seated in a restaurant, she realizes that she has only $19.00. If she must pay 7% tax and wants to leave a 15% tip on the price of the meal before tax, what is the price range of meals that she can order?

52. *Making a Profit* For a business to realize a profit, its revenue, R, must be greater than its costs, C; that is, a profit will only result if $R > C$ (the company breaks even when $R = C$). A book publishing company has a weekly cost equation of $C = 2x + 2000$ and a weekly revenue equation of $R = 12x$, where x is the number of books produced and sold in a week. How many books must be sold weekly for the company to make a profit?

53. *Finding Velocity* The velocity, v, in feet per second, t sec after a tennis ball is projected directly upward is given by the formula $v = 84 - 32t$. How many seconds after being projected upward will the velocity be between 36 ft/sec and 68 ft/sec?

54. *Speed Limit* The minimum speed for vehicles on a highway is 40 mph, and the maximum speed is 55 mph. If Philip Rowe has been driving nonstop along the highway for 4 hr, what range in miles could he have legally traveled?

55. *A Grade of B* In Example 9 on page 332, what range of grades on the fifth test would result in Devon receiving a grade of B if his grades on the first four tests were 78%, 64%, 88%, and 76%?

56. *Tent Rental* The Cuhayoga Community College Planning Committee wants to rent tents for the spring job fair. Rent-a-Tent charges $325 for setup and delivery of its tents. This fee is charged regardless of the number of tents delivered and set up. In addition, Rent-a-Tent charges $125 for each tent rented. If the minimum amount the planning committee wishes to spend is $950 and the maximum amount they wish to spend is $1200, determine the minimum and maximum number of tents they can rent.

Challenge Problems/Group Activities

57. *Painting a House* J. B. Davis is painting the exterior of his house. The instructions on the paint can indicate that 1 gal covers from 250 to 400 ft^2. The total surface of the house to be painted is 2750 ft^2. Determine the number of gallons of paint he could use and express the answer as an inequality.

58. *Final Exam* Teresa's five test grades for the semester are 86%, 74%, 68%, 96%, and 72%. Her final exam counts one-third of her final grade. What range of grades on her final exam would result in Teresa receiving a final grade of B in the course? (See Example 9.)

59. A student multiplied both sides of the inequality $-\frac{1}{3}x \leq 4$ by -3 and forgot to reverse the direction of the inequality symbol. What is the relation between the student's incorrect solution set and the correct solution set? Is any number in both the correct solution set and the student's incorrect solution set? If so, what is it?

Internet/Research Activity

60. Find a newspaper or a magazine article that contains the mathematical concept of inequality.

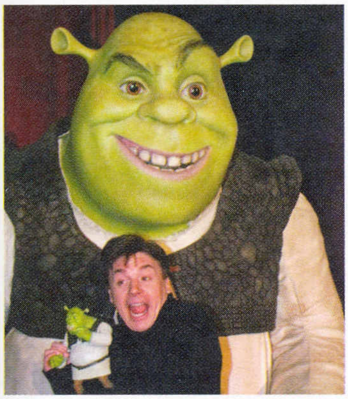
a) From the information in the article write a statement of inequality.

b) Summarize the article and explain how you arrived at the inequality statement in part (a).

6.7 GRAPHING LINEAR EQUATIONS

MATHEMATICS *Everywhere*

Hollywood's Four Quadrants



Mike Myers, the voice of *Shrek*

Hollywood studio executives use four "quadrants" to divide up the movie-going audience. Men 25 years and older, men younger than 25, women 25 years and older, and women younger than 25 are the age groups represented by Hollywood's four quadrants. If a studio produces a movie that appeals to all four quadrants, it is sure to have a hit movie. If the movie appeals to none of the four quadrants, the movie is sure to fail. A challenge for studio executives is to determine the core quadrant and then try to make sure no other movie geared toward the same quadrant debuts at the same time. One of the biggest hit movies of the summer of 2001 was *Shrek*, a movie that appealed to all four quadrants. On the other hand, the target or core audience for the movie *A.I.* Artificial Intelligence, also released in the summer of 2000, was unclear. The logo and previews for *A.I.,* featured the child star, whereas everything else suggested a movie more appropriate for adults. As a result, the marketing campaign confused both audiences, and *A.I.* was much more a critical success than a box office success.

In Section 6.2, we solved equations with a single variable. Real-world problems, however, often involve two or more unknowns. For example, the profit, p, of a company may depend on the amount of sales, s; or the cost, c, of mailing a package may depend on the weight, w, of the package. Thus, it is helpful to be able to work with equations with two variables (for example, $x + 2y = 6$). Doing so requires understanding the ***Cartesian*** (or ***rectangular***) ***coordinate system***, named after the French mathematician René Descartes (1596–1650).

The rectangular coordinate system consists of two perpendicular number lines (Fig. 6.6). The horizontal line is the ***x-axis***, and the vertical line is the ***y-axis***. The point of intersection of the x-axis and y-axis is called the ***origin***. The numbers on the axes to the right and above the origin are positive. The numbers on the axes on the left and below the origin are negative. The axes divide the plane into four parts: the first, second, third, and fourth ***quadrants***.

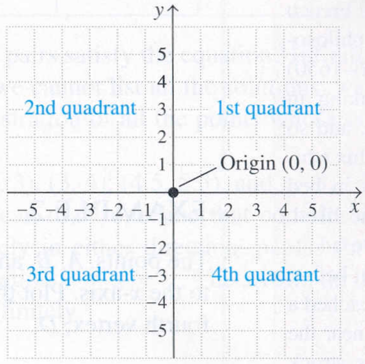


Figure 6.6

We indicate the location of a point in the rectangular coordinate system by means of an ***ordered pair*** of the form (x, y). The x-coordinate is always placed first and the y-coordinate is always placed second in the ordered pair. Consider the point illustrated in Fig. 6.7. Since the x-coordinate of the point is 5 and the y-coordinate is 3, the ordered pair that represents this point is $(5, 3)$.

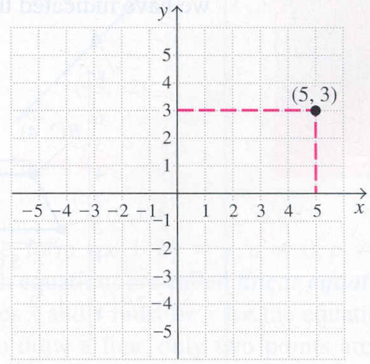
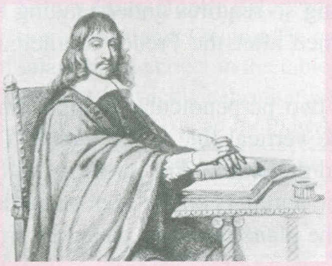


Figure 6.7

The origin is represented by the ordered pair $(0, 0)$. Every point on the plane can be represented by one and only one ordered pair (x, y), and every ordered pair (x, y) represents one and only one point on the plane.

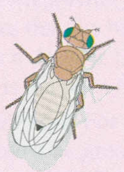
PROFILE IN MATHEMATICS

RENÉ DESCARTES



The Mathematician and the Fly

According to legend, the French mathematician and philosopher René Descartes (1596–1650) did some of his best thinking in bed. He was a sickly child, and so the Jesuits who undertook his education allowed him to stay in bed each morning as long as he liked. This practice he carried into adulthood, seldom getting up before noon. One morning as he watched a fly crawl about the ceiling, near the corner of his room, he was struck with the idea that the fly's position could best be described by the connecting distances from it to the two adjacent walls. These became the coordinates of his rectangular coordinate system and were appropriately named after him (Cartesian coordinates) and not the fly.



EXAMPLE 1 *Plotting Points*

Plot the points $A(-2, 4)$, $B(3, -4)$, $C(6, 0)$, $D(4, 1)$, and $E(0, 3)$.

SOLUTION: Point A has an x-coordinate of -2 and a y-coordinate of 4. Project a vertical line up from -2 on the x-axis and a horizontal line to the left from 4 on the y-axis. The two lines intersect at the point denoted A (Fig. 6.8). The other points are plotted in a similar manner.

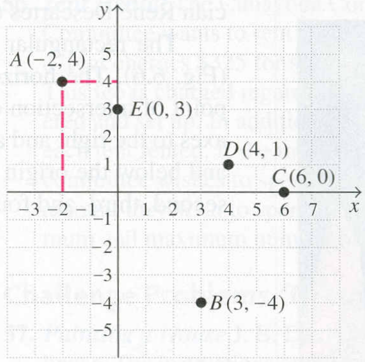


Figure 6.8

EXAMPLE 2 *A Parallelogram*

The points, A, B, and C are three vertices of a parallelogram with two sides parallel to the x-axis. Plot the three points below and determine the coordinates of the fourth vertex, D.

$$A(1, 2) \qquad B(2, 4) \qquad C(7, 4)$$

SOLUTION: A parallelogram is a figure that has opposite sides that are of equal length and are parallel. (Parallel lines are two lines in the same plane that do not intersect.) The horizontal distance between points B and C is 5 units (see Fig. 6.9). Therefore, the horizontal distance between points A and D must also be 5 units. This problem has two possible solutions, as illustrated in Fig. 6.9. In each figure, we have indicated the given points in red.

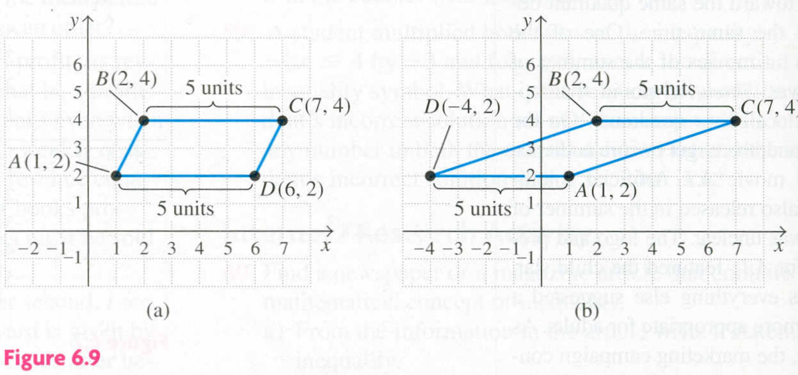


Figure 6.9

The solutions are the points $(6, 2)$ and $(-4, 2)$.

Graphing Linear Equations by Plotting Points

Consider the following equation in two variables: $y = x + 1$. Every ordered pair that makes the equation a true statement is a solution to, or satisfies, the equation. We can mentally find some ordered pairs that satisfy the equation $y = x + 1$ by picking some values of x and solving the equation for y. For example, suppose we let $x = 1$; then $y = 1 + 1 = 2$. The ordered pair (1, 2) is a solution to the equation $y = x + 1$. We can make a chart of other ordered pairs that are solutions to the equation.

x	y	**Ordered Pair**
1	2	(1, 2)
2	3	(2, 3)
3	4	(3, 4)
4.5	5.5	(4.5, 5.5)
−3	−2	(−3, −2)

How many other ordered pairs satisfy the equation? Infinitely many ordered pairs satisfy the equation. Since we cannot list all the solutions, we show them by means of a graph. A ***graph*** is an illustration of all the points whose coordinates satisfy an equation.

The points (1, 2), (2, 3), (3, 4), (4.5, 5.5), and (−3, −2) are plotted in Fig. 6.10. With a straightedge we can draw one line that contains all these points. This line, when extended indefinitely in either direction, passes through all the points in the plane that satisfy the equation $y = x + 1$. The arrows on the ends of the line indicate that the line extends indefinitely.

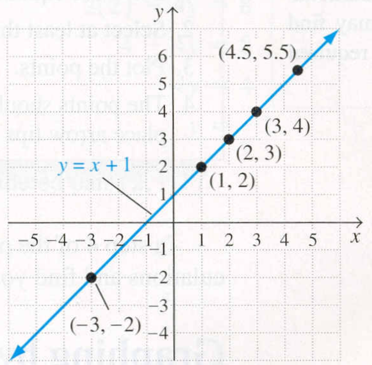


Figure 6.10

All equations of the form $ax + by = c$, $a \neq 0$, $b \neq 0$, will be straight lines when graphed. Thus, such equations are called ***linear equations in two variables***. The exponents on the variables x and y must be 1 for the equation to be linear. Since only two points are needed to draw a line, only two points are needed to graph a linear equation. It is always a good idea to plot a third point as a checkpoint. If no error has been made, all three points will be in a line, or ***collinear***. One method that can be used to obtain points is to solve the equation for y, substitute values for x, and find the corresponding values of y.

DID YOU KNOW

Graphing Calculators

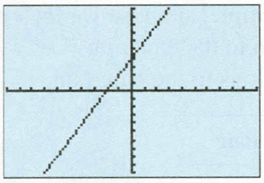


Figure 6.12

On page 308, we illustrated the graph of an exponential equation we obtained from the screen (or window) of a TI-83 Plus graphing calculator. Figure 6.12 shows the graph of the equation $y = 2x + 4$ as illustrated on the screen of that calculator.

Graphing calculators are being used more and more in many mathematics courses. Other makers of graphing calculators include Casio, Sharp, and Hewlett-Packard. Graphing calculators can do a great many more things than a scientific calculator can. If you plan on taking additional mathematics courses, you may find that a graphing calculator is required for those courses.

EXAMPLE 3 *Graphing an Equation by Plotting Points*

Graph $y = 2x + 4$.

SOLUTION: Since the equation is already solved for y, select values for x and find the corresponding values for y. The table indicates values arbitrarily selected for x and the corresponding values for y. The ordered pairs are (0, 4), (1, 6), and (−2, 0). The graph is shown in Fig. 6.11.

x	y
0	4
1	6
−2	0

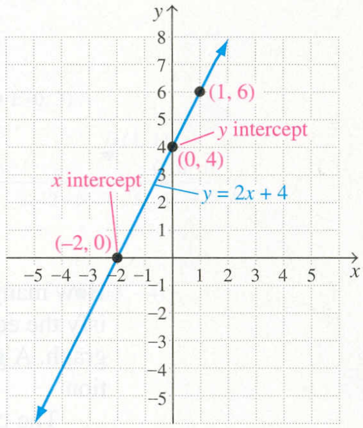


Figure 6.11

To Graph Equations by Plotting Points

1. Solve the equation for y.
2. Select at least three values for x and find their corresponding values of y.
3. Plot the points.
4. The points should be in a straight line. Draw a line through the set of points and place arrow tips at both ends of the line.

In step 4 of the procedure, if the points are not in a straight line, recheck your calculations and find your error.

Graphing by Using Intercepts

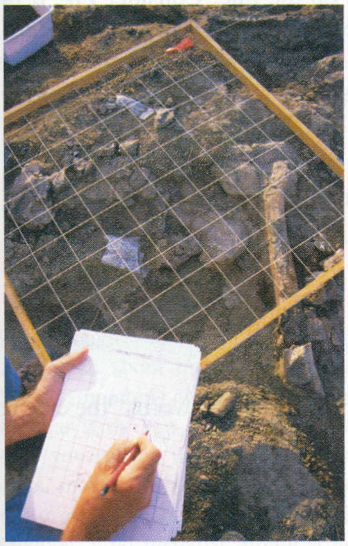
Example 3 contained two special points on the graph, (−2, 0) and (0, 4). At these points, the line crosses the x-axis and the y-axis, respectively. The ordered pairs (−2, 0) and (0, 4) represent the ***x-intercept*** and the ***y-intercept***, respectively. Another method that can be used to graph linear equations is to find the x- and y-intercepts of the graph.

Finding the x- and y-Intercepts

To find the x-intercept, set $y = 0$ and solve the equation for x.
To find the y-intercept, set $x = 0$ and solve the equation for y.

DID YOU KNOW

Use of Grids



Grids have long been used in mapping. In archaeological digs, a rectangular coordinate system may be used to chart the location of each find.

An equation may be graphed by finding the x- and y-intercepts, plotting the intercepts, and drawing a straight line through the intercepts. When graphing by this method, you should always plot a checkpoint before drawing your graph. To obtain a checkpoint, select a nonzero value for x and find the corresponding value of y. The checkpoint should be collinear with the x- and y-intercepts.

EXAMPLE 4 *Graphing Using Intercepts*

Graph $2x - 4y = 8$ by using the x- and y-intercepts.

SOLUTION: To find the x-intercept, set $y = 0$ and solve for x.

$$\begin{aligned} 2x - 4y &= 8 \\ 2x - 4(0) &= 8 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

The x-intercept is $(4, 0)$. To find the y-intercept, set $x = 0$ and solve for y.

$$\begin{aligned} 2x - 4y &= 8 \\ 2(0) - 4y &= 8 \\ -4y &= 8 \\ y &= -2 \end{aligned}$$

The y-intercept is $(0, -2)$. As a checkpoint, try $x = 2$ and find the corresponding value for y.

$$\begin{aligned} 2x - 4y &= 8 \\ 2(2) - 4y &= 8 \\ 4 - 4y &= 8 \\ -4y &= 4 \\ y &= -1 \end{aligned}$$

The checkpoint is the ordered pair $(2, -1)$.

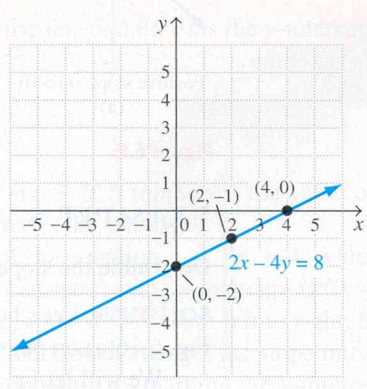


Figure 6.13

Since all three points in Fig. 6.13 are collinear, draw a line through the three points to obtain the graph. ▲

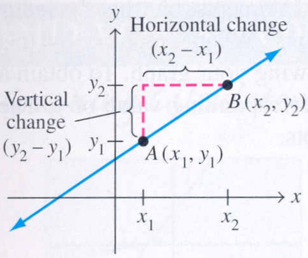


Figure 6.14

Slope

Another useful concept when you are working with straight lines is slope, which is a measure of the "steepness" of a line. The ***slope of a line*** is a ratio of the vertical change to the horizontal change for any two points on the line. Consider Fig. 6.14. Point A has coordinates (x_1, y_1), and point B has coordinates (x_2, y_2). The vertical change between points A and B is $y_2 - y_1$, and the horizontal change between points A and B is $x_2 - x_1$. Thus, the slope, which is often symbolized with the letter m, can be found as follows.

$$\textbf{Slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The Greek capital letter delta, Δ, is often used to represent the words "the change in." Therefore, slope may be defined as

$$m = \frac{\Delta y}{\Delta x}$$

A line may have a positive slope, a negative slope, zero slope, or the slope may be undefined, as indicated in Fig. 6.15. A line with a positive slope rises from left to right, as shown in Fig. 6.15(a). A line with a negative slope falls from left to right, as shown in Fig. 6.15(b). A horizontal line, which neither rises nor falls, has a slope of zero, as shown in Fig. 6.15(c). Since a vertical line does not have any horizontal change (the x value remains constant) and since we cannot divide by 0, the slope of a vertical line is undefined, as shown in Fig. 6.15(d).

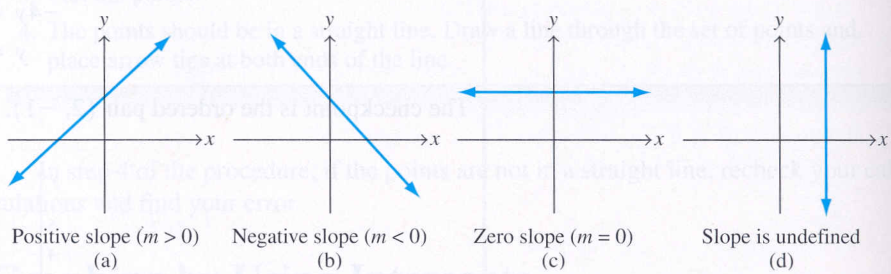


Figure 6.15

EXAMPLE 5 *Finding the Slope of a Line*

Determine the slope of the line that passes through the points $(-1, -3)$ and $(1, 5)$.

SOLUTION: Let's begin by drawing a sketch, illustrating the points and the line. See Fig. 6.16(a) on page 341.

We will let (x_1, y_1) be $(-1, -3)$ and (x_2, y_2) be $(1, 5)$. Then

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-3)}{1 - (-1)} = \frac{5 + 3}{1 + 1} = \frac{8}{2} = \frac{4}{1} = 4$$

DID YOU KNOW

Up Up and Away

Simon Ammann

Although we may not think much about it, the slope of a line is something we are altogether familiar with. You confront it every time you run up the stairs, late for class, moving 8 inches horizontally for every 6 inches up. The 2002 Olympic gold medalist Simon Ammann of Switzerland is familiar with the concept of slope. He speeds down a steep 120 meters of a ski ramp at speeds of over 60 mph before he takes flight.

The slope of 4 means that there is a vertical change of 4 units for each horizontal change of 1 unit; see Fig. 6.16(b). The slope is positive, and the line rises from left to right. Note that we would have obtained the same results if we let (x_1, y_1) be $(1, 5)$ and (x_2, y_2) be $(-1, -3)$. Try this now and see.

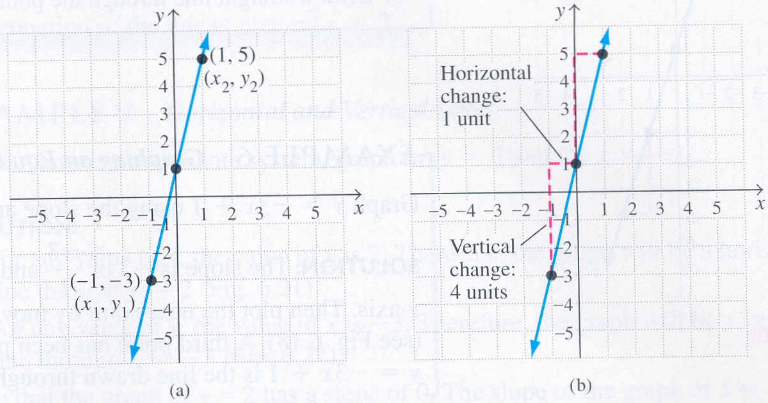


Figure 6.16

Graphing Equations by Using the Slope and *y*-Intercept

A linear equation given in the form $y = mx + b$ is said to be in slope–intercept form.

Slope–Intercept Form of the Equation of a Line

$$y = mx + b$$

where m is the slope of the line and $(0, b)$ is the y-intercept of the line.

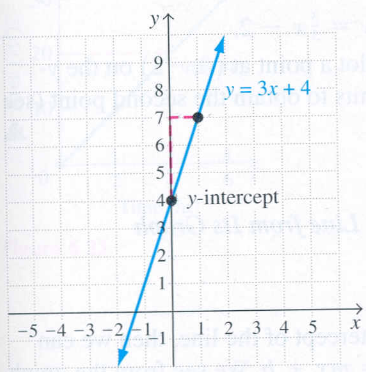


Figure 6.17

In the equation $y = mx + b$, b represents the value of y where the graph of the equation $y = mx + b$ crosses the y-axis.

Consider the graph of the equation $y = 3x + 4$, which appears in Fig. 6.17. By examining the graph, we can see that the y-intercept is $(0, 4)$. We can also see that the graph has a positive slope, since it rises from left to right. Since the vertical change is 3 units for every 1 unit of horizontal change, the slope must be $\frac{3}{1}$ or 3.

We could graph this equation by marking the y-intercept at $(0, 4)$ and then moving *up* 3 units and to the *right* 1 unit to get another point. If the slope were -3, which means $\frac{-3}{1}$, we could start at the y-intercept and move *down* 3 units and to the *right* 1 unit. Thus, if we know the slope and y-intercept of a line, we can graph the line.

To Graph Equations by Using the Slope and y-Intercept

1. Solve the equation for y to place the equation in slope–intercept form.
2. Determine the slope and y-intercept from the equation.
3. Plot the y-intercept.
4. Obtain a second point using the slope.
5. Draw a straight line through the points.

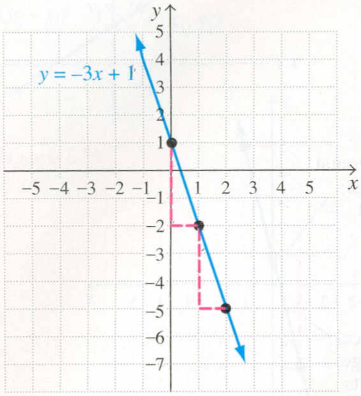


Figure 6.18

EXAMPLE 6 *Graphing an Equation Using the Slope and y-Intercept*

Graph $y = -3x + 1$ using the slope and y-intercept.

SOLUTION: The slope is -3 or $\frac{-3}{1}$ and the y-intercept is (0, 1). Plot (0, 1) on the y-axis. Then plot the next point by moving *down* 3 units and to the *right* 1 unit (see Fig. 6.18). A third point has been plotted in the same way. The graph of $y = -3x + 1$ is the line drawn through these three points.

EXAMPLE 7 *Write an Equation in Slope–Intercept Form*

a) Write $3x - 5y = 10$ in slope–intercept form.

b) Graph the equation.

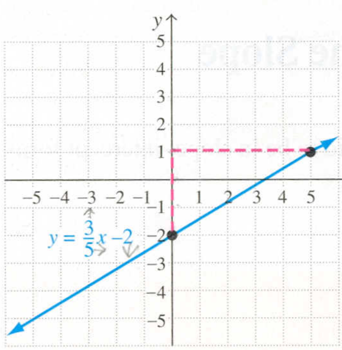


Figure 6.19

SOLUTION:

a) To write $3x - 5y = 10$ in slope–intercept form, we solve the given equation for y.

$$3x - 5y = 10$$
$$3x - 3x - 5y = -3x + 10$$
$$-5y = -3x + 10$$
$$\frac{-5y}{-5} = \frac{-3x + 10}{-5}$$
$$y = \frac{-3x}{-5} + \frac{10}{-5} \quad \text{or} \quad y = \frac{3}{5}x - 2$$

Thus, in slope–intercept form, the equation is $y = \frac{3}{5}x - 2$.

b) The y-intercept is $(0, -2)$ and the slope is $\frac{3}{5}$. Plot a point at $(0, -2)$ on the y-axis, then move *up* 3 units and to the *right* 5 units to obtain the second point (see Fig. 6.19). Draw a line through the two points.

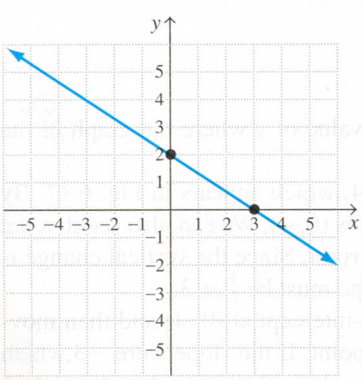


Figure 6.20

EXAMPLE 8 *Determine the Equation of a Line from Its Graph*

Determine the equation of the line in Fig. 6.20.

SOLUTION: If we determine the slope and the y-intercept of the line, then we can write the equation using slope–intercept form, $y = mx + b$. We see from the graph that the y-intercept is (0, 2); thus, $b = 2$. The slope of the line is negative because the graph falls from left to right. The change in y is 2 units for every 3-unit change in x. Thus, m, the slope of the line, is $-\frac{2}{3}$.

$$y = mx + b$$

$$y = -\frac{2}{3}x + 2$$

The equation of the line is $y = -\frac{2}{3}x + 2$. ▲

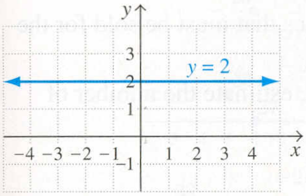


Figure 6.21

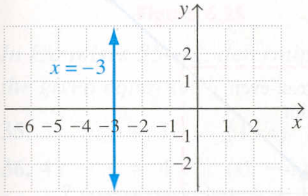


Figure 6.22

EXAMPLE 9 *Horizontal and Vertical Lines*

In the Cartesian coordinate system, graph (a) $y = 2$ and (b) $x = -3$.

SOLUTION:

a) For any value of x, the value of y is 2. Therefore, the graph will be a horizontal line through $y = 2$ (Fig. 6.21).

b) For any value of y, the value of x is -3. Therefore, the graph will be a vertical line through $x = -3$ (Fig. 6.22).

Note that the graph of $y = 2$ has a slope of 0. The slope of the graph of $x = -3$ is undefined. ▲

In graphing the equations in this section, we labeled the horizontal axis the x-axis and the vertical axis the y-axis. For each equation, we can determine values for y by substituting values for x. Since the value of y depends on the value of x, we refer to y as the ***dependent variable*** and x as the ***independent variable***. We label the *vertical axis* with the *dependent variable* and the *horizontal axis* with the *independent variable*. For the equation $C = 3n + 5$, the C is the dependent variable and n is the independent variable. Thus, to graph this equation, we label the vertical axis C and the horizontal axis n.

In many graphs, the values to be plotted on one axis are much greater than the values to be plotted on the other axis. When that occurs, we can use different scales on the horizontal and the vertical axes, as illustrated in Examples 10 and 11. The next two examples illustrate applications of graphing.

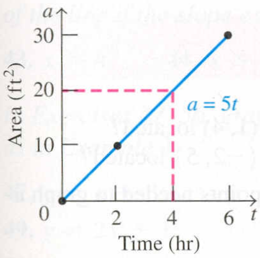


Figure 6.23

EXAMPLE 10 *Using a Graph to Determine Area*

The Professional Patio Company installs brick patios. The area of brick, a, in square feet, the company can install in t hours can be approximated by the formula $a = 5t$.

a) Graph $a = 5t$, for $t \leq 6$.

b) Use the graph to estimate the area of brick the company can install in 4 hours.

SOLUTION:

a) Since $a = 5t$ is a linear equation, its graph will be a straight line. Select three values for t, find the corresponding values for a, and then draw the graph (Fig. 6.23).

$$a = 5t$$

Let $t = 0$, $\quad a = 5(0) = 0$

Let $t = 2$, $\quad a = 5(2) = 10$

Let $t = 6$, $\quad a = 5(6) = 30$

t	a
0	0
2	10
6	30

b) By drawing a vertical line from $t = 4$ on the time axis up to the graph and then drawing a horizontal line across to the area axis, we can determine that the area installed in 4 hours is 20 ft^2.

EXAMPLE 11 *Using a Graph to Determine Profits*

Jonathan Cwirko owns a small business that manufactures compact discs. He believes that the profit (or loss) from the compact discs produced can be estimated by the formula $P = 3.5S - 200{,}000$, where S is the number of compact discs sold.

a) Graph $P = 3.5S - 200{,}000$, for $S \leq 500{,}000$ compact discs.

b) From the graph, estimate the number of compact discs that must be sold for the company to break even.

c) If the profit from selling compact discs is \$1 million, estimate the number of compact discs sold.

SOLUTION:

a) Select values for S and find the corresponding values of P.

S	P
0	−200,000
100,000	150,000
500,000	1,550,000

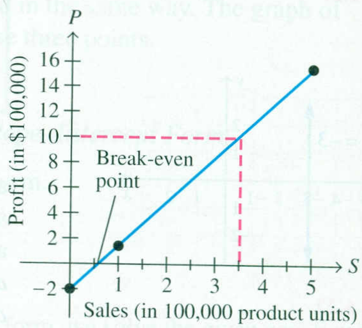


Figure 6.24

b) On the graph (Fig. 6.24), note that the break-even point is about 0.6, or 60,000 compact discs.

c) We can obtain the answer by drawing a horizontal line from 10 on the profit axis. Since the horizontal line cuts the graph at about 3.4 on the S axis, approximately 340,000 compact discs were sold.

SECTION 6.7 EXERCISES

Concept/Writing Exercises

1. What is a graph?
2. Explain how to find the x-intercept of a linear equation.
3. Explain how to find the y-intercept of a linear equation.
4. What is the slope of a line?
5. a) Explain in your own words how to find the slope of a line between two points.
   b) Based on your explanation in part (a), find the slope of the line through the points (6, 2) and (−3, 5).
6. Describe the three methods used to graph a linear equation in this section.
7. a) In which quadrant is the point (1, 4) located?
   b) In which quadrant is the point (−2, 5) located?
8. What is the minimum number of points needed to graph a linear equation?

Practice the Skills

In Exercises 9–16, plot all the points on the same axes.

9. (−3, 2) **10.** (2, 3) **11.** (−5, −1)
12. (4, 0) **13.** (0, 2) **14.** (0, 0)
15. (0, −5) **16.** $(3\frac{1}{2}, 4\frac{1}{2})$

In Exercises 17–24, plot all the points on the same axes.

17. (5, 1) **18.** (0, −3) **19.** (−6, −1)
20. (1, 0) **21.** (−3, 0) **22.** (−3, 1)
23. (4, −1) **24.** (4.5, 3.5)

In Exercises 25–34 (indicated on Fig. 6.25), write the coordinates of the corresponding point.

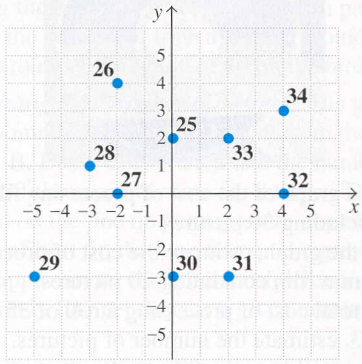


Figure 6.25

In Exercises 35–42, determine which ordered pairs satisfy the given equation.

35. $3x + y = 7$ $(1, 3)$ $(1, 4)$ $(-1, 10)$

36. $4x - y = 4$ $(0, -4)$ $(1, 0)$ $(2, -3)$

37. $2x - 3y = 10$ $(5, 0)$ $(0, 3)$ $(0, -\frac{10}{3})$

38. $3y = 4x + 2$ $(2, 1)$ $(1, 2)$ $(0, \frac{2}{3})$

39. $7y = 3x - 5$ $(1, -1)$, $(-3, -2)$, $(2, 5)$

40. $\frac{x}{2} + 3y = 4$ $(0, \frac{4}{3})$, $(8, 0)$, $(10, -2)$

41. $\frac{x}{2} + \frac{3y}{4} = 2$ $(0, \frac{8}{3})$, $(1, \frac{11}{4})$, $(4, 0)$

42. $2x - 5y = -7$ $(2, 1)$, $(-1, 1)$, $(4, 3)$

In Exercises 43–46, graph the equation and state the slope of the line if the slope exists (see Example 9).

43. $x = 4$ **44.** $x = -2$ **45.** $y = 3$ **46.** $y = -5$

In Exercises 47–56, graph the equation by plotting points, as in Example 3.

47. $y = x + 3$ **48.** $y = x - 2$
49. $y = 2x - 1$ **50.** $y = -x + 4$
51. $y + 3x = 6$ **52.** $y - 4x = 8$
53. $y = \frac{1}{2}x + 4$ **54.** $3y = 2x - 3$
55. $2y = -x + 6$ **56.** $y = -\frac{3}{4}x$

In Exercises 57–66, graph the equation, using the x- and y-intercepts, as in Example 4.

57. $x + y = 1$ **58.** $-x + y = 7$
59. $3x + y = 6$ **60.** $4x - 2y = 12$
61. $2x = -4y - 8$ **62.** $y = 4x + 4$
63. $y = 3x + 5$ **64.** $3x + 6y = 9$
65. $3x - y = 5$ **66.** $4y = 2x + 12$

In Exercises 67–76, find the slope of the line through the given points. If the slope is undefined, so state.

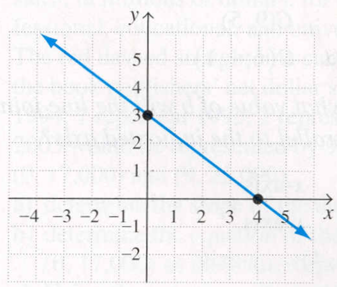
67. (3, 7) and (10, 21) **68.** (4, 1) and (1, 4)
69. (2, 6) and (−5, −9) **70.** (−5, 6) and (7, −9)
71. (5, 2) and (−3, 2) **72.** (−3, −5) and (−1, −2)
73. (8, −3) and (8, 3) **74.** (2, 6) and (2, −3)
75. (−2, 3) and (1, −1) **76.** (−7, −5) and (5, −6)

In Exercises 77–86, graph the equation using the slope and y-intercept, as in Examples 6 and 7.

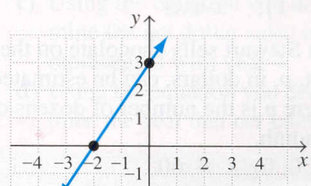
77. $y = x + 3$ **78.** $y = 3x + 2$
79. $y = -x - 4$ **80.** $y = -2x + 1$
81. $y = -\frac{1}{3}x + 2$ **82.** $y = -x - 2$
83. $7y = 4x - 7$ **84.** $3x + 2y = 6$
85. $3x - 2y + 6 = 0$ **86.** $3x + 4y - 8 = 0$

In Exercises 87–90, determine the equation of the graph.

87.



88.



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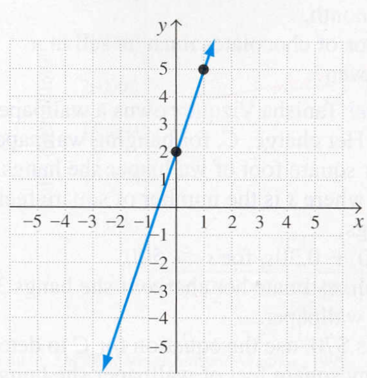
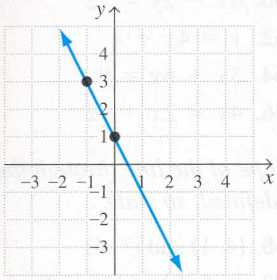


90.



Problem Solving

In Exercises 91 and 92, points A, B, and C are three vertices of a rectangle (the points where two sides meet). Plot the three points. (a) Find the coordinates of the fourth point, D, to complete the rectangle. (b) Find the area of the rectangle; use $A = lw$.

91. $A(-1, 4)$, $B(4, 4)$, $C(4, 2)$

92. $A(-4, 2)$, $B(7, 2)$, $C(7, 8)$

In Exercises 93 and 94, points A, B, and C are three vertices of a parallelogram with sides parallel to the x-axis. Plot the three points. Find the coordinates of the fourth point, D, to complete the parallelogram. Note: There are two possible answers for point D.

93. $A(3, 2)$, $B(5, 5)$, $C(9, 5)$

94. $A(-2, 2)$, $B(3, 2)$, $C(6, -1)$

In Exercises 95–98, for what value of b will the line joining the points P and Q be parallel to the indicated axis?

95. $P(-1, 3)$, $Q(4, b)$; x-axis

96. $P(5, 6)$, $Q(b, -2)$; y-axis

97. $P(3b - 1, 5)$, $Q(8, 4)$; y-axis

98. $P(-6, 2b + 3)$, $Q(7, -1)$; x-axis

99. ***Selling Chocolates*** Ryan Stewart sells chocolate on the Internet. His monthly profit, p, in dollars, can be estimated by $p = 15n - 300$, where n is the number of dozens of chocolates he sells in a month.

a) Graph $p = 15n - 300$, for $n \leq 60$.

b) From the graph, estimate his profit if he sells 40 dozen chocolates in a month.

c) How many dozens of chocolates must he sell in a month to break even?

100. ***Hanging Wallpaper*** Tanisha Vizquez owns a wallpaper hanging business. Her charge, C, for hanging wallpaper is \$40 plus \$0.30 per square foot of wallpaper she hangs, or $C = 40 + 0.30s$, where s is the number of square feet of wallpaper she hangs.

a) Graph $C = 40 + 0.30s$, for $s \leq 500$.

b) From the graph, estimate her charge if she hangs 300 square feet of wallpaper.

c) If her charge is \$70, use the equation for C to determine how many square feet of wallpaper she hung.

101. ***Photo Processing*** The charge, C, for processing a roll of 35-millimeter (mm) film onto a picture compact disc at Costco's 1 Hour Photo is \$8.95 plus \$0.33 per picture, or $C = 8.95 + 0.33n$, where n is the number of pictures printed.

a) Draw a graph of the cost of processing film for up to and including 36 pictures.

b) From the graph, estimate the cost of processing a roll of 35 mm film containing 20 pictures.

c) If the total cost of processing a roll of 35 mm film is \$20.83, estimate the number of pictures.

102. ***Earning Simple Interest*** When \$1000 is invested in a savings account paying simple interest for a year, the interest, i in dollars, earned can be found by the formula $i = 1000r$, where r is the rate in decimal form.

a) Graph $i = 1000r$, for r up to and including a rate of 15%.

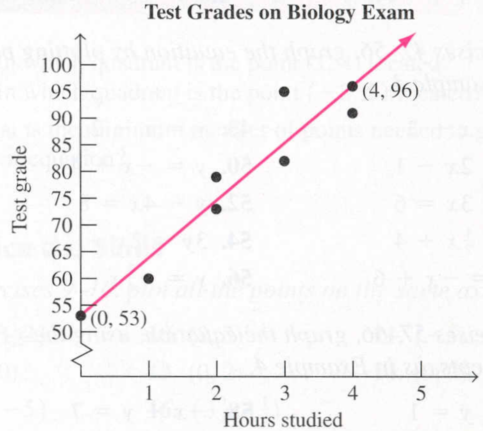
b) If the rate is 4%, what is the simple interest?

c) If the rate is 6%, what is the simple interest?

In Exercises 103 and 104, a set of points is plotted. Also shown is a straight line through the set of points that is called the line of best fit *(or a* regression line, *as will be discussed in Chapter 13, Statistics.)*

103. ***Determining a Test Grade*** The graph shows the hours studied and the test grades on a biology test for six students. (The two points indicated on the line do not represent any of the six students.) The line of best fit, the red line on the graph, can be used to approximate the test grade the average student receives for the number of hours he or she studies.

Test Grades on Biology Exam



a) Determine the slope of the line of best fit using the two points indicated.

b) Using the slope determined in part (a) and the y-intercept, (0, 53), determine the equation of the line of best fit.

c) Using the equation you determined in part (b), determine the approximate test grade for a student who studied for 3 hours.

d) Using the equation you determined in part (b), determine the amount of time a student would need to study to receive a grade of 80 on the biology test.

104. ***Determining the Number of Defects*** The graph shows the daily number of workers absent from the assembly line at J. B. Davis Corporation and the number of defects coming off the assembly line for 8 days. (The two points indicated on the line do not represent any of the 8 days.) The line of best fit, the blue line on the graph, can be used to approximate the number of defects coming off the assembly line per day for a given number of workers absent.

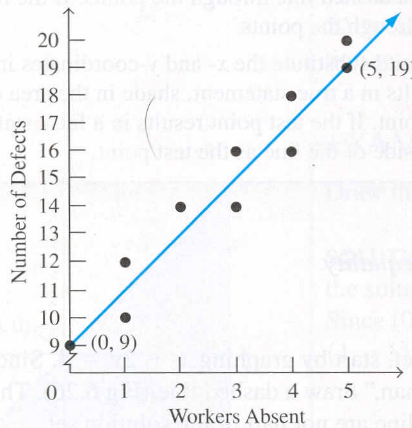
a) Determine the slope of the line of best fit using the two points indicated.

b) Using the slope determined in part (a) and the y-intercept, (0, 9), determine the equation of the line of best fit.

c) Using the equation you determined in part (b), determine the approximate number of defects for a day if 3 workers are absent.

d) Using the equation you determined in part (b), approximate the number of workers absent for a day if there are 17 defects that day.

Daily Number of Defects at J.B. Davis Corporation



105. ***Married Householders with Children*** The blue graph shows the percent of married householders with children. The red dashed straight line can be used to approximate the percent of married householders with children. If we let 0 represent 1970, 10 represent 1980, and so on, then 2000 would be represented by 30. Using the ordered pairs (0, 40) and (30, 24),

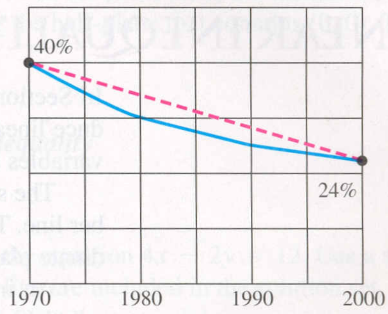
a) determine the slope of the dashed line.

b) determine the equation of the dashed line using (0, 40) as the y-intercept.

c) Using the equation you determined in part (b), determine the percent of married householders with children in 1985, which would be represented by 15.

d) Using the equation you determined in part (b), determine the year in which the percent of married householders with children was 30.

Married House-holders with Kids



Source: *Newsweek*

106. ***Book Sales*** The green graph shows book publishers' net sales, in millions of dollars, for trade, mass market, professional, educational, and university press publishers. The red dashed straight line can be used to approximate the book publishers' net dollar sales. If we let 0 represent 1994, 1 represent 1995, 2 represent 1996, and so on, then 2003 would be represented by 9. Using the ordered pairs (0, 17,000) and (9, 25,000),

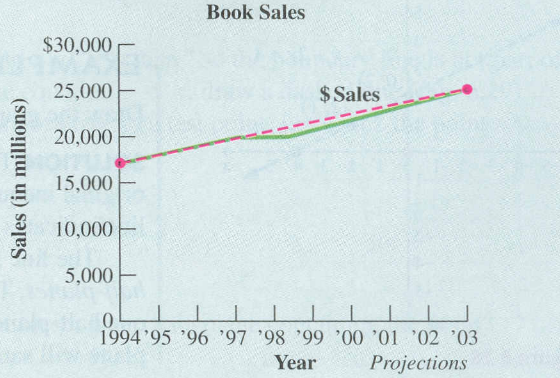
a) determine the slope of the dashed line.

b) determine the equation of the dashed line using (0, 17,000) as the y-intercept of the graph.

c) Using the equation you determined in part (b), determine the net dollar sales in 1998, which would be represented with year 4.

d) Using the equation you determined in part (b), determine the year that net sales were $20,000 (in millions).

Book Sales



Source: *The Wall Street Journal Almanac*, 1999.

Challenge Problems/Group Activities

107. a) Two lines are parallel when they do not intersect no matter how far they are extended. Explain how you can determine, without graphing the equations, whether two equations will be parallel lines when graphed.
b) Determine whether the graphs of the equations $2x - 3y = 6$ and $4x = 6y + 6$ are parallel lines.

108. In which quadrants will the set of points that satisfy the equation $x + y = 1$ lie? Explain.

Internet/Research Activity

109. René Descartes is known for his contributions to algebra. Write a paper on his life and his contributions to algebra.

6.8 LINEAR INEQUALITIES IN TWO VARIABLES

In Section 6.6, we introduced linear inequalities in one variable. Now we will introduce linear inequalities in two variables. Some examples of linear inequalities in two variables are $2x + 3y \leq 7$, $x + 7y \geq 5$, and $x - 3y < 6$.

The solution set of a linear inequality in one variable may be indicated on a number line. The solution set of a linear inequality in two variables is indicated on a coordinate plane.

An inequality that is strictly less than ($<$) or greater than ($>$) will have as its solution set a ***half-plane***. A half-plane is the set of all the points on one side of a line. An inequality that is less than or equal to ($\leq$) or greater than or equal to ($\geq$) will have as its solution set the set of points that consists of a half-plane and a line. To indicate that the line is part of the solution set, we draw a solid line. To indicate that the line is not part of the solution set, we draw a dashed line.

To Graph Inequalities in Two Variables

1. Mentally substitute the equal sign for the inequality sign and plot points as if you were graphing the equation.
2. If the inequality is $<$ or $>$, draw a dashed line through the points. If the inequality is $\leq$ or $\geq$, draw a solid line through the points.
3. Select a test point not on the line and substitute the x- and y-coordinates into the inequality. If the substitution results in a true statement, shade in the area on the same side of the line as the test point. If the test point results in a false statement, shade in the area on the opposite side of the line as the test point.

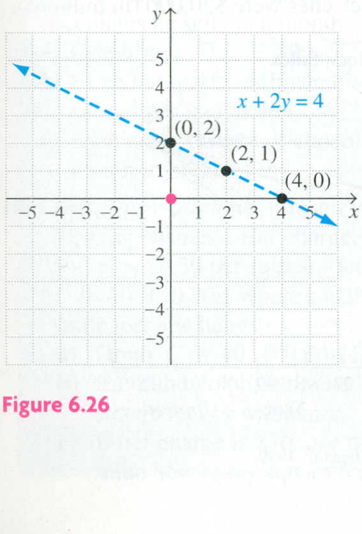


Figure 6.26

EXAMPLE 1 *Graphing an Inequality*

Draw the graph of $x + 2y < 4$.

SOLUTION: To obtain the solution set, start by graphing $x + 2y = 4$. Since the original inequality is strictly "less than," draw a dashed line (Fig 6.26). The dashed line indicates that the points on the line are not part of the solution set.

The line $x + 2y = 4$ divides the plane into three parts, the line itself and two *half-planes*. The line is the boundary between the two half-planes. The points in one half-plane will satisfy the inequality $x + 2y < 4$. The points in the other half-plane will satisfy the inequality $x + 2y > 4$.

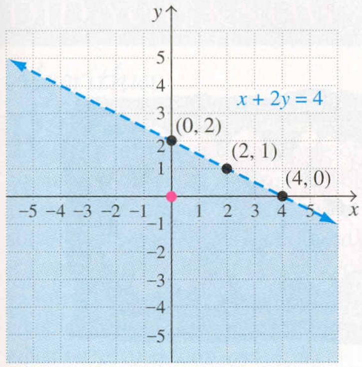


Figure 6.27

To determine the solution set to the inequality $x + 2y < 4$, pick any point on the plane that is not on the line. The simplest point to work with is the origin, (0, 0). Substitute $x = 0$ and $y = 0$ into $x + 2y < 4$.

$$x + 2y < 4$$
$$\text{Is } 0 + 2(0) < 4?$$
$$0 + 0 < 4$$
$$0 < 4 \quad \text{True}$$

Since 0 is less than 4, the point (0, 0) is part of the solution set. All the points on the same side of the graph of $x + 2y = 4$ as the point (0, 0) are members of the solution set. We indicate this by shading the half-plane that contains (0, 0). The graph is shown in Fig. 6.27.

EXAMPLE 2 *Graphing an Inequality*

Draw the graph of $4x - 2y \geq 12$.

SOLUTION: First draw the graph of the equation $4x - 2y = 12$. Use a solid line because the points on the boundary line are included in the solution set. Now pick a point that is not on the line. Take (0, 0) as the test point.

$$4x - 2y \geq 12$$
$$\text{Is } 4(0) - 2(0) \geq 12?$$
$$0 \geq 12 \quad \text{False}$$

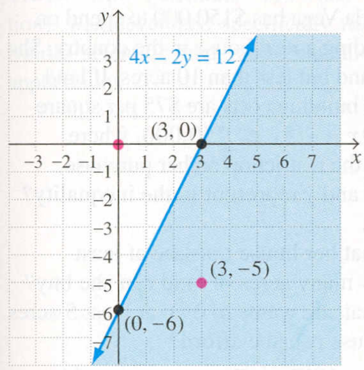


Figure 6.28

Since 0 is not greater than or equal to 12 ($0 \not\geq 12$), the solution set is the line and the half-plane that does not contain the point (0, 0). The graph is shown in Fig. 6.28.

If you had arbitrarily selected the test point $(3, -5)$ from the other half-plane, you would have found that the inequality would be true: $4(3) - 2(-5) \geq 12$, or $22 \geq 12$. Thus, the point $(3, -5)$ would be in the half-plane containing the solution set.

EXAMPLE 3 *Graphing an Inequality*

Draw the graph of $y < x$.

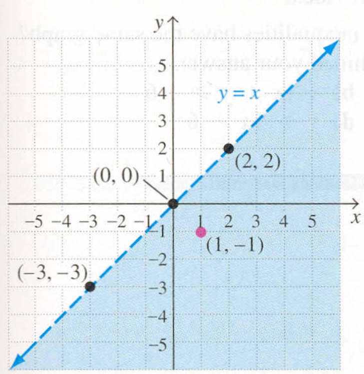


Figure 6.29

SOLUTION: The inequality is strictly "less than," so the boundary line is not part of the solution set. In graphing the equation $y = x$, draw a dashed line (Fig. 6.29). Since (0, 0) is *on* the line, it cannot serve as a test point. Let's pick the point $(1, -1)$.

$$y < x$$
$$-1 < 1 \quad \text{True}$$

Since $-1 < 1$ is true, the solution set is the half-plane containing the point $(1, -1)$.

SECTION 6.8 EXERCISES

Concept/Writing Exercises

1. Outline the procedure used to graph inequalities in two variables.
2. Explain why we use a solid line when graphing an inequality containing $\leq$ or $\geq$ and we use a dashed line when graphing an inequality containing $<$ or $>$.

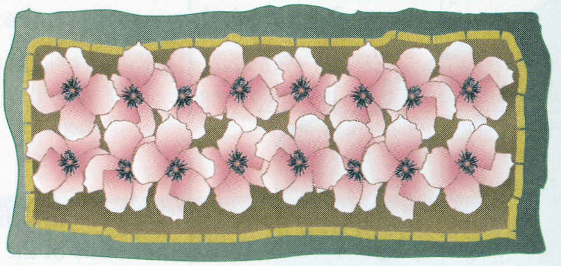
Practice the Skills

In Exercises 3–24, draw the graph of the inequality.

3. $x \leq 1$
4. $y \geq -2$
5. $y > x + 3$
6. $y < x - 5$
7. $y \geq 2x - 6$
8. $y < -2x + 2$
9. $3x - 4y > 12$
10. $x + 2y > 4$
11. $3x - 4y \leq 9$
12. $4y - 3x \geq 9$
13. $3x + 2y < 6$
14. $-x + 2y < 2$
15. $x + y > 0$
16. $x + 2y \leq 0$
17. $5x - 2y \leq 10$
18. $y \geq -2x + 1$
19. $3x + 2y > 12$
20. $y \leq 3x - 4$
21. $\frac{2}{5}x - \frac{1}{2}y \leq 1$
22. $0.1x + 0.3y \leq 0.4$
23. $0.2x + 0.5y \leq 0.3$
24. $\frac{1}{3}x + \frac{3}{4}y \geq 1$

Problem Solving

25. ***Gas Grills*** A manufacturer of gas grills must produce and ship x gas grills to one outlet and y gas grills to a second outlet. The maximum number of gas grills the manufacturer can produce and ship is 300.
    a) Write an inequality in two variables that represents this problem.
    b) Graph the inequality.
26. ***Flower Garden*** Jim Lawler has 40 ft of landscape edging to place around a new rectangular flower garden.



    a) Write an inequality illustrating all possible dimensions of the rectangular garden. $P = 2l + 2w$ is the formula for the perimeter of a rectangle.
    b) Graph the inequality.

Challenge Problems

27. ***Building a House*** Yolanda Vega has \$150,000 to spend on purchasing land and building a new house in the country. She wants at least 1 acre of land but less than 10 acres. If land costs \$1500 per acre and building costs are \$75 per square foot, the inequality $1500x + 75y \leq 150{,}000$, where $1 \leq x < 10$, describes the restriction on her purchase.
    a) What quantities do x and y represent in the inequality?
    b) Graph the inequality.
    c) If Yolanda decides that her house must be at least 1950 ft^2 in size, how many acres of land can she buy?
    d) If Yolanda decides that she wants to own at least 5 acres of land, what size house can she afford?
28. ***Men's Shirts*** The Tommy Hilfiger Company must ship x men's shirts to one outlet and y men's shirts to a second outlet. The maximum number of shirts the manufacturer can produce and ship is 250. We can represent this situation with the inequality $x + y \leq 250$.
    a) Can x or y be negative? Explain.
    b) Graph the inequality.
    c) Write one or two paragraphs interpreting the information that the graph provides.
29. Which of the following inequalities have the same graph? Explain how you determined your answer.
    a) $3x - y < 6$ b) $-3x + y > -6$
    c) $3x - 2y < 12$ d) $y > 3x - 6$

6.9 SOLVING QUADRATIC EQUATIONS BY USING FACTORING AND BY USING THE QUADRATIC FORMULA

We begin this section by discussing multiplication of binomials and factoring trinomials. After we discuss factoring trinomials, we will explain how to solve quadratic equations using factoring.

DID YOU KNOW

Algorithm

Recall that the series of steps taken to solve a certain type of equation is called an *algorithm*. Because mathematical procedures to solve equations can be generalized, it is possible to program computers to solve problems. For example, the equation $4x^2 + 7x + 3 = 0$ is an equation of the form $ax^2 + bx + c = 0$. If you can find a procedure to solve an equation of the general form, it can be used to solve all equations of that form. One of the first computer languages, FORTRAN (for FORmula TRANslation), was developed specifically to handle scientific and mathematical applications.

We will now look at the ***FOIL method*** of multiplying two binomials. A ***binomial*** is an expression that contains two terms, in which each exponent that appears on a variable is a whole number.

Examples of binomials

$x + 3$	$x - 5$
$3x + 5$	$4x - 2$

To multiply two binomials, we can use the FOIL method. The name of the method, FOIL, is an acronym to help its users remember it as a method that obtains the products of the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials.

$$(a+b)(c+d) = \overset{\text{First}}{a\cdot c} + \overset{\text{Outer}}{a\cdot d} + \overset{\text{Inner}}{b\cdot c} + \overset{\text{Last}}{b\cdot d}$$

After multiplying the first, outer, inner, and last terms, combine all like terms.

EXAMPLE 1 *Multiplying Binomials*

Multiply $(x + 3)(x + 5)$.

SOLUTION: The FOIL method of multiplication yields

$$\begin{aligned}(x+3)(x+5) &= \overset{\text{F}}{x\cdot x} + \overset{\text{O}}{x\cdot 5} + \overset{\text{I}}{3\cdot x} + \overset{\text{L}}{3\cdot 5}\\ &= x^2 + 5x + 3x + 15\\ &= x^2 + 8x + 15\end{aligned}$$

Note that the like terms $5x$ and $3x$ were combined to get $8x$.

EXAMPLE 2 *Multiplying Binomials*

Multiply $(2x - 1)(x + 4)$.

SOLUTION:

$$\begin{aligned}(2x-1)(x+4) &= \overset{\text{F}}{2x\cdot x} + \overset{\text{O}}{2x\cdot 4} + \overset{\text{I}}{(-1)\cdot x} + \overset{\text{L}}{(-1)\cdot 4}\\ &= 2x^2 + 8x - x - 4\\ &= 2x^2 + 7x - 4\end{aligned}$$

Factoring Trinomials of the Form $x^2 + bx + c$

The expression $x^2 + 8x + 15$ is an example of a trinomial. A ***trinomial*** is an expression containing three terms in which each exponent that appears on a variable is a whole number.

Example 1 showed that

$$(x + 3)(x + 5) = x^2 + 8x + 15$$

Since the product of $x + 3$ and $x + 5$ is $x^2 + 8x + 15$, we say that $x + 3$ and $x + 5$ are ***factors*** of $x^2 + 8x + 15$. To ***factor*** an expression means to write the expression as a product of its factors. For example, to factor $x^2 + 8x + 15$ we write

$$x^2 + 8x + 15 = (x + 3)(x + 5)$$

Let's look at the factors more closely.

$$3 + 5 = 8$$

$$3 \cdot 5 = 15$$

$$x^2 + 8x + 15 = (x + 3)(x + 5)$$

Note that the sum of the two numbers in the factors is $3 + 5$ or 8. The 8 is the coefficient of the x-term. Also note that the product of the numbers in the two factors is $3 \cdot 5$, or 15. The 15 is the constant in the trinomial. In general, when factoring an expression of the form $x^2 + bx + c$, we need to find two numbers whose product is c and whose sum is b. When we determine the two numbers, the factors will be of the form

$$(x + \underset{\substack{\uparrow \\ \text{One} \\ \text{number}}}{\blacksquare})\ (x + \underset{\substack{\uparrow \\ \text{Other} \\ \text{number}}}{\blacksquare})$$

EXAMPLE 3 *Factoring a Trinomial*

Factor $x^2 + 5x + 6$.

SOLUTION: We need to find two numbers whose product is 6 and whose sum is 5. Since the product is +6, the two numbers must both be positive or both be negative. Because the coefficient of the x-term is positive, only the positive factors of 6 need to be considered. Can you explain why? We begin by listing the positive numbers whose product is 6.

Factors of 6	Sum of Factors
1(6)	$1 + 6 = 7$
2(3)	$2 + 3 = 5$

Since $2 \cdot 3 = 6$ and $2 + 3 = 5$, 2 and 3 are the numbers we are seeking. Thus, we write

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Note that $(x + 3)(x + 2)$ is also an acceptable answer. ▲

To Factor Trinomial Expressions of the Form $x^2 + bx + c$

1. Find two numbers whose product is c and whose sum is b.
2. Write factors in the form

$$(x + \blacksquare)\ (x + \blacksquare)$$

One number from step 1 — Other number from step 1

3. Check your answer by multiplying the factors using the FOIL method.

If, for example, the numbers found in step 1 of the above procedure were 6 and -4, the factors would be written $(x + 6)(x - 4)$.

EXAMPLE 4 *Factoring a Trinomial*

Factor $x^2 - 6x - 16$.

SOLUTION: We must find two numbers whose product is -16 and whose sum is -6. Begin by listing the factors of -16.

Factors of -16	Sum of Factors
$-16(1)$	$-16 + 1 = -15$
$-8(2)$	$-8 + 2 = -6$
$-4(4)$	$-4 + 4 = 0$
$-2(8)$	$-2 + 8 = 6$
$-1(16)$	$-1 + 16 = 15$

The table lists all the factors of -16. The only factors listed whose product is -16 and whose sum is -6 are -8 and 2. We listed all factors in this example so that you could see, for example, that $-8(2)$ is a different set of factors than $-2(8)$. Once you find the factors you are looking for, there is no need to go any further. The trinomial can be written in factored form as

$$x^2 - 6x - 16 = (x - 8)(x + 2)$$

Factoring Trinomials of the Form $ax^2 + bx + c,\ a \neq 1$

Now we discuss how to factor an expression of the form $ax^2 + bx + c$, where a, the coefficient of the squared term, is not equal to 1.

Consider the multiplication problem $(2x + 1)(x + 3)$.

$$\begin{aligned}(2x + 1)(x + 3) &= 2x \cdot x + 2x \cdot 3 + 1 \cdot x + 1 \cdot 3\\ &= 2x^2 + 6x + x + 3\\ &= 2x^2 + 7x + 3\end{aligned}$$

Since $(2x + 1)(x + 3) = 2x^2 + 7x + 3$, the factors of $2x^2 + 7x + 3$ are $2x + 1$ and $x + 3$.

Let's study the coefficients more closely.

$$\underset{\downarrow}{\text{F}}\quad\underset{\downarrow}{\text{O + I}}\quad\underset{\downarrow}{\text{L}}$$

$$2x^2 + 7x + 3 \qquad (2x + 1)(1x + 3)$$

F L I O

$$\textbf{F} = 2 \cdot 1 = 2 \qquad \textbf{O} + \textbf{I} = (2 \cdot 3) + (1 \cdot 1) = 7 \qquad \textbf{L} = 1 \cdot 3 = 3$$

Note that the product of the coefficient of the first terms in the multiplication of the binomials equals 2, the coefficient of the squared term. The sum of the products of the coefficients of the outer and inner terms equals 7, the coefficient of the x-term. The product of the last terms equals 3, the constant.

A procedure to factor expressions of the form $ax^2 + bx + c$, $a \neq 1$, follows.

To Factor Trinomial Expressions of the Form $ax^2 + bx + c, a \neq 1$

1. Write all pairs of factors of the coefficient of the squared term, a.
2. Write all pairs of factors of the constant, c.
3. Try various combinations of these factors until the sum of the products of the outer and inner terms is bx.
4. Check your answer by multiplying the factors using the FOIL method.

EXAMPLE 5 *Factoring a Trinomial, $a \neq 1$*

Factor $3x^2 + 17x + 10$.

SOLUTION: The only positive factors of 3 are 3 and 1. Therefore, we write

$$3x^2 + 17x + 10 = (3x \qquad)(x \qquad)$$

The number 10 has both positive and negative factors. However, since both the constant, 10, and the sum of the products of the outer and inner terms, 17, are positive, the two factors must be positive. Why? The positive factors of 10 are 1(10) and 2(5). The following is a list of the possible factors.

Possible Factors	Sum of Products of Outer and Inner Terms
$(3x + 1)(x + 10)$	$31x$
$(3x + 10)(x + 1)$	$13x$
$(3x + 2)(x + 5)$	$17x$ ← Correct middle term
$(3x + 5)(x + 2)$	$11x$

Thus, $3x^2 + 17x + 10 = (3x + 2)(x + 5)$.

Note that factoring problems of this type may be checked by using the FOIL method of multiplication. We will check the results to Example 5:

$$\begin{aligned}(3x + 2)(x + 5) &= 3x \cdot x + 3x \cdot 5 + 2 \cdot x + 2 \cdot 5 \\ &= 3x^2 + 15x + 2x + 10 \\ &= 3x^2 + 17x + 10\end{aligned}$$

Since we obtained the expression we started with, our factoring is correct.

EXAMPLE 6 *Factoring a Trinomial, $a \neq 1$*

Factor $6x^2 - 11x - 10$.

SOLUTION: The factors of 6 will be either $6 \cdot 1$ or $2 \cdot 3$. Therefore, the factors may be of the form $(6x \quad)(x \quad)$ or $(2x \quad)(3x \quad)$. When there is more than one set of factors for the first term, we generally try the medium-sized factors first. If that does not work, we try the other factors. Thus, we write

$$6x^2 - 11x - 10 = (2x \quad)(3x \quad)$$

The factors of -10 are $(-1)(10)$, $(1)(-10)$, $(-2)(5)$, and $(2)(-5)$. There will be eight different pairs of possible factors of the trinomial $6x^2 - 11x - 10$. Can you list them?

The correct factoring is $6x^2 - 11x - 10 = (2x - 5)(3x + 2)$.

Note that in Example 6 we first tried factors of the form $(2x \quad)(3x \quad)$. If we had not found the correct factors using them, we would have tried $(6x \quad)(x \quad)$.

Solving Quadratic Equations by Factoring

In Section 6.2, we solved linear, or first-degree, equations. In those equations, the exponent on all variables was 1. Now we deal with the *quadratic equation*. The standard form of a quadratic equation in one variable is shown in the box.

Standard Form of a Quadratic Equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

Note that in the standard form of a quadratic equation, the greatest exponent on x is 2 and the right side of the equation is equal to zero. *To solve a quadratic equation* means to find the value or values that make the equation true. In this section, we will solve quadratic equations by factoring and by the quadratic formula.

To solve a quadratic equation by factoring, set one side of the equation equal to 0 and then use the *zero-factor* property.

Zero-Factor Property

If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

The zero-factor property indicates that, if the product of two factors is 0, then one (or both) of the factors must have a value of 0.

EXAMPLE 7 *Using the Zero-Factor Property*

Solve the equation $(x + 3)(x - 6) = 0$.

SOLUTION: When we use the zero-factor property, either $(x + 3)$ or $(x - 6)$ must equal 0 for the product to equal 0. Thus, we set each individual factor equal to 0 and solve each resulting equation for x.

$$(x + 3)(x - 6) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x - 6 = 0$$
$$x = -3 \qquad\qquad x = 6$$

Thus, the solutions are -3 and 6.

Check: $x = -3$ $\qquad$ $x = 6$

$$(x + 3)(x - 6) = 0 \qquad (x + 3)(x - 6) = 0$$
$$(-3 + 3)(-3 - 6) = 0 \qquad (6 + 3)(6 - 6) = 0$$
$$0(-9) = 0 \qquad 9(0) = 0$$
$$0 = 0 \quad \text{True} \qquad 0 = 0 \quad \text{True}$$

To Solve a Quadratic Equation by Factoring

1. Use the addition or subtraction property to make one side of the equation equal to 0.
2. Factor the side of the equation not equal to 0.
3. Use the zero-factor property to solve the equation.

Examples 8 and 9 illustrate this procedure.

EXAMPLE 8 *Solving a Quadratic Equation by Factoring*

Solve the equation $x^2 - 8x = -15$.

SOLUTION: First add 15 to both sides of the equation to make the right side of the equation equal to 0.

$$x^2 - 8x = -15$$
$$x^2 - 8x + 15 = -15 + 15$$
$$x^2 - 8x + 15 = 0$$

Factor the left side of the equation. The object is to find two numbers whose product is 15 and whose sum is -8. Since the product of the numbers is positive and the sum of the numbers is negative, the two numbers must both be negative. The numbers are -3 and -5. Note that $(-3)(-5) = 15$ and $-3 + (-5) = -8$.

$$x^2 - 8x + 15 = 0$$
$$(x - 3)(x - 5) = 0$$

Now use the zero-factor property to find the solution.

$$x - 3 = 0 \quad \text{or} \quad x - 5 = 0$$
$$x = 3 \qquad\qquad x = 5$$

The solutions are 3 and 5.

EXAMPLE 9 ***Solve a Quadratic Equation by Factoring***

Solve the equation $3x^2 - 13x + 4 = 0$.

SOLUTION: $3x^2 - 13x + 4$ factors into $(3x - 1)(x - 4)$. Thus, we write

$$3x^2 - 13x + 4 = 0$$
$$(3x - 1)(x - 4) = 0$$
$$3x - 1 = 0 \quad \text{or} \quad x - 4 = 0$$
$$3x = 1 \qquad\qquad x = 4$$
$$x = \frac{1}{3}$$

The solutions are $\frac{1}{3}$ and 4.

TIMELY TIP As stated on page 355, every factoring problem can be checked by multiplying the factors. If you have factored correctly, the product of the factors will be identical to the original expression that was factored. If we wished to check the factoring of Example 9, we would multiply $(3x - 1)(x - 4)$. Since the product of the factors is $3x^2 - 13x + 4$, the expression we started with, our factoring is correct.

Solving Quadratic Equations by Using the Quadratic Formula

Not all quadratic equations can be solved by factoring. When a quadratic equation cannot be easily solved by factoring, we can solve the equation with the *quadratic formula*. The quadratic formula can be used to solve any quadratic equation.

Quadratic Formula

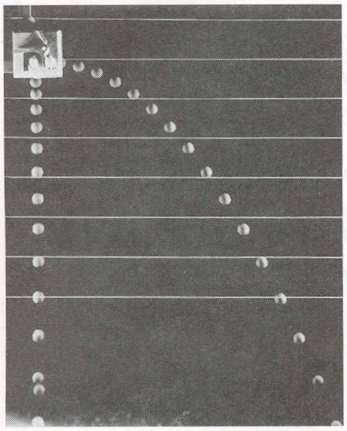
For a quadratic equation in standard form, $ax^2 + bx + c = 0$, $a \neq 0$, the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the quadratic formula, the plus or minus symbol, $\pm$, is used. If, for example, $x = 2 \pm 3$, then $x = 2 + 3 = 5$ or $x = 2 - 3 = -1$.

DID YOU KNOW

The Mathematics of Motion



The free fall of an object is something that has interested scientists and mathematicians for centuries. It is described by a quadratic equation. Shown here is a time-lapse photo that shows the free fall of a ball in equal-time intervals. What you see can be described verbally this way: The rate of change in velocity in each interval is the same; therefore, velocity is continuously increasing and acceleration is constant.

It is possible for a quadratic equation to have no real solution. In solving an equation, if the radicand (the expression inside the square root) is a negative number, then the quadratic equation has ***no real solution***.

To use the quadratic formula, first write the quadratic equation in standard form. Then determine the values for a (the coefficient of the squared term), b (the coefficient of the x-term), and c (the constant). Finally, substitute the values of a, b, and c into the quadratic formula and evaluate the expression.

EXAMPLE 10 *Solve a Quadratic Equation Using the Quadratic Formula*

Solve the equation $x^2 + 2x - 15 = 0$ using the quadratic formula.

SOLUTION: In this equation, $a = 1$, $b = 2$, and $c = -15$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(1)(-15)}}{2(1)}$$
$$= \frac{-2 \pm \sqrt{4 + 60}}{2}$$
$$= \frac{-2 \pm \sqrt{64}}{2}$$
$$= \frac{-2 \pm 8}{2}$$
$$x = \frac{-2 + 8}{2} = \frac{6}{2} = 3 \quad \text{or} \quad x = \frac{-2 - 8}{2} = \frac{-10}{2} = -5$$

The solutions are 3 and -5.

Note that Example 10 can also be solved by factoring. We suggest that you do so now.

EXAMPLE 11 *Irrational Solutions to a Quadratic Equation*

Solve $4x^2 - 8x = -1$ using the quadratic formula.

SOLUTION: Begin by writing the equation in standard form by adding 1 to both sides of the equation, which gives the following.

$$4x^2 - 8x + 1 = 0$$
$$a = 4, \quad b = -8, \quad c = 1$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)}$$
$$= \frac{8 \pm \sqrt{64 - 16}}{8}$$
$$= \frac{8 \pm \sqrt{48}}{8}$$

Since $\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$ (see Section 5.4), we write

$$\frac{8 \pm \sqrt{48}}{8} = \frac{8 \pm 4\sqrt{3}}{8} = \frac{\overset{1}{\cancel{4}}(2 \pm \sqrt{3})}{\underset{2}{\cancel{8}}} = \frac{2 \pm \sqrt{3}}{2}$$

The solutions are $\dfrac{2 + \sqrt{3}}{2}$ and $\dfrac{2 - \sqrt{3}}{2}$.

Note that the solutions to Example 11 are irrational numbers.

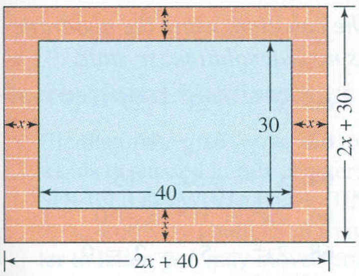


Figure 6.30

EXAMPLE 12 ***Brick Border***

Diane Cecero and her husband recently installed an inground rectangular swimming pool measuring 40 ft by 30 ft. They want to add a brick border of uniform width around all sides of the pool. How wide can they make the brick border if they purchased enough brick to cover 296 ft^2?

SOLUTION: Let's make a diagram of the pool and the brick border (Fig. 6.30) Let x = the uniform width of the brick border. Then the total length of the larger rectangular area, the pool plus the border, is $2x + 40$. The total width of the larger rectangular area is $2x + 30$.

The area of the brick border can be found by subtracting the area of the pool from the area of the pool plus the brick border.

$$\begin{aligned}
\text{Area of pool} &= l \cdot w = (40)(30) = 1200 \text{ ft}^2 \\
\text{Area of pool plus brick border} &= l \cdot w = (2x + 40)(2x + 30) \\
&= 4x^2 + 140x + 1200 \\
\text{Area of the brick border} &= \text{area of pool plus brick border} - \text{area of pool} \\
&= (4x^2 + 140x + 1200) - 1200 \\
&= 4x^2 + 140x
\end{aligned}$$

The total brick border must be 296 ft^2. Therefore,

$$296 = 4x^2 + 140x$$

or

$$\begin{aligned}
4x^2 + 140x - 296 &= 0 \\
4(x^2 + 35x - 74) &= 0 && \text{Factor out 4 from each term.} \\
\frac{4}{4}(x^2 + 35x - 74) &= \frac{0}{4} && \text{Divide both sides of the equation by 4.} \\
x^2 + 35x - 74 &= 0 \\
(x + 37)(x - 2) &= 0 && \text{Factor trinomial.}
\end{aligned}$$

$$x + 37 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -37 \qquad\qquad x = 2$$

Since lengths are positive, the only possible answer is $x = 2$. Thus, they can make a brick border 2 ft wide all around the pool.

SECTION 6.9 EXERCISES

Concept/Writing Exercises

1. What is a *binomial*? Give three examples of binomials.
2. What is a *trinomial*? Give three examples of trinomials.
3. In your own words, explain the FOIL method used to multiply two binomials.
4. In your own words, state the zero-factor property.
5. Give the standard form of a quadratic equation.
6. Have you memorized the quadratic formula? If not, you need to do so. Without looking at the book, write the quadratic formula.

Practice the Skills

In Exercises 7–22, factor the trinomial. If the trinomial cannot be factored, so state.

7. $x^2 + 9x + 18$ **8.** $x^2 + 5x + 4$
9. $x^2 - x - 6$ **10.** $x^2 + x - 6$
11. $x^2 + 2x - 24$ **12.** $x^2 - 6x + 8$
13. $x^2 - 2x - 3$ **14.** $x^2 - 5x - 6$
15. $x^2 - 10x + 21$ **16.** $x^2 - 81$
17. $x^2 - 25$ **18.** $x^2 - x - 20$
19. $x^2 + 3x - 28$ **20.** $x^2 + 4x - 32$
21. $x^2 + 2x - 63$ **22.** $x^2 - 2x - 48$

In Exercises 23–34, factor the trinomial. If the trinomial cannot be factored, so state.

23. $2x^2 - x - 10$ **24.** $3x^2 - 2x - 5$
25. $4x^2 + 13x + 3$ **26.** $2x^2 - 11x - 21$
27. $5x^2 + 12x + 4$ **28.** $2x^2 - 9x + 10$
29. $4x^2 + 11x + 6$ **30.** $4x^2 + 20x + 21$
31. $4x^2 - 11x + 6$ **32.** $6x^2 - 11x + 4$
33. $3x^2 - 14x - 24$ **34.** $6x^2 + 5x + 1$

In Exercises 35–38, solve each equation, using the zero-factor property.

35. $(x - 1)(x + 2) = 0$ **36.** $(2x + 5)(x - 1) = 0$
37. $(3x + 4)(2x - 1) = 0$ **38.** $(x - 6)(5x - 4) = 0$

In Exercises 39–58, solve each equation by factoring.

39. $x^2 + 10x + 21 = 0$ **40.** $x^2 + 4x - 5 = 0$
41. $x^2 - 4x + 3 = 0$ **42.** $x^2 - 5x - 24 = 0$
43. $x^2 - 15 = 2x$ **44.** $x^2 - 7x = -6$
45. $x^2 = 4x - 3$ **46.** $x^2 - 13x + 40 = 0$
47. $x^2 - 81 = 0$ **48.** $x^2 - 64 = 0$
49. $x^2 + 5x - 36 = 0$ **50.** $x^2 + 12x + 20 = 0$
51. $3x^2 + 10x = 8$ **52.** $3x^2 - 5x = 2$
53. $5x^2 + 11x = -2$ **54.** $2x^2 = -5x + 3$
55. $3x^2 - 4x = -1$ **56.** $5x^2 + 16x + 12 = 0$
57. $4x^2 - 9x + 2 = 0$ **58.** $6x^2 + x - 2 = 0$

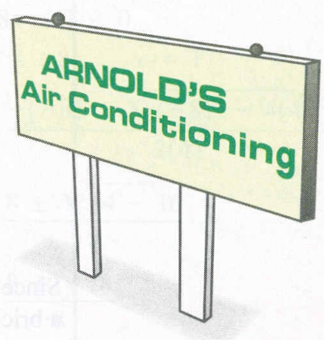
In Exercises 59–78, solve the equation, using the quadratic formula. If the equation has no real solution, so state.

59. $x^2 + 2x - 15 = 0$ **60.** $x^2 + 12x + 27 = 0$
61. $x^2 - 3x - 18 = 0$ **62.** $x^2 - 6x - 16 = 0$
63. $x^2 - 8x = 9$ **64.** $x^2 = -8x + 15$
65. $x^2 - 2x + 3 = 0$ **66.** $2x^2 - x - 3 = 0$
67. $x^2 - 4x + 2 = 0$ **68.** $2x^2 - 5x - 2 = 0$
69. $3x^2 - 8x + 1 = 0$ **70.** $2x^2 + 4x + 1 = 0$
71. $4x^2 - x - 1 = 0$ **72.** $4x^2 - 5x - 3 = 0$
73. $2x^2 + 7x + 5 = 0$ **74.** $3x^2 = 9x - 5$
75. $3x^2 - 10x + 7 = 0$ **76.** $4x^2 + 7x - 1 = 0$
77. $4x^2 - 11x + 13 = 0$ **78.** $5x^2 + 9x - 2 = 0$

Challenge Problems/Group Activities

79. *Flower Garden* Karen and Kurt Ohliger's backyard has a width of 20 meters and a length of 30 meters. Karen and Kurt want to put a flower garden in the middle of the backyard leaving a strip of grass of uniform width around all sides of the flower garden. If they want to have 336 square meters of grass, what will be the width and length of the garden?

80. *Air Conditioning* The yearly profit p of Arnold's Air Conditioning is given by $p = x^2 + 15x - 100$, where x is the number of air conditioners produced and sold. How many air conditioners must be produced and sold to have a yearly profit of \$45,000?



81. a) Explain why solving $(x-4)(x-7)=6$ by setting each factor equal to 6 is not correct.
b) Determine the correct solution to $(x-4)(x-7)=6$.

82. The radicand in the quadratic formula, b^2-4ac, is called the **discriminant.** How many real number solutions will the quadratic equation have if the discriminant is (a) greater than 0, (b) equal to 0, or (c) less than zero? Explain your answer.

83. Write an equation that has solutions -1 and 3.

Recreational Mathematics

84. Hidden in the grid above and to the right are the following words discussed in this chapter: ALGEBRA, FORMULA, SOLUTION, VARIATION, BINOMIAL. You will find these words by going letter to letter. As you move from letter to letter, you may move vertically or horizontally. A letter can be used only once when spelling out each particular word. Find the words listed above in the grid. One example, ALGEBRA, is shown.

R	I	A	A	S	N	E	Y
A	V	T	R	R	I	U	P
U	I	I	O	N	R	A	B
R	R	L	U	A	L	G	D
U	S	O	T	N	A	E	C
A	L	U	I	O	R	B	U
X	S	M	S	E	I	I	N
F	O	R	M	M	O	N	S
C	G	E	M	I	A	L	P

Internet/Research Activity

85. Italian mathematician Girolamo Cardano (1501–1576) is recognized for his skill in solving equations. Write a paper about his life and his contributions to mathematics, in particular his contribution to solving equations.

86. Chinese mathematician Foo Ling Awong, who lived during the Pong dynasty, developed a technique, other than trial and error, to factor trinomials of the form ax^2+bx+c, $a \neq 1$. Write a paper about his life and his contributions to mathematics, in particular his technique for factoring trinomials in the form ax^2+bx+c, $a \neq 1$. (References include history of mathematics books, encyclopedias, and the Internet.)

6.10 FUNCTIONS AND THEIR GRAPHS

The concepts of relations and functions are extremely important in mathematics. A ***relation*** is any set of ordered pairs. Therefore, every graph will be a relation. A function is a special type of relation. Suppose you are purchasing oranges at a supermarket where each orange costs \$0.20. Then one orange would cost \$0.20, two oranges $2 \times \$0.20 = \$0.40$, three oranges \$0.60, and so on. We can indicate this relation in a table of values.

Number of Oranges	Cost
0	0.00
1	0.20
2	0.40
3	0.60
⋮	⋮
10	2.00
⋮	⋮

In general, the cost for purchasing n oranges will be 20 cents times the number of oranges, or $0.20n$. We can represent the cost, c, of n items by the equation $c = 0.20n$.

Since the value of c depends on the value of n, we refer to c as the *dependent variable* and n as the *independent variable. Note for each value of the independent variable, n, there is one and only one value of the dependent variable, c.* Such an equation is called a ***function***. In the equation $c = 0.20n$, the value of c depends on the value of n, so we say that "c is a function of n."

A **function** is a special type of relation where each value of the independent variable corresponds to a unique value of the dependent variable.

The set of values that can be used for the independent variable is called the ***domain*** of the function, and the resulting set of values obtained for the dependent variable is called the ***range***. The domain and range for the function $c = 0.20n$ are illustrated in Fig. 6.31.

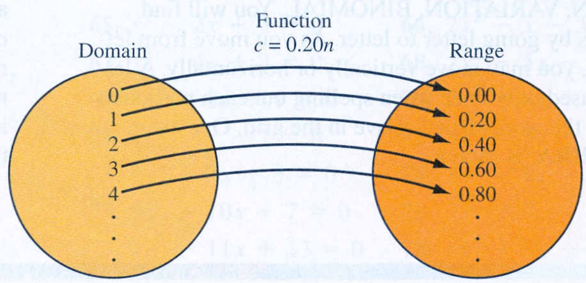


Figure 6.31

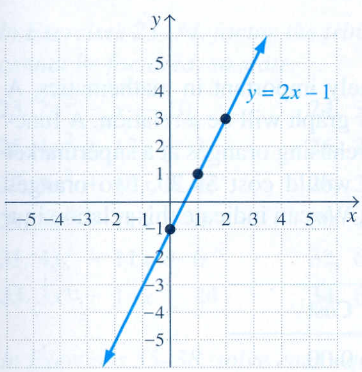


Figure 6.32

When we graphed equations of the form $ax + by = c$ in Section 6.7, we found that they were straight lines. For example, the graph of $y = 2x - 1$ is illustrated in Fig. 6.32.

Is the equation $y = 2x - 1$ a function? To answer this question, we must ask, "Does each value of x correspond to a unique value of y?" The answer is yes; therefore, this equation is a function.

For the equation $y = 2x - 1$, we say that "y is a function of x" and write $y = f(x)$. The notation $f(x)$ is read "f of x." When we are given an equation that is a function, we may replace the y in the equation with $f(x)$, since $f(x)$ represents y. Thus, $y = 2x - 1$ may be written $f(x) = 2x - 1$.

To evaluate a function for a specific value of x, replace each x in the function with the given value, then evaluate. For example, to evaluate $f(x) = 2x - 1$ when $x = 8$, we do the following.

$$f(x) = 2x - 1$$
$$f(8) = 2(8) - 1 = 16 - 1 = 15$$

Thus, $f(8) = 15$. Since $f(x) = y$, when $x = 8$, $y = 15$. What is the domain and range of $f(x) = 2x - 1$? Because x can be any real number, the domain is the set of real numbers, symbolized $\mathbb{R}$. The range is also $\mathbb{R}$.

We can determine whether a graph represents a function by using the ***vertical line test***: If a vertical line can be drawn so that it intersects the graph at more than one point, then each x does not have a unique y and the graph does not represent a function. If a vertical line cannot be made to intersect the graph in at least two different places, then the graph represents a function.

EXAMPLE 1 *Using the Vertical Line Test*

Use the vertical line test to determine which of the graphs in Figure 6.33 represent functions.

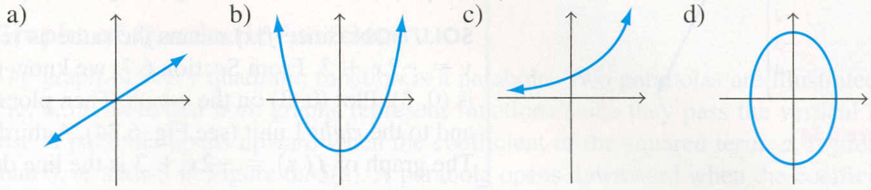
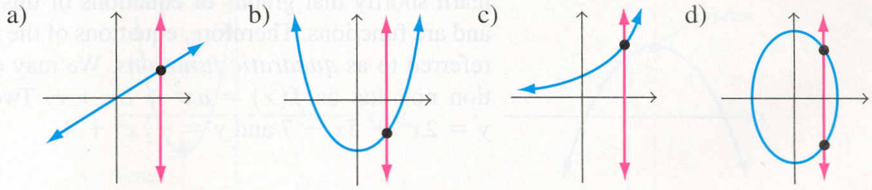


Figure 6.33

SOLUTION: (a), (b), and (c) represent functions, but (d) does not.



There are many real-life applications of functions. In fact, all the applications illustrated in Sections 6.2 through 6.4 are functions.

In this section, we will discuss three types of functions: linear functions, quadratic functions, and exponential functions.

Linear Functions

In Section 6.7, we graphed linear equations. The graph of any linear equation of the form $y = ax + b$ will pass the vertical line test, and so equations of the form $y = ax + b$ are ***linear functions***. If we wished, we could write the linear function as $f(x) = ax + b$ since $f(x)$ means the same as y.

EXAMPLE 2 *Cost as a Linear Function*

Adam Finiteri's weekly cost of operating a taxi, c, is given by the function $c(m) = 52 + 0.18m$, where m is the number of miles driven per week. What is his weekly cost if he drives 200 miles in a week?

SOLUTION: Substitute 200 for m in the function.

$$c(m) = 52 + 0.18m$$
$$c(200) = 52 + 0.18(200)$$
$$c(200) = 52 + 36 = 88$$

Thus, if Adam drives 200 miles in a week, his weekly cost will be \$88.

Graphs of Linear Functions

The graphs of linear functions are straight lines that will pass the vertical line test. In Section 6.7, we discussed how to graph linear equations. Linear functions can be graphed by plotting points, by using intercepts, or by using the slope and y-intercept.

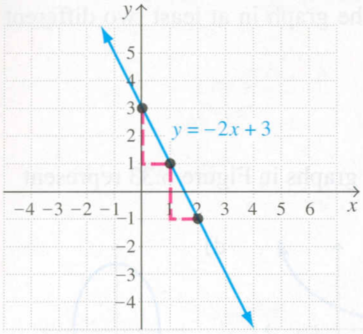


Figure 6.34

EXAMPLE 3 *Graphing a Linear Function*

Graph $f(x) = -2x + 3$ by using the slope and y-intercept.

SOLUTION: Since $f(x)$ means the same as y, we can rewrite this function as $y = -2x + 3$. From Section 6.7, we know that the slope is -2 and the y-intercept is $(0, 3)$. Plot $(0, 3)$ on the y-axis. Then plot the next point by moving *down* 2 units and to the *right* 1 unit (see Fig. 6.34). A third point has been plotted in the same way. The graph of $f(x) = -2x + 3$ is the line drawn through these three points.

Quadratic Functions

The standard form of a quadratic equation is $y = ax^2 + bx + c, a \neq 0$. We will learn shortly that graphs of equations of this form always pass the vertical line test and are functions. Therefore, equations of the form $y = ax^2 + bx + c, a \neq 0$, may be referred to as ***quadratic functions***. We may express quadratic functions using function notation as $f(x) = ax^2 + bx + c$. Two examples of quadratic functions are $y = 2x^2 + 5x - 7$ and $y = -\frac{1}{2}x^2 + 4$.

EXAMPLE 4 *Landing on the Moon*

On July 20, 1969, Neil Armstrong became the first person to walk on the moon. The velocity, v, of his spacecraft, the *Eagle,* in meters per second, was a function of time before touchdown, t, given by

$$v = f(t) = 3.2t + 0.45$$

The height of the spacecraft, h, above the moon's surface, in meters, was also a function of time before touchdown, given by

$$h = g(t) = 1.6t^2 + 0.45t$$

What was the velocity of the spacecraft and its distance from the surface of the moon

a) at 3 seconds before touchdown? b) at touchdown (0 seconds)?

SOLUTION:

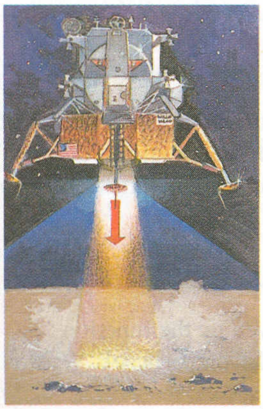
a) $v = f(t) = 3.2t + 0.45$, $\quad h = g(t) = 1.6t^2 + 0.45t$

$$f(3) = 3.2(3) + 0.45 \qquad g(3) = 1.6(3)^2 + 0.45(3)$$
$$= 9.6 + 0.45 \qquad = 1.6(9) + 1.35$$
$$= 10.05 \qquad = 14.4 + 1.35$$
$$= 15.75$$

The velocity 3 seconds before touchdown was 10.05 meters per second and the height 3 seconds before touchdown was 15.75 meters.

DID YOU KNOW

Sea of Tranquillity



Apollo II touched down at Mare Tranqillitatis, the Sea of Tranquillity. The rock samples taken there placed the age of the rocks at 3.5 billion years old, as old as the oldest known Earth rocks.

b) $v = f(t) = 3.2t + 0.45, \quad h = g(t) = 1.6t^2 + 0.45t$

$$\begin{aligned} f(0) &= 3.2(0) + 0.45 & g(0) &= 1.6(0)^2 + 0.45(0) \\ &= 0 + 0.45 & &= 0 + 0 \\ &= 0.45 & &= 0 \end{aligned}$$

The touchdown velocity was 0.45 meter per second. At touchdown, the *Eagle* is on the moon, and therefore the distance from the surface of the moon is 0 meter.

Graphs of Quadratic Functions

The graph of every quadratic function is a parabola. Two parabolas are illustrated in Fig. 6.35. Note that both graphs represent functions since they pass the vertical line test. A parabola opens upward when the coefficient of the squared term, a, is greater than 0, as shown in Figure 6.35(a). A parabola opens downward when the coefficient of the squared term, a, is less than 0, as shown in Fig. 6.35(b).

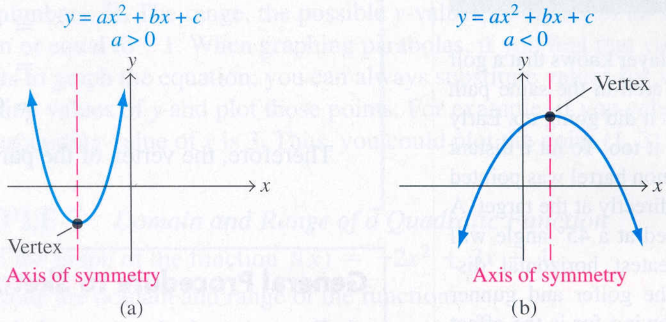


Figure 6.35

The *vertex* of a parabola is the lowest point on a parabola that opens upward and the highest point on a parabola that opens downward. Every parabola is *symmetric* with respect to a vertical line through its vertex. This line is called the *axis of symmetry* of the parabola. The x-coordinate of the vertex and the equation of the axis of symmetry can be found by using the following equation.

Axis of Symmetry of a Parabola

$$x = \frac{-b}{2a}$$

Once the x-coordinate of the vertex has been determined, the y-coordinate can be found by substituting the value found for the x-coordinate into the quadratic equation and evaluating the equation. This procedure is illustrated in Example 5.

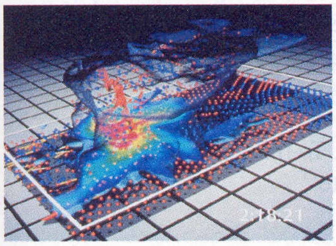
EXAMPLE 5 *Describing the Graph of a Quadratic Equation*

Consider the equation $y = -2x^2 + 8x + 1$.

a) Determine whether the graph will be a parabola that opens upward or downward.

DID YOU KNOW

Computing Reality with Mathematical Models



It is a relatively easy matter for scientists and mathematicians to describe and predict a simple motion like that of a falling object. When the phenomenon is complicated, such as making an accurate prediction of the weather, the mathematics becomes much more difficult. The National Weather Service has devised an algorithm that takes the temperature, pressure, moisture content, and wind velocity of more than 250,000 points in Earth's atmosphere and applies a set of equations that, they believe, will reasonably predict what will happen at each point over time. Researchers at the National Center for Supercomputing Applications are working on a computer model that simulates thunderstorms. Researchers want to know why some thunderstorms can turn severe, even deadly.

DID YOU KNOW

Gravity and the Parabola

Any golf player knows that a golf ball will arch in the same path going down as it did going up. Early gunners knew it too. To hit a distant target, the cannon barrel was pointed skyward, not directly at the target. A cannonball fired at a 45° angle will travel the greatest horizontal distance. What the golfer and gunner alike were allowing for is the effect of gravity on a projectile. Projectile motion follows a parabolic path. Galileo was neither a gunner nor a golfer, but he gave us the formula that effectively describes that motion and the distance traveled by an object if it is projected at a specific angle with a specific initial velocity.

b) Determine the equation of the axis of symmetry of the parabola.

c) Determine the vertex of the parabola.

SOLUTION:

a) Since $a = -2$, which is less than 0, the parabola opens downward.

b) To find the axis of symmetry, we use the equation $x = -\frac{b}{2a}$. In the equation $y = -2x^2 + 8x + 1$, $a = -2$, $b = 8$, and $c = 1$, so

$$x = \frac{-b}{2a} = \frac{-(8)}{2(-2)} = \frac{-8}{-4} = 2$$

The equation of the axis of symmetry is $x = 2$.

c) The x-coordinate of the vertex is 2, from part (b). To find the y-coordinate, we substitute 2 for x in the equation and then evaluate.

$$\begin{aligned} y &= -2x^2 + 8x + 1 \\ &= -2(2)^2 + 8(2) + 1 \\ &= -2(4) + 16 + 1 \\ &= -8 + 16 + 1 \\ &= 9 \end{aligned}$$

Therefore, the vertex of the parabola is located at the point (2, 9) on the graph.

General Procedure to Sketch the Graph of a Quadratic Equation

1. Determine whether the parabola opens upward or downward.
2. Determine the equation of the axis of symmetry.
3. Determine the vertex of the parabola.
4. Determine the y-intercept by substituting $x = 0$ into the equation.
5. Determine the x-intercepts (if they exist) by substituting $y = 0$ into the equation and solving for x.
6. Draw the graph, making use of the information gained in steps 1 through 5. Remember the parabola will be symmetric with respect to the axis of symmetry.

In step 5, to determine the x-intercepts, you may use either factoring or the quadratic formula.

EXAMPLE 6 *Graphing a Quadratic Equation*

Sketch the graph of the equation $y = x^2 - 6x + 8$.

SOLUTION: We follow the steps outlined in the general procedure.

1. Since $a = 1$, which is greater than 0, the parabola opens upward.
2. Axis of symmetry: $x = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$

   Thus, the axis of symmetry is $x = 3$.
3. y-coordinate of vertex: $y = x^2 - 6x + 8$

   $y = (3)^2 - 6(3) + 8 = 9 - 18 + 8 = -1$

   Thus, the vertex is at $(3, -1)$.

4. y-intercept: $y = x^2 - 6x + 8$

$$y = 0^2 - 6(0) + 8 = 8$$

Thus, the y-intercept is at (0, 8).

5. x-intercepts: $0 = x^2 - 6x + 8$, or $x^2 - 6x + 8 = 0$

We can solve this equation by factoring.

$$x^2 - 6x + 8 = 0$$
$$(x - 4)(x - 2) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 4 \qquad\qquad x = 2$$

Thus, the x-intercepts are (4, 0) and (2, 0).

6. Plot the vertex $(3, -1)$, the y-intercept (0, 8), and the x-intercepts (4, 0) and (2, 0). Then sketch the graph (Fig. 6.36).

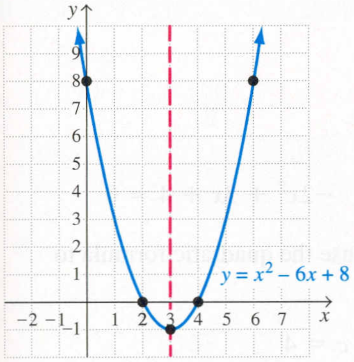


Figure 6.36

Note that the domain of the graph in Example 6, the possible x-values, is the set of all real numbers, $\mathbb{R}$. The range, the possible y-values, is the set of all real numbers greater than or equal to -1. When graphing parabolas, if you feel that you need additional points to graph the equation, you can always substitute values for x and find the corresponding values of y and plot those points. For example, if you substituted 1 for x, the corresponding value of y is 3. Thus, you could plot the point (1, 3).

EXAMPLE 7 *Domain and Range of a Quadratic Function*

a) Sketch the graph of the function $f(x) = -2x^2 + 3x + 4$.

b) Determine the domain and range of the function.

SOLUTION:

a) Since $f(x)$ means y, we can replace $f(x)$ with y to obtain $y = -2x^2 + 3x + 4$. Now graph $y = -2x^2 + 3x + 4$ using the steps outlined in the general procedure.

1. Since $a = -2$, which is less than 0, the parabola opens downward.

2. Axis of symmetry: $x = \dfrac{-b}{2a} = \dfrac{-(3)}{2(-2)} = \dfrac{-3}{-4} = \dfrac{3}{4}$

Thus, the axis of symmetry is $x = \dfrac{3}{4}$.

3. y-coordinate of vertex:

$$\begin{aligned} y &= -2x^2 + 3x + 4 \\ &= -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) + 4 \\ &= -2\left(\frac{9}{16}\right) + \frac{9}{4} + 4 \\ &= -\frac{9}{8} + \frac{9}{4} + 4 \\ &= -\frac{9}{8} + \frac{18}{8} + \frac{32}{8} = \frac{41}{8} \quad \text{or} \quad 5\frac{1}{8} \end{aligned}$$

Thus, the vertex is at $(\frac{3}{4}, 5\frac{1}{8})$.

4. y-intercept: $y = -2x^2 + 3x + 4$

$$= -2(0)^2 + 3(0) + 4 = 4$$

Thus, the y-intercept is (0, 4).

5. x-intercepts: $y = -2x^2 + 3x + 4$

$$0 = -2x^2 + 3x + 4 \quad \text{or} \quad -2x^2 + 3x + 4 = 0$$

This equation cannot be factored, so we will use the quadratic formula to solve it.

$$a = -2, \qquad b = 3, \qquad c = 4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{3^2 - 4(-2)(4)}}{2(-2)}$$

$$= \frac{-3 \pm \sqrt{9 + 32}}{-4}$$

$$= \frac{-3 \pm \sqrt{41}}{-4}$$

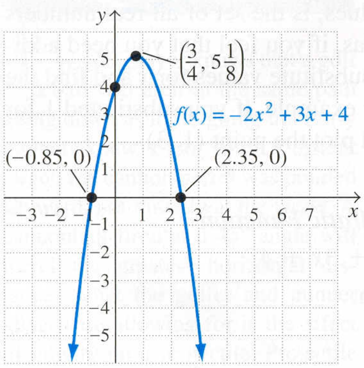


Figure 6.37

Since $\sqrt{41} \approx 6.4$,

$$x \approx \frac{-3 + 6.4}{-4} \approx \frac{3.4}{-4} \approx -0.85 \quad \text{or} \quad x \approx \frac{-3 - 6.4}{-4} \approx \frac{-9.4}{-4} \approx 2.35$$

Thus, the x-intercepts are $(-0.85, 0)$ and $(2.35, 0)$.

6. Plot the vertex $(\frac{3}{4}, 5\frac{1}{8})$, the y-intercept (0, 4), and the x-intercepts $(-0.85, 0)$ and (2.35, 0). Then sketch the graph (Fig. 6.37).

b) The domain, the values that can be used for x, is the set of all real numbers, $\mathbb{R}$. The range, the values of y, is $y \leq 5\frac{1}{8}$.

When we use the quadratic formula to find the x-intercepts of a graph, if the radicand, $b^2 - 4ac$, is a negative number, the graph has no x-intercepts. The graph will lie totally above or below the x-axis.

Exponential Functions

In Section 6.3, we discussed exponential equations. Recall that exponential equations are of the form $y = a^x$, $a > 0$, $a \neq 1$. The graph of every exponential equation will pass the vertical line test, and so every exponential equation is also an exponential function. Exponential functions may be written as $f(x) = a^x$, $a > 0$, $a \neq 1$.

In Section 6.3, we also introduced the natural exponential formula $P = P_0e^{kt}$. We can write this formula in function notation as $P(t) = P_0e^{kt}$. This expression is referred to as the ***natural exponential function***. In Example 8, we use the natural exponential function.

EXAMPLE 8 *Evaluating an Exponential Decay Function*

The power supply of a satellite is a radioisotope. The power output, p, in watts remaining in the power supply is a function of the time the satellite is in space. If there are originally 100 grams of the radioisotope, the power remaining after t days is $p(t) = 100e^{-0.001t}$. What will be the remaining power after 1 year (or 365 days) in space?

SOLUTION: Substitute 365 days for t in the function, and then evaluate using a calculator as described in Section 6.3.

$$\begin{aligned} p(t) &= 100e^{-0.001t} \\ p(365) &= 100e^{-0.001(365)} \\ &= 100e^{-0.365} \\ &\approx 100(0.6941966509) \\ &\approx 69.4 \text{ watts} \end{aligned}$$

Thus, after 365 days, the power remaining will be about 69.4 watts. ▲

EXAMPLE 9 *Evaluating an Exponential Decay Function*

Carbon 14 is used by scientists to find the age of fossils and other artifacts. If an object originally had 25 grams of carbon 14, the amount present after t years is $f(t) = 25e^{-0.00012010t}$. How much carbon 14 will be found after 350 years?

SOLUTION: Substitute 350 for t in the function, and then evaluate using a calculator as described in Section 6.3.

$$\begin{aligned} f(t) &= 25e^{-0.00012010t} \\ f(350) &= 25e^{-0.00012010(350)} \\ &= 25e^{-0.042035} \\ &\approx 25(0.9588362207) \\ &\approx 23.97090552 \\ &\approx 24 \text{ grams} \end{aligned}$$

Thus, after 350 years, about 24 grams of carbon 14 will remain. ▲

Graphs of Exponential Functions

What does the graph of an ***exponential function*** of the form $y = a^x$, $a > 0$, $a \neq 1$, look like? Examples 10 and 11 illustrate graphs of exponential functions.

EXAMPLE 10 *Graphing an Exponential Function, $a > 1$*

a) Graph $y = 2^x$.

b) Determine the domain and range of the function.

DID YOU KNOW

Population Growth

Population growth during certain time periods can be described by an exponential function. Whether it is a population of bacteria, fish, flowers, or people, the same general trend emerges: a period of rapid (exponential) growth, which is then followed by a leveling-off period.

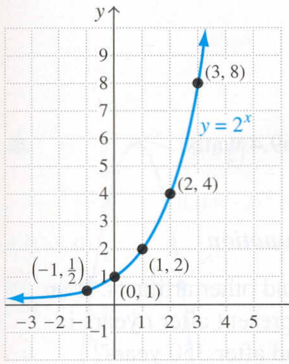


Figure 6.38

SOLUTION:

a) Substitute values for x and find the corresponding values of y. The graph is shown in Fig. 6.38.

$$y = 2^x$$

$$x = -3, \qquad y = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

$$x = -2, \qquad y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$x = -1, \qquad y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$

$$x = 0, \qquad y = 2^0 = 1$$

$$x = 1, \qquad y = 2^1 = 2$$

$$x = 2, \qquad y = 2^2 = 4$$

$$x = 3, \qquad y = 2^3 = 8$$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

b) The domain is all real numbers, $\mathbb{R}$. The range is $y > 0$. Note that y can never have a value of 0.

All exponential functions of the form $y = a^x$, $a > 1$, will have the general shape of the graph illustrated in Fig. 6.38. Since $f(x)$ is the same as y, the graphs of functions of the form $f(x) = a^x$, $a > 1$, will also have the general shape of the graph illustrated in Fig. 6.38. Can you now predict the shape of the graph of $y = e^x$? Remember: e has a value of about 2.7183.

EXAMPLE 11 *Graphing an Exponential Function,* $0 < a < 1$

a) Graph $y = (\frac{1}{2})^x$.

b) Determine the domain and range of the function.

SOLUTION:

a) We begin by substituting values for x and calculating values for y. We then plot the ordered pairs and use these points to sketch the graph. To evaluate a fraction with a negative exponent, we use the fact that

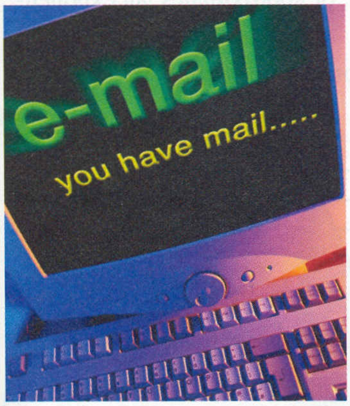
$$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$$

For example,

$$\left(\frac{1}{2}\right)^{-3} = \left(\frac{2}{1}\right)^3 = 8$$

DID YOU KNOW

The Electronic Superhighway



Electronic mail, or e-mail, is fast becoming one of the most popular and easiest methods of communicating. The number of e-mail messages is increasing exponentially. Although 45% of the U.S. population now uses e-mail, according to Jupiter Research, 93% of people online frequently use e-mail. Researchers expect this percentage to increase to 98% by 2007. Nearly one-quarter of e-mail users maintain three or more personal e-mail accounts. Thirty-five percent of e-mail messages received is considered "spam mail" or unsolicited e-mail, whereas e-mail from friends and family makes up 34% of messages received. E-mail is also becoming a valuable marketing tool. According to the Kelsey Group, annual spending by U.S. small business on localized e-mail marketing will exceed \$2.2 billion by 2005.

Then

$$y = \left(\frac{1}{2}\right)^x$$

$$x = -3, \quad y = \left(\frac{1}{2}\right)^{-3} = 2^3 = 8$$

$$x = -2, \quad y = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$$

$$x = -1, \quad y = \left(\frac{1}{2}\right)^{-1} = 2^1 = 2$$

$$x = 0, \quad y = \left(\frac{1}{2}\right)^{0} = 1$$

$$x = 1, \quad y = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$$

$$x = 2, \quad y = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$$

$$x = 3, \quad y = \left(\frac{1}{2}\right)^{3} = \frac{1}{8}$$

x	y
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

The graph is illustrated in Fig. 6.39.

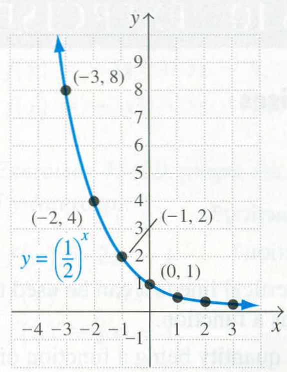


Figure 6.39

b) The domain is the set of all real numbers, $\mathbb{R}$. The range is $y > 0$. ▲

All exponential functions of the form $y = a^x$ or $f(x) = a^x$, $0 < a < 1$, will have the general shape of the graph illustrated in Fig. 6.39.

EXAMPLE 12 ***Is the Growth Exponential?***

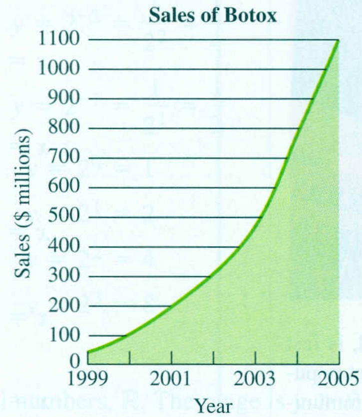
Sales of Botox, a drug injected under the skin to smooth out wrinkles, have increased tremendously since the late 1990s. The graph on page 372 shows the sales of Botox from 1999 through 2001 and projected though 2005.

a) Does the graph approximate the graph of an exponential function?
b) Estimate the sales of Botox in 2004.

SOLUTION:

a) Yes, the graph has the approximate shape of an exponential function. A function that increases rapidly with this general shape, is, or approximates, an exponential function.

b) From the graph, we see that in 2004, sales of Botox were about $800 million.



Source: American Society of Plastic Surgeons, Allergan

SECTION 6.10 EXERCISES

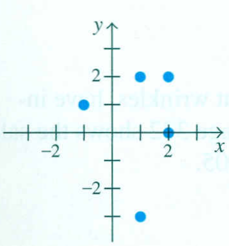
Concept/Writing Exercises

1. What is a function?
2. What is a relation?
3. What is the domain of a function?
4. What is the range of a function?
5. Explain how and why the vertical line test can be used to determine whether a graph is a function.
6. Give three examples of one quantity being a function of another quantity.

Practice the Skills

In Exercises 7–24, determine whether the graph represents a function. If it does represent a function, give its domain and range.

7.



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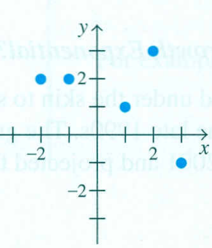

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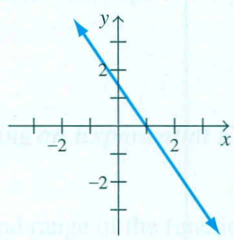

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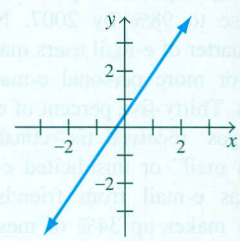

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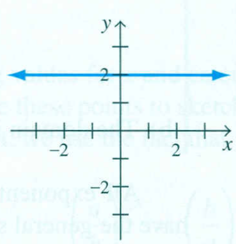

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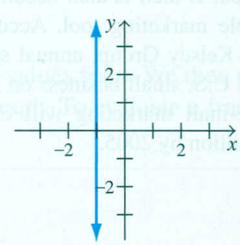

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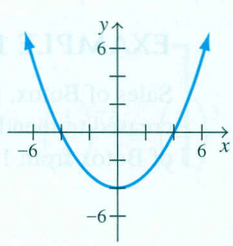

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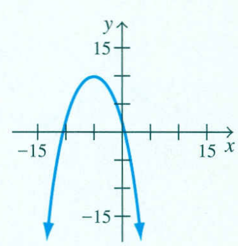

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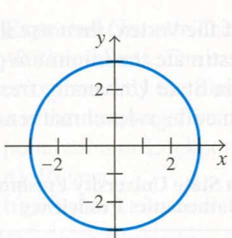

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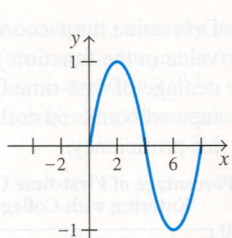

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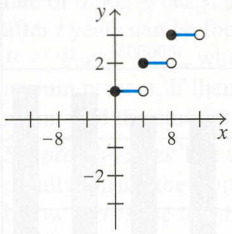

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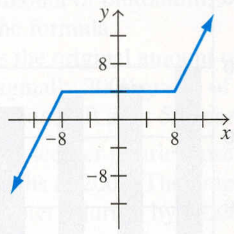

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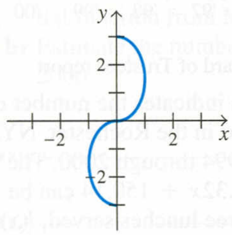

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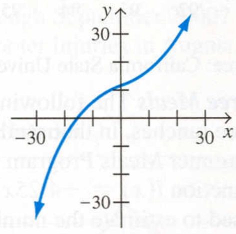

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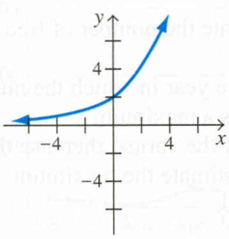

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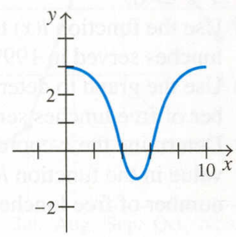

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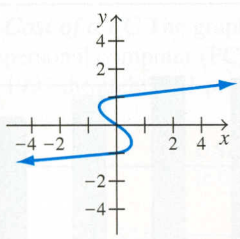

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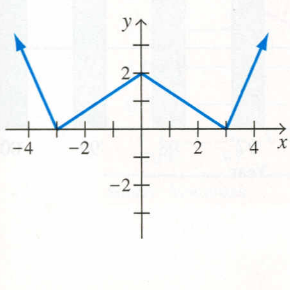


In Exercises 25–30, determine whether the set of ordered pairs is a function.

25. $\{(2, 9), (3, 6), (7, 11), (9, 15)\}$

26. $\{(1, -3)(2, -5), (3, -7), (4, -9)\}$

27. $\{(4, 4), (5, 1), (4, 0)\}$

28. $\{(3, 1), (3, 3), (3, 5)\}$

29. $\{(7, 1), (6, 1), (5, 1)\}$

30. $\{(1, 7), (1, 6), (1, 5)\}$

In Exercises 31–44, evaluate the function for the given value of x.

31. $f(x) = x + 3, \quad x = 2$

32. $f(x) = 2x + 5, \quad x = 4$

33. $f(x) = -2x - 7, \quad x = -4$

34. $f(x) = 5x + 3, \quad x = -1$

35. $f(x) = 10x - 6, \quad x = 0$

36. $f(x) = 7x - 6, \quad x = 4$

37. $f(x) = x^2 - 3x + 1, \quad x = 4$

38. $f(x) = x^2 - 5, \quad x = 7$

39. $f(x) = 2x^2 - 2x - 8, \quad x = -2$

40. $f(x) = -x^2 + 3x + 7, \quad x = 3$

41. $f(x) = -3x^2 + 5x + 4, \quad x = -3$

42. $f(x) = 5x^2 + 2x + 5, \quad x = 4$

43. $f(x) = -5x^2 + 3x - 9, \quad x = -1$

44. $f(x) = -3x^2 - 6x + 10, \quad x = -2$

In Exercises 45–50, graph the function by using the slope and y-intercept.

45. $f(x) = 2x - 1$ **46.** $f(x) = -x + 3$

47. $f(x) = -4x + 2$ **48.** $f(x) = 2x + 5$

49. $f(x) = \frac{3}{2}x - 1$ **50.** $f(x) = -\frac{1}{2}x + 3$

In Exercises 51–66,

a) *determine whether the parabola will open upward or downward.*

b) *find the equation of the axis of symmetry.*

c) *find the vertex.*

d) *find the y-intercept.*

e) *find the x-intercepts if they exist.*

f) *sketch the graph.*

g) *find the domain and range of the function.*

51. $y = x^2 - 16$ **52.** $y = x^2 - 9$

53. $y = -x^2 + 4$ **54.** $y = -x^2 + 16$

55. $f(x) = -x^2 - 4$ **56.** $y = -2x^2 - 8$

57. $y = 2x^2 - 3$ **58.** $f(x) = -3x^2 - 6$

59. $f(x) = x^2 + 2x + 6$ **60.** $y = x^2 - 8x + 1$

61. $y = x^2 + 5x + 6$ **62.** $y = x^2 - 7x - 8$

63. $y = -x^2 + 4x - 6$ **64.** $y = -x^2 + 8x - 8$

65. $y = -3x^2 + 14x - 8$ **66.** $y = 2x^2 - x - 6$

In Exercises 67–78, draw the graph of the function and state the domain and range.

67. $y = 3^x$ **68.** $f(x) = 4^x$

69. $y = (\frac{1}{3})^x$ **70.** $y = (\frac{1}{4})^x$

71. $f(x) = 2^x + 1$ **72.** $y = 3^x - 1$

73. $y = 4^x + 1$ **74.** $y = 2^x - 1$

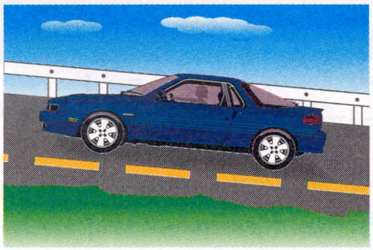
75. $y = 3^{x-1}$ **76.** $y = 3^{x+1}$

77. $f(x) = 4^{x+1}$ **78.** $y = 4^{x-1}$

Problem Solving

79. ***Monthly Salary*** Chet Rogalski is part owner of a newly opened hardware store. Chet's monthly salary is given by the function $m(s) = 300 + 0.10s$, where s is the store's monthly sales in dollars. If sales for the month of July are \$20,000, determine Chet's monthly salary for July.

80. ***Finding Distances*** The distance a car travels, $d(t)$, at a constant 60 mph is given by the function $d(t) = 60t$, where t is the time in hours. Find the distance traveled in



a) 3 hours.
b) 7 hours.

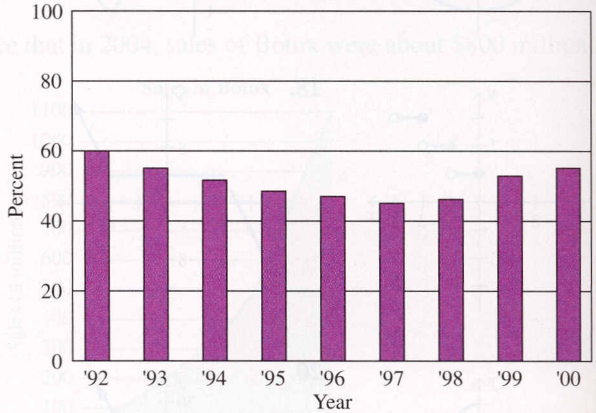
81. The following graph indicates the percent of first-time California State University freshmen who entered college with college-level mathematics proficiency for the years 1992 through 2001. The function $f(x) = 0.56x^2 - 5.43x + 59.83$ can be used to estimate the percent of first-time California State University freshmen who entered college with college-level mathematics proficiency, $f(x)$, where x is the number of years since 1992 and $0 \leq x \leq 9$.

a) Use the function $f(x)$ to estimate the percent of first-time California State University freshmen who entered college with college-level mathematics proficiency in 2000.

b) Use the graph to determine the year in which the percent of first-time California State University freshmen who entered college with college-level mathematics proficiency was a minimum.

c) Determine the x-coordinate of the vertex, then use this value in the function $f(x)$ to estimate the minimum percentage of first-time California State University freshmen who entered college with college-level mathematics proficiency.

Percentage of First-time California State University Freshmen Entering with College-level Mathematics Proficiency



Source: California State University Board of Trustees report

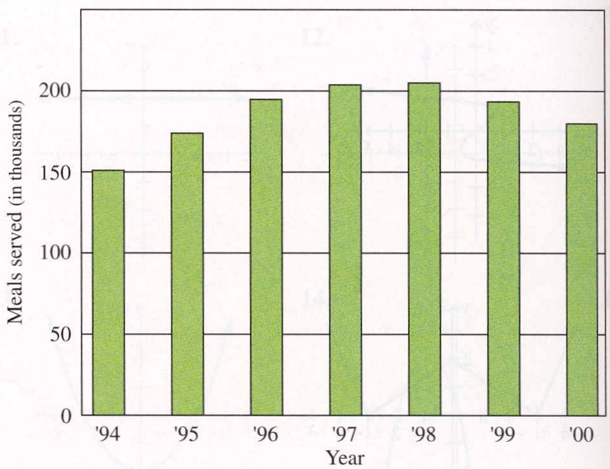
82. ***Free Meals*** The following graph indicates the number of free lunches, in thousands, served in the Rochester, NY, Summer Meals Program from 1994 through 2000. The function $l(x) = -4.25x^2 + 30.32x + 150.14$ can be used to estimate the number of free lunches served, $l(x)$, where x is the number of years since 1994 and $0 \leq x \leq 6$.

a) Use the function $l(x)$ to estimate the number of free lunches served in 1999.

b) Use the graph to determine the year in which the number of free lunches served was a maximum.

c) Determine the x-coordinate of the vertex, then use this value in the function $l(x)$ to estimate the maximum number of free lunches served.

Free Lunches



Source: YWCA

83. ***Expected Growth*** The town of Lockport currently has 4000 residents. The expected future population can be approximated by the function $P(x) = 4000(1.3)^{0.1x}$, where x is the number of years in the future. Find the expected population of Lockport in

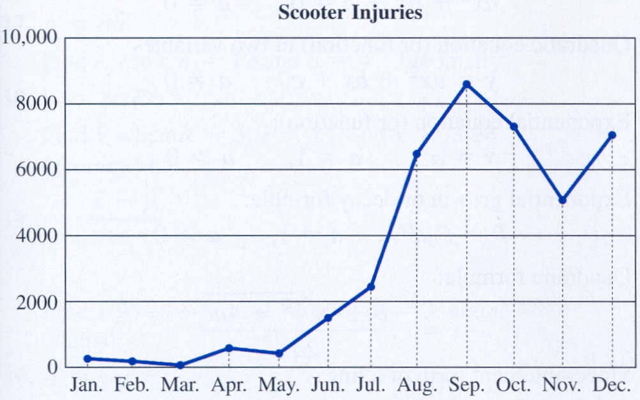
a) 10 years.
b) 50 years.

84. ***Decay of Plutonium*** Plutonium, a radioactive material used in most nuclear reactors, decays exponentially at a rate of 0.003% per year. The amount of plutonium, P, left after t years can be found by the formula $P = P_0e^{-0.00003t}$, where P_0 is the original amount of plutonium present. If there are originally 2000 grams of plutonium, find the amount of plutonium left after 50 years.

85. ***Scooter Injuries*** The number of scooter injuries rose rapidly during the summer months in 2000. The graph below shows the number of scooter injuries, by month, in 2000.

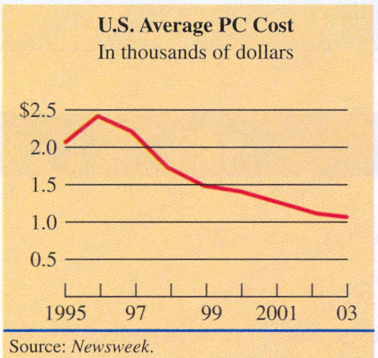
a) Does the graph approximate the graph of an exponential function from May through September 2000?
b) Estimate the number of scooter injuries in August 2000.

Scooter Injuries



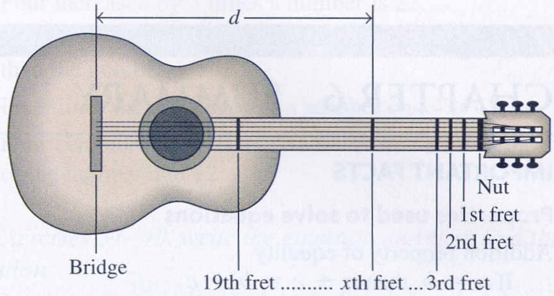
Source: Consumer Product Safety Commission

86. ***Cost of a PC*** The graph shows the U.S. average cost of a personal computer (PC), in thousands of dollars, from 1995 through 2001 projected through 2003.



a) From 1995 through 2001 projected through 2003, does the graph approximate the graph of an exponential function? Explain.
b) Estimate the U.S. average cost of a PC in 2002.

87. The spacing of the frets on the neck of a classical guitar is determined from the equation $d = (21.9)(2)^{(20-x)/12}$, where x = the fret number and d = the distance in centimeters of the xth fret from the bridge.



a) Determine how far the 19th fret should be from the bridge (rounded to one decimal place).
b) Determine how far the 4th fret should be from the bridge (rounded to one decimal place).
c) The distance of the nut from the bridge can be found by letting $x = 0$ in the given exponential equation. Find the distance from the nut to the bridge (rounded to one decimal place).

Challenge Problems/Group Activities

88. ***Appreciation of a House*** A house initially cost \$85,000. The value, V, of the house after n years if it appreciates at a constant rate of 4% per year can be determined by the function $V = f(n) = \$85{,}000(1.04)^n$.

a) Determine $f(8)$ and explain its meaning.
b) After how many years is the value of the house greater than \$153,000? (Find by trial and error.)

89. ***Target Heart Rate*** While exercising, a person's recommended target heart rate is a function of age. The recommended number of beats per minute, y, is given by the function $y = f(x) = -0.85x + 187$, where x represents a person's age in years. Determine the number of recommended heart beats per minute for the following ages and explain the results.

a) 20
b) 30
c) 50
d) 60
e) What is the age of a person with a recommended target heart rate of 85?

90. ***Speed of Light*** Light travels at about 186,000 miles per second through space. The distance, d, in miles that light travels in t seconds can be determined by the function $d(t) = 186{,}000t$.

a) Light reaches the moon from Earth in about 1.3 sec. Determine the approximate distance from Earth to the moon.
b) Express the distance in miles, d, traveled by light in t minutes as a function of time, t.
c) Light travels from the sun to Earth in about 8.3 min. Determine the approximate distance from the sun to Earth.

Internet/Research Activity

91. The idea of using variables in algebraic equations was introduced by the French mathematician François Viète (1540–1603). Write a paper about his life and his contributions to mathematics. In particular, discuss his work with algebra equations. (References include history of mathematics books, encyclopedias, and the Internet.)

CHAPTER 6 SUMMARY

IMPORTANT FACTS

Properties used to solve equations

Addition property of equality
If $a = b$, then $a + c = b + c$

Subtraction property of equality
If $a = b$, then $a - c = b - c$

Multiplication property of equality
If $a = b$, then $ac = bc$

Division property of equality
If $a = b$, then $a/c = b/c$, $c \neq 0$

Variation

Direct: $y = kx$

Inverse: $y = \dfrac{k}{x}$

Joint: $y = kxz$

Inequality symbols

$a < b$ means that a is less than b.

$a \leq b$ means that a is less than or equal to b.

$a > b$ means that a is greater then b.

$a \geq b$ means that a is greater than or equal to b.

Intercepts

To find the x-intercept, set $y = 0$ and solve the resulting equation for x.

To find the y-intercept, set $x = 0$ and solve the resulting equation for y.

Slope

Slope (m): $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Equations and formulas

Linear equation in two variables:
$$ax + by = c, \quad a \neq 0, \quad b \neq 0$$

Quadratic equation in one variable:
$$ax^2 + bx + c = 0, \quad a \neq 0$$

Quadratic equation (or function) in two variables:
$$y = ax^2 + bx + c, \quad a \neq 0$$

Exponential equation (or function):
$$y = a^x, \quad a \neq 1, \quad a > 0$$

Exponential growth or decay formula:
$$P = P_0a^{kt}, \quad a \neq 1, \quad a > 0$$

Quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Slope–intercept form of a line:
$$y = mx + b$$

Axis of symmetry of a parabola:
$$x = \frac{-b}{2a}$$

Zero-factor property

If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

CHAPTER 6 REVIEW EXERCISES

6.1

In Exercises 1–6, evaluate the expression for the given value(s) of the variable.

1. $x^2 + 12, \quad x = 3$
2. $-x^2 - 9, \quad x = -1$
3. $4x^2 - 2x + 5, \quad x = 2$
4. $-x^2 + 7x - 3, \quad x = \frac{1}{2}$
5. $4x^3 - 7x^2 + 3x + 1, \quad x = -2$
6. $3x^2 - xy + 2y^2, \quad x = 1, \quad y = -2$

6.2

In Exercises 7–9, combine like terms.

7. $3x - 4 + x + 5$ **8.** $3x + 4(x - 2) + 6x$

9. $4(x - 1) + \frac{1}{3}(9x + 3)$

In Exercises 10–14, solve the equation for the given variable.

10. $4s + 10 = -30$ **11.** $3t + 8 = 6t - 13$

12. $\dfrac{x+5}{6} = \dfrac{x-3}{3}$ **13.** $4(x - 2) = 3 + 5(x + 4)$

14. $\dfrac{x}{4} + \dfrac{3}{5} = 7$

15. ***Making Oatmeal*** A recipe for Hot Oats Cereal calls for 2 cups of water and for $\frac{1}{3}$ cup of dry oats. How many cups of dry oats would be used with 3 cups of water?

16. ***Laying Blocks*** A mason lays 120 blocks in 1 hr 40 min. How long will it take her to lay 300 blocks?

6.3

In Exercises 17–20, use the formula to find the value of the indicated variable for the values given.

17. $A = bh$
Find A when $b = 12$ and $h = 4$ (geometry).

18. $V = 2\pi R^2 r^2$
Find V when $R = 3$, $r = 1\frac{3}{4}$, and $\pi = 3.14$ (geometry).

19. $z = \dfrac{\bar{x} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$
Find $\bar{x}$ when $z = 2$, $\mu = 100$, $\sigma = 3$, and $n = 16$ (statistics).

20. $K = \frac{1}{2}mv^2$
Find m when $v = 30$ and $k = 4500$ (physics).

In Exercises 21–24, solve for y.

21. $3x - 9y = 18$ **22.** $2x + 5y = 12$

23. $2x - 3y + 52 = 30$ **24.** $-3x - 4y + 5z = 4$

In Exercises 25–28, solve for the variable indicated.

25. $A = lw$, for w

26. $P = 2l + 2w$, for w

27. $L = 2(wh + lh)$, for l

28. $a_n = a_1 + (n - 1)d$, for d

6.4

In Exercises 29–32, write the phrase in mathematical terms.

29. 8 increased by 2 times x

30. 3 times y decreased by 7

31. 10 increased by 3 times r

32. The difference between 8 divided by q and 11

In Exercises 33–36, write an equation that can be used to solve the problems. Solve the equation and find the desired value(s).

33. Four increased by 3 times a number is 22.

34. The product of 3 and a number, increased by 8, is 6 less than the number.

35. Five times the difference of a number and 4 is 45.

36. Fourteen more than 10 times a number is 8 times the sum of the number and 12.

In Exercises 37–40, write the equation and then find the solution.

37. **MODELING -** ***Investing*** Jim Lawton received an inheritance of \$15,000. If he wants to invest twice as much money in mutual funds as in bonds, how much should he invest in mutual funds?

38. **MODELING -** ***Lawn Chairs*** Larry's Lawn Chair Company has fixed costs of \$15,000 per month and variable costs of \$9.50 per lawn chair manufactured. The company has \$95,000 available to meet its total monthly expenditures. What is the maximum number of lawn chairs the company can manufacture in a month? (Fixed costs, such as rent and insurance, are those that occur regardless of the level of production. Variable costs, such as those for materials, depend on the level of production.)

39. **MODELING -** ***Zoo Animals*** According to the 2003 *World Almanac*, the number of species at the San Diego Zoo is 140 more than two times the number of species at the Philadelphia Zoo. The sum of the species at the San Diego Zoo and the Philadelphia Zoo is 1130. Determine the number of species at each zoo.

40. **MODELING -** ***Restaurant Profit*** John Smith owns two restaurants. His profit for a year at restaurant A is \$12,000 greater than his profit at restaurant B. The total profit from both restaurants is \$68,000. Determine the profit at each restaurant.

6.5

In Exercises 41–44, find the quantity indicated.

41. s is inversely proportional to t. If $s = 10$ when $t = 3$, find s when $t = 5$.

42. J is directly proportional to the square of A. If $J = 32$ when $A = 4$, find J when $A = 7$.

43. W is directly proportional to L and inversely proportional to A. If $W = 80$ when $L = 100$ and $A = 20$, find W when $L = 50$ and $A = 40$.

44. z is jointly proportional to x and y and inversely proportional to the square of r. If $z = 12$ when $x = 20$, $y = 8$, and $r = 8$, find z when $x = 10$, $y = 80$, and $r = 3$.

45. ***Buying Fertilizer***
a) A 30 lb bag of fertilizer will cover an area of 2500 ft^2. How many pounds of fertilizer are needed to cover an area of 12,500 ft^2?
b) How many bags of fertilizer are needed?

46. ***Map Reading*** The scale of a map is 1 in. to 30 mi. What distance on the map represents 120 mi?

47. ***Electric Bill*** An electric company charges \$0.162 per kilowatt-hour (kWh). What is the electric bill if 740 kWh are used in a month?

48. ***A Falling Object*** The distance, d, an object drops in free fall is directly proportional to the square of the time, t. If an object falls 16 ft in 1 sec, how far will an object fall in 5 sec?

6.6

In Exercises 49–52, graph the solution set for the set of real numbers.

49. $5 + 9x \leq 7x - 7$ **50.** $2x + 8 \geq 4x + 10$

51. $3(x + 9) \leq 4x + 11$ **52.** $-3 \leq x + 1 < 7$

In Exercises 53–56, graph the solution set for the set of integers.

53. $2 + 5x > -8$ **54.** $5x + 13 \geq -22$

55. $-1 < x \leq 9$ **56.** $-8 \leq x + 2 \leq 7$

6.7

In Exercises 57–60, graph the ordered pair in the Cartesian coordinate system.

57. $(1, 4)$ **58.** $(-2, 5)$ **59.** $(-4, 3)$ **60.** $(5, -3)$

In Exercises 61 and 62, points A, B, and C are vertices of a rectangle. Plot the points. Find the coordinates of the fourth point, D, to complete the rectangle. Find the area of the rectangle.

61. $A(-3, 3)$, $B(2, 3)$, $C(2, -1)$

62. $A(-3, 1)$, $B(-3, -2)$, $C(4, -2)$

In Exercises 63–66, graph the equation by plotting points.

63. $x - y = 4$ **64.** $2x + 3y = 12$

65. $x = y$ **66.** $x = 3$

In Exercises 67–70, graph the equation, using the x- and y-intercepts.

67. $x - 2y = 6$ **68.** $x + 3y = 6$

69. $4x - 3y = 12$ **70.** $2x + 3y = 9$

In Exercises 71–74, find the slope of the line through the given points.

71. $(1, 3)$, $(6, 5)$ **72.** $(3, -1)$, $(5, -4)$

73. $(-1, -4)$, $(2, 3)$ **74.** $(6, 2)$, $(6, -2)$

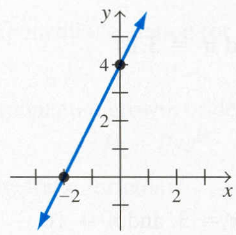
In Exercises 75–78, graph the equation by plotting the y-intercept and then plotting a second point by making use of the slope.

75. $y = 2x - 5$ **76.** $2y - 4 = 3x$

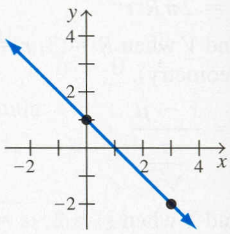
77. $2y + x = 8$ **78.** $y = -x - 1$

In Exercises 79 and 80, determine the equation of the graph.

79.



80.



81. ***Disability Income*** The monthly disability income, I, that Nadja Muhidin receives is $I = 460 - 0.5m$, where m is her monthly earnings for her part-time job for the previous month.
a) Draw a graph of disability income versus earnings for earnings up to and including \$920.
b) If Nadja earns \$600 in January, how much disability income will she receive in February?
c) If she received \$380 disability income in November, how much did she earn in October?

82. ***Business Space Rental*** The monthly rental cost, C, in dollars, for space in the Galleria Mall can be approximated by the equation $C = 1.70A + 3000$, where A is the area, in square feet, of space rented.
a) Draw a graph of monthly rental cost versus square feet for up to and including 12,000 ft^2.
b) Determine the monthly rental cost if 2000 ft^2 are rented.
c) If the rental cost is \$10,000 per month, how many square feet are rented?

6.8

In Exercises 83–86, graph the inequality.

83. $4x + 3y \le 12$

84. $3x + 2y \ge 12$

85. $2x - 3y > 12$

86. $-7x - 2y < 14$

6.9

In Exercises 87–92, factor the trinomial. If the trinomial cannot be factored, so state.

87. $x^2 + 9x + 18$

88. $x^2 + x - 20$

89. $x^2 - 10x + 24$

90. $x^2 - 9x + 20$

91. $6x^2 + 7x - 3$

92. $2x^2 + 13x - 7$

In Exercises 93–96, solve the equation by factoring.

93. $x^2 + 3x + 2 = 0$

94. $x^2 - 5x = -4$

95. $3x^2 - 17x + 10 = 0$

96. $3x^2 = -7x - 2$

In Exercises 97–100, solve the equation, using the quadratic formula. If the equation has no real solution, so state.

97. $x^2 - 4x - 1 = 0$

98. $x^2 - 3x + 2 = 0$

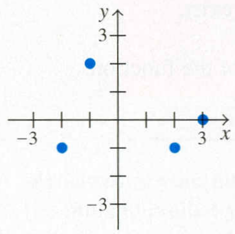
99. $2x^2 - 3x + 4 = 0$

100. $2x^2 - x - 3 = 0$

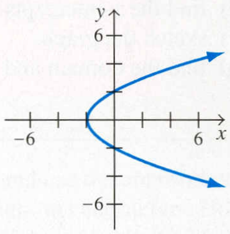
6.10

In Exercises 101–104, determine whether the graph represents a function. If it does represent a function, give its domain and range.

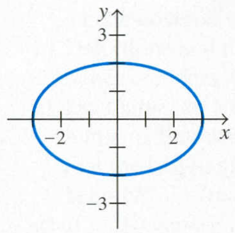
101.



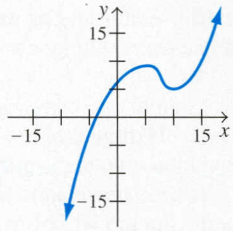
102.



103.



104.



In Exercises 105–108, evaluate f(x) for the given value of x.

105. $f(x) = 5x - 2,\quad x = 4$

106. $f(x) = -2x + 7,\quad x = -3$

107. $f(x) = 2x^2 - 3x + 4,\quad x = 5$

108. $f(x) = -4x^2 + 7x + 9,\quad x = 4$

In Exercises 109 and 110, for each function

a) *determine whether the parabola will open upward or downward.*
b) *find the equation of the axis of symmetry.*
c) *find the vertex.*
d) *find the y-intercept.*
e) *find the x-intercepts if they exist.*
f) *sketch the graph.*
g) *find the domain and range.*

109. $y = -x^2 - 4x + 21$

110. $f(x) = 2x^2 - 8x + 10$

In Exercises 111 and 112, draw the graph of the function and state the domain and range.

111. $y = 2^{2x}$

112. $y = (\frac{1}{2})^x$

6.2, 6.3, 6.10

113. ***Gas Mileage*** The gas mileage, m, of a specific car can be estimated by the equation (or function)

$$m = 30 - 0.002n^2, \qquad 20 \le n \le 80$$

where n is the speed of the car in miles per hour. Estimate the gas mileage when the car travels at 60 mph.

114. ***Auto Accidents*** The approximate number of accidents in one month, n, involving drivers between 16 and 30 years of age inclusive can be approximated by the equation

$$n = 2a^2 - 80a + 5000, \qquad 16 \le a \le 30$$

where a is the age of the driver. Approximate the number of accidents in one month that involved

a) 18-year-olds
b) 25-year-olds.

115. ***Filtered Light*** The percent of light filtering through Swan Lake, P, can be approximated by the function $P(x) = 100(0.92)^x$, where x is the depth in feet. Find the percent of light filtering through at a depth of 4.5 ft.

CHAPTER 6 TEST

1. Evaluate $3x^2 + 4x - 1$, when $x = -2$.

In Exercises 2 and 3, solve the equation.

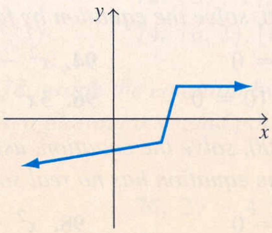
2. $3x + 5 = 2(4x - 7)$
3. $-2(x - 3) + 6x = 2x + 3(x - 4)$

In Exercises 4 and 5, write an equation to represent the problem. Then solve the equation.

4. The product of a number and 2, increased by 7 is 25.
5. **Buying a Car** The cost of a car including a 7% sales tax is \$26,750. Determine the cost of the car before tax.
6. Evaluate $L = ah + bh + ch$ when $a = 3, b = 4, c = 5$, and $h = 7$.
7. Solve $3x + 5y = 11$ for y.
8. L varies jointly as M and N and inversely as P. If $L = 12$ when $M = 8$, $N = 3$, and $P = 2$, find L when $M = 10$, $N = 5$, and $P = 15$.
9. For a constant area, the length, l, of a rectangle varies inversely as the width, w. If $l = 15$ ft when $w = 9$ ft, find the length of a rectangle with the same area if the width is 20 ft.
10. Graph the solution set of $-3x + 11 \le 5x + 35$ on the real number line.
11. Determine the slope of the line through the points $(-3, 5)$ and $(7, 12)$.

In Exercises 12 and 13, graph the equation.

12. $y = 2x - 4$
13. $2x - 3y = 15$
14. Graph the inequality $3y \ge 5x - 12$.
15. Solve the equation $x^2 - 3x = 28$ by factoring.
16. Solve the equation $3x^2 + 2x = 8$ by using the quadratic formula.
17. Determine whether the graph is a function. Explain your answer.



18. Evaluate $f(x) = -4x^2 - 11x + 5$ when $x = -2$.
19. For the equation $y = x^2 - 2x + 4$,
    a) determine whether the parabola will open upward or downward.
    b) find the equation of the axis of symmetry.
    c) find the vertex.
    d) find the y-intercept.
    e) find the x-intercepts if they exist.
    f) sketch the graph.
    g) find the domain and range of the function.

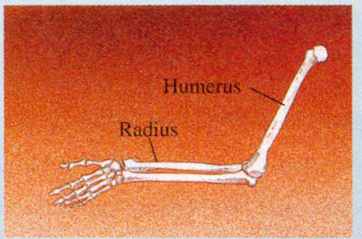
GROUP PROJECTS

Archeology: Gaining Information from Bones

1. Archeologists have developed formulas to predict the height and, in some cases, the age at death of the deceased by knowing the lengths of certain bones in the body. The long bones of the body grow at approximately the same rate. Thus, a linear relationship exists between the length of the bones and the person's height. If the length of one of these major bones—the femur (F), the tibia (T), the humerus (H), and the radius (R)— is known, the height, h, of a person can be calculated with one of the following formulas. The relationship between bone length and height is different for males and females.

Male	Female
$h = 2.24F + 69.09$	$h = 2.23F + 61.41$
$h = 2.39T + 81.68$	$h = 2.53T + 72.57$
$h = 2.97H + 73.57$	$h = 3.14H + 64.98$
$h = 3.65R + 80.41$	$h = 3.88R + 73.51$

All measurements are in centimeters.



a) Measure your humerus and use the appropriate formula to predict your height in centimeters. How close is this predicted height to your actual height? (The result is an approximation because measuring a bone covered with flesh and muscle is difficult.)

b) Determine and describe where the femur and tibia bones are located.

c) Dr. Juarez, an archeologist, had one female humerus that was 29.42 cm in length. He concluded that the height of the entire skeleton would have been 157.36 cm. Was his conclusion correct?

d) If a 21-year-old woman is 167.64 cm tall, about how long should her tibia be?

e) Sometimes the age of a person may be determined by using the fact that the height of a person, and the length of his or her long bones, decreases at the rate of 0.06 cm per year after the age of 30.
   i. At age 30, Jolene is 168 cm tall. Estimate the length of her humerus.
   ii. Estimate the length of Jolene's humerus when she is 60 years old.

f) Select six people of the same gender and measure their height and one of the bones for which an equation is given (the same bone on each person). Each measurement should be made to the nearest 0.5 cm. For each person, you will have two measurements, which can be considered an ordered pair (bone length, height). Plot the ordered pairs on a piece of graph paper, with the bone length on the horizontal axis and the height on the vertical axis. Start the scale on both axes at zero. Draw a straight line that you feel is the best approximation, or best fit, through these points. Determine where the line crosses the y-axis and the slope of the line. Your y-intercept and slope should be close to the values in the given equation for that bone. (Reference: M. Trotter and G. C. Gleser, "Estimation of Stature from Long Bones of American Whites and Negroes," *American Journal of Physical Anthropology*, 1952, 10:463–514.)

Graphing Calculator

2. The functions that we graphed in this chapter can be easily graphed with a graphing calculator (or grapher). If you do not have a graphing calculator, borrow one from your instructor or a friend.

a) Explain how you would set the domain and range. The calculator key to set the domain and range may be labeled *range* or *window*. Set the grapher with the following range or window settings:
   Xmin = -12, Xmax = 12, Xscl = 1,
   Ymin = -13, Ymax = 6, and Yscl = 1.

b) Explain how to enter a function in the graphing calculator. Enter the function $y = 3x^2 - 7x - 8$ in the calculator.

c) Graph the function you entered in part (b).

d) Learn how to use the *trace feature*. Then use it to estimate the x-intercepts. Record the estimated values for the x-intercepts.

e) Learn how to use the *zoom feature* to obtain a better approximation of the x-intercepts. Use the zoom feature twice and record the x-intercepts each time.