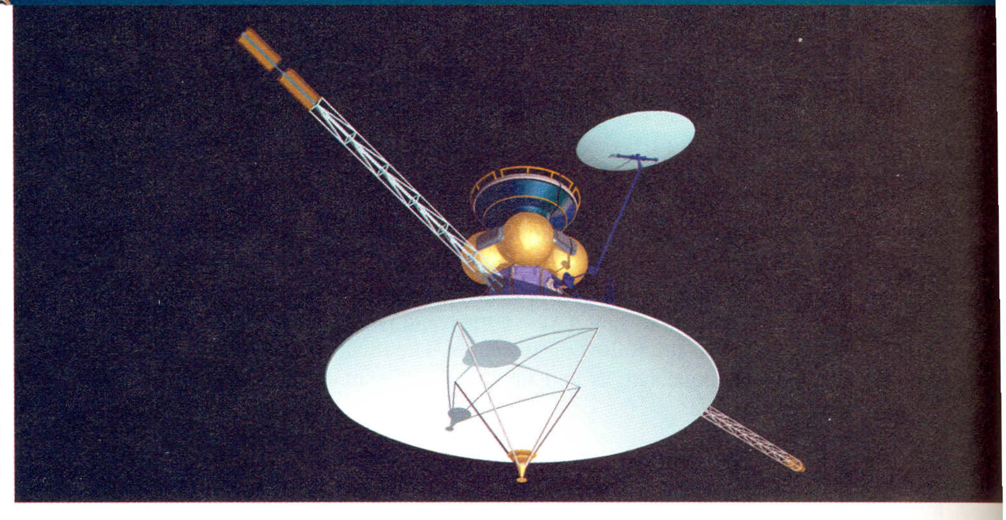
CHAPTER 7



Linear programming provides businesses and governments with a mathematical form of decision making that makes the most efficient use of time and resources. Telecommunications companies use it to route calls through satellites, like this one being tested, so that few of their customers will reach a "no circuits available" message.

SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

Imagine that you and a friend have a great idea for a new T-shirt. First you make a few to give away to friends, using your own money. Then other students see the shirt, and soon everyone on campus wants one. Suddenly, you have entered the T-shirt business. To make a profit in your business, you need to keep track of the cost of your materials, the quantity of shirts sold, and the price at which you sell them, a relatively straightforward calculation.

Now suppose you come up with three other designs, and you want to put them on sweatshirts as well as T-shirts, and you want to offer a variety of colors: black, white, blue, and maroon. The equation for finding the profitability of your venture becomes more complicated because there are more variables. To track your profits, you may need to develop and solve systems of equations.

For most business owners, numerous factors must be considered to determine not only whether the business is profitable, but also how much they should charge their customers, which production method is most efficient, what return they can expect by placing advertisements, and so on. Many small-business owners routinely make these calculations based on their own experience, mathematics, and sometimes computer programs. Larger companies often employ inventory analysts, quality control engineers, and efficiency experts to help them, along with computers, keep track of vast quantities of data.

7.1 SYSTEMS OF LINEAR EQUATIONS

In Chapter 6, we discussed linear equations in two variables. In algebra, it is often necessary to find the common solution to two or more such equations. We refer to the equations in this type of problem as a ***system of linear equations*** or as ***simultaneous linear equations***. A ***solution to a system of equations*** is the ordered pair or pairs that satisfy *all* equations in the system. A system of linear equations may have exactly one solution, no solution, or infinitely many solutions.

The solution to a system of linear equations may be found by a number of different techniques. In this section, we illustrate how a system of linear equations may be solved by graphing. In Section 7.2, we illustrate two algebraic methods, the substitution method and the addition method, for solving a system of linear equations.

EXAMPLE 1 *Is the Ordered Pair a Solution?*

Determine which of the ordered pairs is a solution to the following system of linear equations.

$$3x + y = -6$$
$$2x - y = -4$$

a) $(1, -9)$ b) $(-2, 0)$ c) $(2, 8)$

SOLUTION: For the ordered pair to be a solution to the system, it must satisfy each equation in the system.

a)
$$\begin{aligned} 3x + y &= -6 \\ 3(1) + (-9) &= -6 \\ -6 &= -6 \quad \text{True} \end{aligned} \qquad \begin{aligned} 2x - y &= -4 \\ 2(1) - (-9) &= -4 \\ 11 &= -4 \quad \text{False} \end{aligned}$$

Since $(1, -9)$ does not satisfy both equations, it is not a solution to the system.

b)
$$\begin{aligned} 3x + y &= -6 \\ 3(-2) + 0 &= -6 \\ -6 &= -6 \quad \text{True} \end{aligned} \qquad \begin{aligned} 2x - y &= -4 \\ 2(-2) - 0 &= -4 \\ -4 &= -4 \quad \text{True} \end{aligned}$$

Since $(-2, 0)$ satisfies both equations, it is a solution to the system.

c)
$$\begin{aligned} 3x + y &= -6 \\ 3(2) + 8 &= -6 \\ 14 &= -6 \quad \text{False} \end{aligned} \qquad \begin{aligned} 2x - y &= -4 \\ 2(2) - 8 &= -4 \\ -4 &= -4 \quad \text{True} \end{aligned}$$

Since $(2, 8)$ does not satisfy both equations, it is not a solution to the system. ▲

To find the solution to a system of linear equations graphically, we graph both of the equations on the same axes. The coordinates of the point or points of intersection of the graphs are the solution or solutions to the system of equations.

$x + y = 4$

x	y
0	4
1	3
4	0

$2x - y = -1$

x	y
0	1
1	3
-2	-3

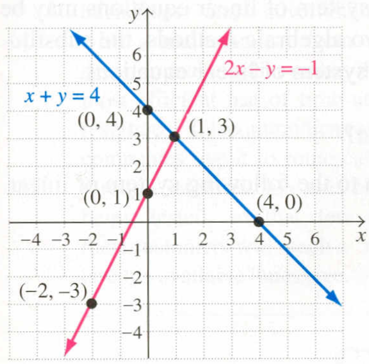


Figure 7.1

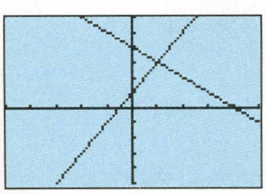


Figure 7.2

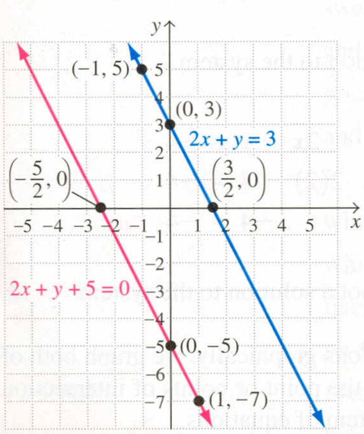


Figure 7.3

Procedure for Solving a System of Equations by Graphing

1. Determine three ordered pairs that satisfy each equation.
2. Plot the ordered pairs and sketch the graphs of both equations on the same axes.
3. The coordinates of the point or points of intersection of the graphs are the solution or solutions to the system of equations.

When two linear equations are graphed, three situations are possible. The two lines may intersect at one point, as in Example 2; or the two lines may be parallel and not intersect, as in Example 3; or the two equations may represent the same line, as in Example 4.

Since the solution to a system of equations may not be integer values, you may not be able to obtain the exact solution by graphing.

EXAMPLE 2 *A System with One Solution*

Find the solution to the following system of equations graphically.

$$x + y = 4$$
$$2x - y = -1$$

SOLUTION: To find the solution, graph both $x + y = 4$ and $2x - y = -1$ on the same axes (Fig. 7.1). Three points that satisfy each equation are shown in the tables above Fig. 7.1. Figure 7.2 shows the system $x + y = 4$ and $2x - y = -1$ graphed on a Texas Instrument TI-83 Plus graphing calculator.

The graphs intersect at (1, 3), which is the solution. This point is the only point that satisfies *both* equations.

Check:

$x + y = 4$	$2x - y = -1$
$1 + 3 = 4$	$2(1) - 3 = -1$
$4 = 4$ True	$2 - 3 = -1$
	$-1 = -1$ True

The system of equations in Example 2 is an example of a ***consistent system of equations***. A consistent system of equations is one that has a solution.

EXAMPLE 3 *A System with No Solution*

Find the solution to the following system of equations graphically.

$$2x + y = 3$$
$$2x + y + 5 = 0$$

SOLUTION: Three ordered pairs that satisfy the equation $2x + y = 3$ are (0, 3), $(\frac{3}{2}, 0)$, and $(-1, 5)$. Three ordered pairs that satisfy the equation $2x + y + 5 = 0$ are $(0, -5)$, $(-\frac{5}{2}, 0)$, and $(1, -7)$. The graphs of both equations are given in Fig. 7.3. Since the two lines are parallel, they do not intersect; therefore, the system has *no solution*.

The system of equations in Example 3 has no solution. A system of equations that has no solution is called an ***inconsistent system***.

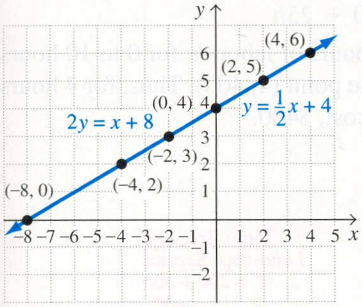


Figure 7.4

EXAMPLE 4 *A System with an Infinite Number of Solutions*

Find the solution to the following system of equations graphically.

$$y = \frac{1}{2}x + 4$$

$$2y = x + 8$$

SOLUTION: Three ordered pairs that satisfy the equation $y = \frac{1}{2}x + 4$ are (0, 4), (2, 5), and (−2, 3). Three ordered pairs that satisfy the equation $2y = x + 8$ are (−8, 0), (4, 6), and (−4, 2). Graph the equations on the same axes (Fig. 7.4). Because all six points are on the same line, the two equations represent the same line. Therefore, every ordered pair that is a solution to one equation is also a solution to the other equation. Every point on the line satisfies both equations; thus, this system has an *infinite number of solutions*. Solving the second equation for y reveals that the equations are equivalent.

When a system of equations has an infinite number of solutions, as in Example 4, it is called a ***dependent system***. Note that a dependent system is also a consistent system, since it has a solution.

Figure 7.5 summarizes the three possibilities for a system of linear equations.

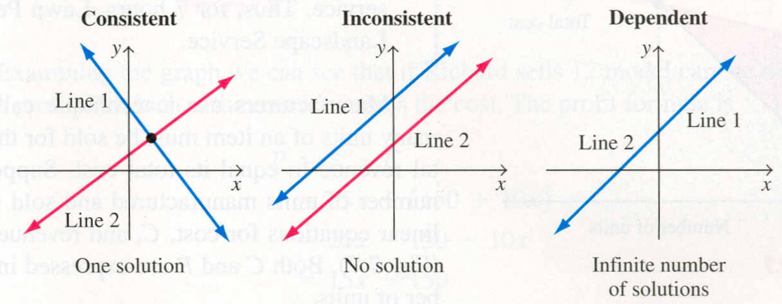


Figure 7.5

In Chapter 6 we introduced *modeling*. Recall that a *mathematical model* is an equation or system of equations that represents a real-life situation. In Examples 5 and 6 we develop equations that model a real-life situation.

EXAMPLE 5 **MODELING** - *A Landscape Service Application*

Tom's Tree and Landscape Service charges a consultation fee of \$200 plus \$50 per hour for labor for landscaping. Lawn Perfect Landscape Service charges a consultation fee of \$300 plus \$25 per hour for labor for landscaping.

a) Write a system of equations to represent the cost, C, of the two landscaping services, each with h hours of labor.

b) Graph both equations on the same axes and determine the number of hours needed for both services to have the same cost.

c) If the Johnsons need 7 hours of landscaping service done at their home, which service is less expensive?

SOLUTION: Let h = the number of hours of labor. The total cost of each service is the consultation fee plus the cost of the labor.

a) Tom's Tree and Landscape Service: $C = 200 + 50h$

Lawn Perfect Landscape Service: $C = 300 + 25h$

b) We graphed the cost, C, versus the number of hours of labor, h, for 0 to 10 hours (Fig. 7.6). On the graph, the lines intersect at the point (4, 400). Thus, for 4 hours of service, both services would have the same cost, $400.

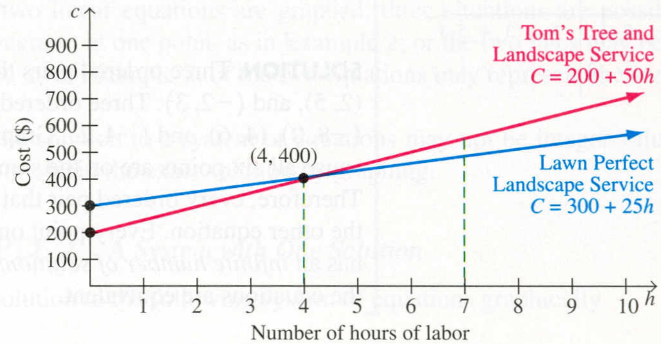


Figure 7.6

c) The graph shows that for more than 4 hours, Lawn Perfect is the least expensive service. Thus, for 7 hours, Lawn Perfect is less expensive than Tom's Tree and Landscape Service.

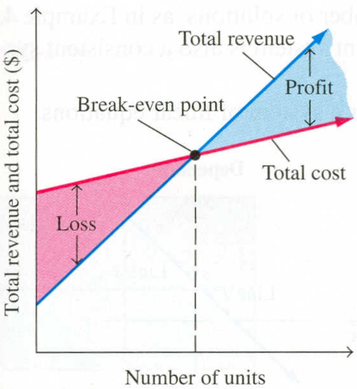


Figure 7.7

Manufacturers use a technique called ***break-even analysis*** to determine how many units of an item must be sold for the business to "break even," that is, for its total revenue to equal its total cost. Suppose we let the horizontal axis represent the number of units manufactured and sold and the vertical axis represent dollars. Then linear equations for cost, C, and revenue, R, can both be sketched on the same axes (Fig. 7.7). Both C and R are expressed in dollars and both are a function of the number of units.

Initially, the cost graph is higher than the revenue graph because of fixed (overhead) costs such as rent and utilities. During low levels of production the manufacturer suffers a loss (the cost graph is greater). During higher levels of production the manufacturer realizes a profit (the profit graph is greater). The point at which the two graphs intersect is called the ***break-even point***. At that number of units sold revenue equals cost, and the manufacturer breaks even.

EXAMPLE 6 MODELING - *Profit and Loss in Business*

At a collectibles show, Richard Lane can sell model cars for $25. The costs for making the cars are a fixed cost of $150 and a production cost of $10 apiece.

a) How many model cars must Richard sell to break even?

b) Determine whether Richard makes a profit if he sells 12 model cars. What is the profit or loss?

c) How many model cars must Richard sell to make a profit of $450?

SOLUTION:

a) Let x denote the number of model cars made and sold. The revenue is given by the equation

$$R = 25x \quad (\$25 \text{ times the number of units})$$

and the cost is given by the equation

$$C = 150 + 10x \quad (\$150 \text{ plus } \$10 \text{ times the number of units})$$

The break-even point is the point at which the revenue and cost graphs intersect. In Fig. 7.8, the graphs intersect at the point (10, 250), which is the break-even point. Thus, for Richard to break even, he must sell 10 model cars. When 10 model cars are made and sold the cost and revenue are both \$250.

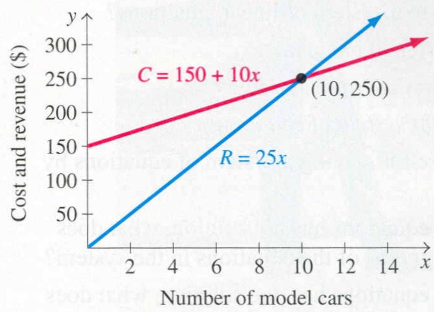


Figure 7.8

b) Examining the graph we can see that if Richard sells 12 model cars he will have a profit, P, which is the revenue minus the cost. The profit formula is

$$\begin{aligned} P &= R - C \\ &= 25x - (150 + 10x) \\ &= 25x - 150 - 10x \\ &= 15x - 150 \end{aligned}$$

Thus, for 12 model cars,

$$\begin{aligned} P &= 15x - 150 \\ &= 15(12) - 150 = 30 \end{aligned}$$

Richard has a profit of \$30 if he sells 12 model cars.

c) We can determine the number of model cars that Richard must sell to have a profit of \$450 by using the profit formula. Substituting 450 for P we have

$$\begin{aligned} P &= 15x - 150 \\ 450 &= 15x - 150 \\ 600 &= 15x \\ 40 &= x \end{aligned}$$

Thus, Richard must sell 40 model cars to make a profit of \$450. ▲

TIMELY TIP Following is a summary of the different types of systems of linear equations.

- A *consistent system of equations* is one that has a solution.
- An *inconsistent system of equations* is one that has no solution.
- A *dependent system of equations* is one that has an infinite number of solutions.

SECTION 7.1 EXERCISES

Concept/Writing Exercises

1. What is a system of linear equations?
2. What is the solution to a system of linear equations?
3. Define a *consistent system of equations.*
4. Define a *dependent system of equations.*
5. Define an *inconsistent system of equations.*
6. Outline the procedure for solving a system of equations by graphing.
7. If a system of linear equations has no solution, what does that mean about the graphs of the equations in the system?
8. If a system of linear equations has one solution, what does that mean about the graphs of the equations in the system?
9. If a system of linear equations has an infinite number of solutions, what does that mean about the graphs of the equations in the system?
10. Can a system of linear equations have exactly two solutions? Explain.

Practice the Skills

In Exercises 11 and 12, determine which ordered pairs are solutions to the given system.

11. $y = 2x - 6$ $\quad (3, 0) \quad (2, -2) \quad (1, 2)$
$y = -x + 3$

12. $x + 2y = 6$ $\quad (-2, 4) \quad (2, 2) \quad (3, -9)$
$x - y = -6$

In Exercises 13–16, solve the system of equations graphically.

13. $x = 1$
$y = 4$

14. $x = -3$
$y = 3$

15. $x = 4$
$y = -3$

16. $x = -5$
$y = -3$

In Exercises 17–32, solve the system of equations graphically. If the system does not have a single ordered pair as a solution, state whether the system is inconsistent or dependent.

17. $x = 2$
$y = -x - 3$

18. $y = 1$
$y = x + 2$

19. $y = 3x - 6$
$y = -x + 6$

20. $x + y = 4$
$-x + y = 2$

21. $x + 2y = 8$
$2x - 3y = 2$

22. $3x - y = 1$
$4x - 3y = 3$

23. $2x + y = 3$
$2y = 6 - 4x$

24. $y = 2x - 4$
$2x + y = 0$

25. $y = x + 3$
$y = -1$

26. $x = 1$
$x + y + 3 = 0$

27. $2x - y = -3$
$2x + y = -9$

28. $3x + 2y = 6$
$6x + 4y = 12$

29. $2x - 3y = 12$
$3y - 2x = 9$

30. $y = \frac{1}{3}x - 4$
$3y - x = 4$

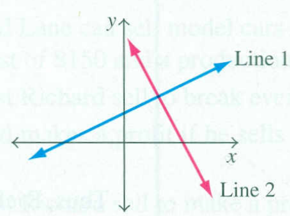
31. $y = -\frac{1}{3}x + 2$
$2x - 2y = 4$

32. $2(x - 1) + 2y = 0$
$3x + 2(y + 2) = 0$

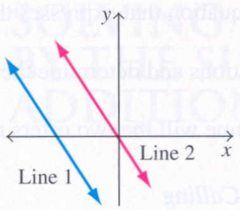
33. a) If the two lines in a system of equations have different slopes, how many solutions will the system have? Explain your answer.
b) If the two lines in a system of equations have the same slope but different y-intercepts, how many solutions will the system have? Explain.
c) If the two lines in a system of equations have the same slope and the same y-intercept, how many solutions will the system have? Explain.

34. Indicate whether the graph shown represents a consistent, inconsistent, or dependent system. Explain your answer.

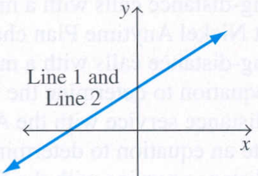
a)



b)



c)



In Exercises 35–46, determine without graphing whether the system of equations has exactly one solution, no solution, or an infinite number of solutions. (Consider your answers to Exercise 33.)

35. $2x - y = 6$
$y = 2x - 6$

36. $3x + 4y = 8$
$8y = -6x + 4$

37. $3x - 4y = 5$
$x - 8y = 10$

38. $x + 3y = 6$
$3x + y = 4$

39. $3x + y = 7$
$y = -3x + 9$

40. $x + 4y = 12$
$x = 4y + 3$

41. $2x - 3y = 6$
$x - \frac{3}{2}y = 3$

42. $x - 2y = 6$
$x + 2y = 4$

43. $3x = 6y + 5$
$y = \frac{1}{2}x - 3$

44. $3y = 6x + 4$
$-2x + y = \frac{4}{3}$

45. $12x - 5y = 4$
$3x + 4y = 6$

46. $4x + 7y = 2$
$4x = 6 + 7y$

Problem Solving

Two lines are ***perpendicular*** *when they meet at a right angle (a 90° angle). Two lines are perpendicular to each other when their slopes are* ***negative reciprocals*** *of each other. The negative reciprocal of 2 is $-\frac{1}{2}$, the negative reciprocal of $\frac{3}{5}$ is $-1/(\frac{3}{5})$ or $-\frac{5}{3}$, and so on. If a represents any real number, except 0, its negative reciprocal is $-1/a$. Note that the product of a number and its negative reciprocal is -1. In Exercises 47–50, determine, by finding the slope of each line, whether the lines will be perpendicular to each other when graphed.*

47. $5y - 2x = 15$
$2y - 5x = 2$

48. $4y - x = 6$
$y = x + 8$

49. $2x + y = 3$
$2y - x = 5$

50. $6x + 5y = 3$
$-10x = 2 + 12y$

In Exercises 51–55, part of the question involves determining a system of equations that models the situation.

51. MODELING - *Landscaping Revisited* In Example 5, assume that Tom's Tree and Landscape Service charges \$200 for a consultation fee plus \$60 per hour for labor and that Lawn Perfect Landscape Service charges \$305 for a consultation fee plus \$25 per hour for labor.

a) Write the system of equations to represent the cost of the two landscaping services.
b) Graph both equations for 0 to 10 hours on the same axes.
c) Determine the number of hours of landscaping that must be used for both services to have the same cost.

52. MODELING - *Security Systems* Tamika Dixon plans to install a security system in her house. She is considering two security companies: ABC Security and SafeHomes Security. ABC's system costs \$3380 to install and their monitoring fee is \$18 per month. SafeHomes equivalent system costs only \$2302 to install but their monitoring fee is \$29 per month.

a) Write a system of equations to represent the cost of each system.
b) Graph both equations (for up to and including 180 months) on the same axes.
c) Determine the number of months the service must be used for both companies to have the same cost.
d) If both companies guarantee not to raise monthly fees for 10 years, and if Tamika plans to use the system for 10 years, which system would be less expensive?

53. MODELING - *Selling Backpacks* Benjamin's Backpacks can sell backpacks for \$25 per backpack. The costs for making the backpacks are a fixed cost of \$400 and a production cost of \$15 per backpack (see Example 6 for an example of cost and revenue equations).

a) Write the cost and revenue equations.

b) Graph both equations, for 0 to 50 backpacks, on the same axes.
c) How many backpacks must Benjamin's Backpacks sell to break even?
d) Write the profit formula.
e) Determine whether Benjamin's Backpacks makes a profit or loss if it sells 30 backpacks. What is the profit or loss?
f) How many backpacks must Benjamin's Backpacks sell to realize a profit of $1000?

54. MODELING - *Purchasing Stocks* When buying or selling stock for a customer, the Mark Demo Agency charges $40 plus 8 cents per share of stock purchased or sold. Andy Harris and Associates charges $15 plus 18 cents per share of stock purchased or sold.
a) Write a system of equations to represent the cost of purchasing or selling stock with each company.
b) Graph both equations (for up to and including 350 shares of stock) on the same axes.
c) Determine the number of shares of stock that must be purchased or sold for the total cost to be the same.
d) If 300 shares of stock are to be purchased, which firm would be less expensive?

55. MODELING - *Manufacturing DVD Players* A manufacturer sells a certain DVD player for $225 per unit. Manufacturing costs consist of a fixed cost of $8400 and a production cost of $155 per unit.
a) Write the cost and revenue equations.
b) Graph both equations (for up to and including 150 units) on the same axes.
c) How many units must the manufacturer sell to break even?
d) Write the profit formula.
e) What is the manufacturer's profit or loss if 100 units are sold?
f) How many units must the manufacturer sell to make a profit of $1260?

56. Explain how you can determine whether a system of two linear equations will be consistent, dependent, or inconsistent without graphing the equations.

Challenge Problems/Group Activities

57. MODELING - *Job Offers* Hubert Hotchkiss had two job offers for sales positions. One pays a salary of $300 per week plus a 15% commission on his dollar sales volume. The second position pays a salary of $450 per week with no commission.
a) For each offer, write an equation that expresses the weekly pay.
b) Graph the system of equations and determine the solution.
c) For what dollar sales volume will the two offers result in the same pay?

58. MODELING- *Long Distance Calling*
a) In March 2002, an AT&T One Rate Plan charged 7 cents per minute for long-distance calls with a monthly fee of $3.95. The Sprint Nickel Anytime Plan charged 5 cents per minute for long-distance calls with a monthly fee of $8.95. Write an equation to determine the monthly cost for long-distance service with the AT&T One Rate Plan, and write an equation to determine the monthly cost for long-distance service with the Sprint Nickel Anytime Plan.
b) Graph the system of equations and determine the solution.
c) After how many minutes will the cost for the two long-distance service plans be the same?

59. *Points of Intersection* a) If two lines have different slopes, what is the maximum possible number of points of intersection?
b) If three lines all have different slopes, what is the maximum possible number of points of intersection?
c) If four lines all have different slopes, what is the maximum possible number of points of intersection?
d) If five lines all have different slopes, what is the maximum possible number of points of intersection?
e) Is there a pattern in the number of points of intersection? If so, explain the pattern. Use the pattern to determine the maximum possible number of points of intersection for six lines.

Recreational Mathematics

60. Connect all the following points using exactly four straight line segments. Do not lift your pencil off the paper.

Internet/Research Activity

61. The Rhind Papyrus indicates that the early Egyptians used linear equations. Do research and write a paper on the symbols used in linear equations and the use of the linear equations by the early Egyptians. (References include history of mathematics books, encyclopedias, and the Internet.)

7.2 SOLVING SYSTEMS OF EQUATIONS BY THE SUBSTITUTION AND ADDITION METHODS

Having solved systems of equations by graphing in Section 7.1, we are now ready to learn two other methods used to solve systems of linear equations: the substitution method and the addition method. We now discuss the substitution method.

Substitution Method

Procedure for Solving a System of Equations Using the Substitution Method

1. Solve one of the equations for one of the variables. If possible, solve for a variable with a numerical coefficient of 1. By doing so, you may avoid working with fractions.
2. Substitute the expression found in step 1 into the other equation. This step yields an equation in terms of a single variable.
3. Solve the equation found in step 2 for the variable.
4. Substitute the value found in step 3 into the equation you rewrote in step 1 and solve for the remaining variable.

Examples 1, 2, and 3 illustrate the *substitution method.* These systems of equations are the same as in Examples 2, 3, and 4 in Section 7.1.

EXAMPLE 1 *A Single Solution, by the Substitution Method*

Solve the following system of equations by substitution.

$$x + y = 4$$
$$2x - y = -1$$

SOLUTION: The numerical coefficients of the x and y terms in the equation $x + y = 4$ are both 1. Thus, we can solve this equation for either x or y. Let's solve for x in the first equation.

STEP 1.

$$x + y = 4$$
$$x + y - y = 4 - y \quad \text{Subtract } y \text{ from both sides of the equation.}$$
$$x = 4 - y$$

STEP 2. Substitute $4 - y$ for x in the other equation.

$$2x - y = -1$$
$$2(4 - y) - y = -1$$

STEP 3. Now solve the equation for y.

$$8 - 2y - y = -1 \quad \text{Distributive property}$$
$$8 - 3y = -1$$
$$8 - 8 - 3y = -1 - 8 \quad \text{Subtract 8 from both sides of the equation.}$$
$$-3y = -9$$
$$\frac{-3y}{-3} = \frac{-9}{-3} \quad \text{Divide both sides of the equation by } -3.$$
$$y = 3$$

STEP 4. Substitute $y = 3$ in the equation solved for x and determine the value of x.

$$x = 4 - y$$
$$x = 4 - 3$$
$$x = 1$$

Thus, the solution is the ordered pair (1, 3). This answer checks with the solution obtained graphically in Section 7.1, Example 2. ▲

TIMELY TIP When solving a system of equations, once you successfully solve for one of the variables, make sure you solve for the other variable. Remember that a solution to a system of equations must contain a numerical value for each variable in the system.

EXAMPLE 2 *No Solution, by the Substitution Method*

Solve the following system of equations by substitution.

$$2x + y = 3$$
$$2x + y + 5 = 0$$

SOLUTION: Solve for y in the first equation.

$$2x + y = 3$$
$$2x - 2x + y = 3 - 2x \quad \text{Subtract } 2x \text{ from both sides of the equation.}$$
$$y = 3 - 2x$$

Now substitute $3 - 2x$ in place of y in the second equation.

$$2x + y + 5 = 0$$
$$2x + (3 - 2x) + 5 = 0$$
$$2x + 3 - 2x + 5 = 0$$
$$8 = 0 \quad \text{False}$$

Since 8 cannot be equal to 0, there is no solution to the system of equations. Thus, the system of equations is inconsistent. This answer checks with the solution obtained graphically in Section 7.1, Example 3. ▲

When solving Example 2, we obtained $8 = 0$ and indicated that the system was inconsistent and that there was no solution. When solving a system of equations, if you obtain a false statement, such as $4 = 0$ or $-2 = 0$, the system is *inconsistent* and has *no solution.*

EXAMPLE 3 *An Infinite Number of Solutions, by the Substitution Method*

Solve the following system of equations by substitution.

$$y = \frac{1}{2}x + 4$$
$$2y = x + 8$$

SOLUTION: The first equation $y = \frac{1}{2}x + 4$ is already solved for y, so we will substitute $\frac{1}{2}x + 4$ for y in the second equation.

$$2y = x + 8$$
$$2\left(\frac{1}{2}x + 4\right) = x + 8$$
$$x + 8 = x + 8 \quad \text{Distributive property}$$
$$x - x + 8 = x - x + 8 \quad \text{Add 8 to both sides of equation}$$
$$8 = 8 \quad \text{True}$$

Since 8 equals 8, the system has an infinite number of solutions. Thus, the system of equations is dependent. This answer checks with the solution obtained in Section 7.1, Example 4.

When solving Example 3, we obtained $8 = 8$ and indicated that the system was dependent and had an infinite number of solutions. When solving a system of equations, if you obtain a true statement, such as $0 = 0$ or $8 = 8$, the system is *dependent* and has an *infinite number of solutions.*

Addition Method

If neither of the equations in a system of linear equations has a variable with a coefficient of 1, it is generally easier to solve the system by using the ***addition*** (or ***elimination***) ***method***.

To solve a system of linear equations by the addition method, it is necessary to obtain two equations whose sum will be a single equation containing only one variable. To achieve this goal, we rewrite the system of equations as two equations where the coefficients of one of the variables are opposites (or additive inverses) of each other. For example, if one equation has a term of $2x$, we might rewrite the other equation so that its x term will be $-2x$. To obtain the desired equations, it might be necessary to multiply one or both equations in the original system by a number. When an equation is to be multiplied by a number, we will place brackets around the equation and place the number that is to multiply the equation before the brackets. For example, $4[2x + 3y = 6]$ means that each term on both sides of the equal sign in the equation $2x + 3y = 6$ is to be multiplied by 4:

$$4[2x + 3y = 6] \quad \text{gives} \quad 8x + 12y = 24.$$

This notation will make our explanations much more efficient and easier for you to follow.

DID YOU KNOW
How to Succeed in Business

Economics, a science dependent on mathematics, dates back to just before the Industrial Revolution of the eighteenth century. Technologies were being invented and applied to the manufacture of cloth, iron, transportation, and agriculture. These new technologies led to the development of mathematically based economic models. French economist Jules Dupuit (1804–1866) suggested a method to calculate the value of railroad bridges; Irish economist Dionysis Larder (1793–1859) showed railroad companies how to structure their rates so as to increase their profits.

Procedure for Solving a System of Equations by the Addition Method

1. If necessary, rewrite the equations so that the variables appear on one side of the equal sign and the constants appear on the other side of the equal sign.
2. If necessary, multiply one or both equations by a constant(s) so that when you add the equations, the result will be an equation containing only one variable.
3. Add the equations to obtain a single equation in one variable.
4. Solve the equation in step 3 for the variable.
5. Substitute the value found in step 4 into either of the original equations and solve for the other variable.

EXAMPLE 4 *Eliminating a Variable by the Addition Method*

Solve the following system of equations by the addition method.

$$x + y = 5$$
$$2x - y = 7$$

SOLUTION: Since the coefficients of the y terms, 1 and -1, are additive inverses, the sum of the y terms will be zero when the equations are added. Thus, the sum of the two equations will contain only one variable, x. Add the two equations to obtain one equation in one variable. Then solve for the remaining variable.

$$\begin{aligned} x + y &= 5 \\ \underline{2x - y} &\underline{= 7} \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Now substitute 4 for x in either of the original equations to find the value of y.

$$\begin{aligned} x + y &= 5 \\ 4 + y &= 5 \\ y &= 1 \end{aligned}$$

The solution to the system is (4, 1).

EXAMPLE 5 *Multiplying by -1 in the Addition Method*

Solve the following system of equations by the addition method.

$$x + 3y = 9$$
$$x + 2y = 5$$

SOLUTION: We want the sum of the two equations to have only one variable. We can eliminate the variable x by multiplying either equation by -1 and then adding. We will multiply the first equation by -1.

$$\begin{aligned} -1[x + 3y = 9] \quad &\text{gives} \quad -x - 3y = -9 \\ x + 2y = 5 \quad & \qquad\quad\; x + 2y = 5 \end{aligned}$$

We now have a system of equations equivalent to the original system. Now add the two equations.

$$\begin{aligned} -x - 3y &= -9 \\ x + 2y &= 5 \\ \hline -y &= -4 \\ y &= 4 \end{aligned}$$

Now we solve for x by substituting 4 for y in either of the original equations.

$$\begin{aligned} x + 3y &= 9 \\ x + 3(4) &= 9 \\ x + 12 &= 9 \\ x &= -3 \end{aligned}$$

The solution is $(-3, 4)$.

EXAMPLE 6 *Multiplying One Equation in the Addition Method*

Solve the following system of equations by the addition method.

$$\begin{aligned} 4x + y &= 6 \\ 3x + 2y &= 7 \end{aligned}$$

SOLUTION: We can multiply the top equation by -2 and then add the two equations to eliminate the variable y.

$$\begin{aligned} -2[4x + y = 6] \quad &\text{gives} \quad -8x - 2y = -12 \\ 3x + 2y = 7 \quad &\phantom{\text{gives}} \quad 3x + 2y = 7 \end{aligned}$$

$$\begin{aligned} -8x - 2y &= -12 \\ 3x + 2y &= 7 \\ \hline -5x &= -5 \\ x &= 1 \end{aligned}$$

Now we find y by substituting 1 for x in either of the original equations.

$$\begin{aligned} 4x + y &= 6 \\ 4(1) + y &= 6 \\ 4 + y &= 6 \\ y &= 2 \end{aligned}$$

The solution is $(1, 2)$.

Note that in Example 6 we could have eliminated the variable x by multiplying the top equation by 3 and the bottom equation by -4, then adding. Try this method now.

EXAMPLE 7 *Multiplying Both Equations*

Solve the following system of equations by the addition method.

$$3x - 4y = 8$$
$$2x + 3y = 9$$

SOLUTION: In this system, we cannot eliminate a variable by multiplying only one equation by an integer value and then adding. To eliminate a variable, we can multiply each equation by a different number. To eliminate the variable x, we can multiply the top equation by 2 and the bottom by -3 (or the top by -2 and the bottom by 3) and then add the two equations. If we want, we can instead eliminate the variable y by multiplying the top equation by 3 and the bottom by 4 and then adding the two equations. Let's eliminate the variable x.

$$\begin{aligned} 2[3x - 4y = 8] &\quad \text{gives} \quad 6x - 8y = 16 \\ -3[2x + 3y = 9] &\quad \text{gives} \quad -6x - 9y = -27 \end{aligned}$$

$$\begin{aligned} 6x - 8y &= 16 \\ -6x - 9y &= -27 \\ \hline -17y &= -11 \\ y &= \frac{11}{17} \end{aligned}$$

We could now find x by substituting $\frac{11}{17}$ for y in either of the original equations. Although it can be done, it gets messy. Instead, let's solve for x by eliminating the variable y from the two original equations. To do so, we multiply the first equation by 3 and the second equation by 4.

$$\begin{aligned} 3[3x - 4y = 8] &\quad \text{gives} \quad 9x - 12y = 24 \\ 4[2x + 3y = 9] &\quad \text{gives} \quad 8x + 12y = 36 \end{aligned}$$

$$\begin{aligned} 9x - 12y &= 24 \\ 8x + 12y &= 36 \\ \hline 17x \qquad &= 60 \\ x &= \frac{60}{17} \end{aligned}$$

The solution to the system is $\left(\frac{60}{17}, \frac{11}{17}\right)$. ▲

When solving a system of linear equations by either the substitution or the addition method, if you obtain the equation $0 = 0$ it indicates that the system is *dependent* (both equations represent the same line; see Fig. 7.4 on page 385), and there are an infinite number of solutions. When solving, if you obtain an equation such as $0 = 6$, or any other equation that is false, it means that the system is *inconsistent* (the two equations represent parallel lines; see Fig. 7.5 on page 385), and there is no solution.

EXAMPLE 8 **MODELING** - *When Are Repair Costs the Same?*

Melinda Melendez needs to purchase a new radiator for her car and have it installed by a mechanic. She is considering two garages: Steve's Repair and Greg's Garage.

At Steve's Repair, the parts cost $200 and the labor cost is $50 per hour. At Greg's Garage, the parts cost $375 and the labor cost is $25 per hour. How many hours would the repair need to take for the total cost at each garage to be the same?

SOLUTION: We are asked to find the number of hours the repair would need to take for each garage to have the same total cost, C. First write a system of equations to represent the total cost for each of the garages. The total cost consists of the cost of the parts and the labor cost. The labor cost depends on the number of hours of labor.

Let x = the number of hours of labor.

$$\text{Total cost} = \text{cost of parts} + \text{labor cost}$$

$$\text{Steve's Repair: } C = 200 + 50x$$

$$\text{Greg's Garage: } C = 375 + 25x$$

We want to determine when the cost will be the same, so we set the two costs equal to each other (substitution method) and solve the resulting equation.

$$200 + 50x = 375 + 25x$$

$$200 - 200 + 50x = 375 - 200 + 25x \quad \text{Subtract 200 from both sides of the equation.}$$

$$50x = 175 + 25x$$

$$50x - 25x = 175 + 25x - 25x \quad \text{Subtract } 25x \text{ from both sides of the equation.}$$

$$25x = 175$$

$$\frac{25x}{25} = \frac{175}{25} \quad \text{Divide both sides of the equation by 25.}$$

$$x = 7$$

Thus, for 7 hours of labor, the cost at both garages would be the same. If we construct a graph (Fig. 7.9) of the two cost equations, the point of intersection is (7, 550). If the repair were to require 7 hours of labor, the total cost at either garage would be $550.

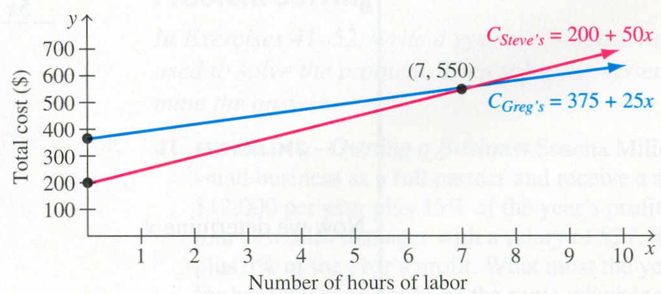


Figure 7.9

EXAMPLE 9 MODELING - *A Mixture Problem*

Karen Guardino, a pharmacist, needs 500 milliliters (mℓ) of a 10% phenobarbital solution. She has only a 5% solution and a 25% solution available. How many milliliters of each solution should she mix to obtain the desired solution?

SOLUTION: First we set up a system of equations. The unknown quantities are the amount of 5% solution and the amount of the 25% solution that must be used. Let

$$x = \text{number of m}\ell \text{ of } 5\% \text{ solution}$$
$$y = \text{number of m}\ell \text{ of } 25\% \text{ solution}$$

We know that 500 mℓ of solution are needed. Thus,

$$x + y = 500$$

The total amount of phenobarbital in a solution is determined by multiplying the percent of phenobarbital by the number of milliliters of solution. The second equation comes from the fact that

$$\begin{pmatrix}\text{Total amount of}\\ \text{phenobarbital in}\\ 5\% \text{ solution}\end{pmatrix} + \begin{pmatrix}\text{total amount of}\\ \text{phenobarbital in}\\ 25\% \text{ solution}\end{pmatrix} = \begin{pmatrix}\text{total amount of}\\ \text{phenobarbital}\\ \text{in } 10\% \text{ mixture}\end{pmatrix}$$

$$0.05x \quad + \quad 0.25y \quad = \quad 0.10(500)$$

$$\text{or} \qquad 0.05x + 0.25y = 50$$

The system of equations is

$$\begin{aligned} x + y &= 500 \\ 0.05x + 0.25y &= 50 \end{aligned}$$

Let's solve this system of equations by using the addition method. There are various ways of eliminating one variable. To obtain integer values in the second equation, we can multiply both sides of the equation by 100. That will result in an x-term of $5x$. If we multiply both sides of the first equation by -5, that will result in an x-term of $-5x$. By following this process, we can eliminate the x-terms from the system.

$$\begin{aligned} -5[x + y = 500] &\quad \text{gives} \quad -5x - 5y = -2500 \\ 100[0.05x + 0.25y = 50] &\quad \text{gives} \quad 5x + 25y = 5000 \end{aligned}$$

$$\begin{aligned} -5x - 5y &= -2500 \\ 5x + 25y &= 5000 \\ \hline 20y &= 2500 \\ \frac{20y}{20} &= \frac{2500}{20} \\ y &= 125 \end{aligned}$$

Now we determine x.

$$\begin{aligned} x + y &= 500 \\ x + 125 &= 500 \\ x &= 375 \end{aligned}$$

Therefore, 375 mℓ of a 5% phenobarbital solution must be mixed with 125 mℓ of a 25% phenobarbital solution to obtain 500 mℓ of a 10% phenobarbital solution. ▲

Example 9 can also be solved by using substitution. Try to do so now.

SECTION 7.2 EXERCISES

Concept/Writing Exercises

1. In your own words, explain how to solve a system of linear equations by using the addition method.
2. In your own words, explain how to solve a system of linear equations by using the substitution method.
3. How will you know, when solving a system of linear equations by either the substitution or the addition method, whether the system is dependent?
4. How will you know, when solving a system of linear equations by either the substitution or the addition method, whether the system is inconsistent?
5. When solving the following system of equations by the substitution method, which variable, in which equation, would you choose to solve for in order to make the solution easier? Explain your answer. Do not solve the system.
$$x + 3y = 3$$
$$3x + 4y = -1$$
6. When solving the following system of equations by the addition method, what will your first step be in solving the system? Explain your answer. Do not solve the system.
$$2x + y = 6$$
$$3x + 3y = 9$$

Practice the Skills

In Exercises 7–24, solve the system of equations by the substitution method. If the system does not have a single ordered pair as a solution, state whether the system is inconsistent or dependent.

7. $y = x + 7$
$y = -x + 5$

8. $y = 3x + 7$
$y = -2x - 3$

9. $2x + 4y = 8$
$2x - y = -2$

10. $y + 3x = 7$
$2x + 3y = 14$

11. $y - x = 4$
$x - y = 3$

12. $x + y = 3$
$y + x = 5$

13. $3y + 2x = 4$
$3y = 6 - x$

14. $x = 5y - 12$
$x - y = 0$

15. $y - 2x = 3$
$2y = 4x + 6$

16. $y = 2$
$y + x + 3 = 0$

17. $x = y + 3$
$x = -3$

18. $x + 2y = 6$
$y = 2x + 3$

19. $y + 3x - 4 = 0$
$2x - y = 7$

20. $x + 4y = 7$
$2x + 3y = 5$

21. $x = 2y + 3$
$y = 3x - 1$

22. $x + 4y = 9$
$2x - y - 6 = 0$

23. $y = -2x + 3$
$4x + 2y = 12$

24. $2x + y = 12$
$x = -\frac{1}{2}y + 6$

In Exercises 25–40, solve the system of equations by the addition method. If the system does not have a single ordered pair as a solution, state whether the system is inconsistent or dependent.

25. $3x + y = 10$
$4x - y = 4$

26. $x + 2y = 9$
$x - 2y = -3$

27. $x + y = 10$
$x - 2y = -2$

28. $3x + y = 10$
$-3x + 2y = -16$

29. $2x - y = -4$
$-3x - y = 6$

30. $x + y = 6$
$-2x + y = -3$

31. $4x + 3y = -1$
$2x - y = -13$

32. $2x + y = 6$
$3x + y = 5$

33. $2x + y = 11$
$x + 3y = 18$

34. $5x - 2y = 11$
$-3x + 2y = 1$

35. $3x - 4y = 11$
$3x + 5y = -7$

36. $4x - 2y = 6$
$4y = 8x - 12$

37. $4x + y = 6$
$-8x - 2y = 13$

38. $2x + 3y = 6$
$5x - 4y = -8$

39. $3x - 4y = 10$
$5x + 3y = 7$

40. $6x + 3y = 7$
$5x + 2y = 9$

Problem Solving

In Exercises 41–52, write a system of equations that can be used to solve the problem. Then solve the system and determine the answer.

41. **MODELING** - *Owning a Business* Sosena Milion can join a small business as a full partner and receive a salary of \$12,000 per year plus 15% of the year's profit, or she can join as a sales manager with a salary of \$27,000 per year plus 5% of the year's profit. What must the year's profit be for her total earnings to be the same whether she joins as a full partner or as a sales manager?

42. **MODELING** - *Mortgage Refinancing* In January 2003, mortgage rates were very low so Wayne Morganstein considered refinancing his mortgage. The cost of refinancing his mortgage would include a one-time charge of \$1600. With the reduced mortgage rate, his monthly interest and principal payments would be \$780. At the higher interest rate he currently has, his interest and principal payments are $980 per month.

a) Determine how many months it will take until both mortgage plans would have the same total cost.

b) If Wayne plans to remain in his house for exactly 6 years, which mortgage plan would result in a lower total cost?

43. **MODELING -** *Pizza Orders* Pizza Corner sells medium and large specialty pizzas. A medium Meat Lovers pizza costs $10.95, and a large Meat Lovers pizza costs $14.95. One Saturday a total of 50 Meat Lovers pizzas were sold, and the receipts from the Meat Lovers pizzas were $663.50. How many medium and how many large Meat Lovers pizzas were sold?

44. **MODELING -** *Basketball Game* The University of Tennessee women's basketball team made 45 field goals in a recent game; some were 2-pointers and some were 3-pointers. How many 2-point baskets were made and how many 3-point baskets were made if Tennessee scored 101 points?

45. **MODELING -** *Chemical Mixture* Antonio Gonzalez is a chemist and needs 10 liters (ℓ) of a 40% hydrochloric acid solution. He discovers he is out of the 40% hydrochloric acid solution and does not have sufficient time to reorder. He checks his supply shelf and finds he has a large supply of both 25% and 50% hydrochloric acid solutions. He decides to use the 25% and 50% solutions to make 10 ℓ of a 40% solution. How many liters of the 25% solution and of the 50% solution should he mix?

46. **MODELING -** *A Milk Mixture* The Guidas own a dairy. They have milk that is 5% butterfat and skim milk without butterfat. How much of the 5% milk and how much of the skim milk should they mix to make 100 gal of milk that is 3.5% butterfat?

47. **MODELING -** *Choosing a Copy Service* Lori Lanier recently purchased a high-speed copier for her home office and wants to purchase a service contract on the copier. She is considering two sources for the contract. The Economy Sales and Service Company charges $18 a month plus 2 cents per copy. Office Superstore charges $24 a month but only 1.5 cents per copy. How many copies would Lori need to make for the monthly costs of both plans to be the same?

48. **MODELING -** *Cellular Phone Plans* Rich Gratien is considering two cellular phone plans. Both plans offer 300 free minutes each month. Cingular Home 300 Plan charges $30 per month plus 45 cents for each additional minute after 300 minutes. Verizon America's Choice Plan charges $35 per month plus 20 cents for each additional minute after 300 minutes.

a) In addition to the 300 free minutes, how long would Rich have to talk on the phone, in a month, for the two plans to have the same total cost?

b) If Rich talks for 350 minutes a month, which plan would be less expensive for him?

49. **MODELING -** *Nut and Pretzel Mix* Dave Chwalik wants to purchase 20 pounds of party mix for a total of $30. To obtain the mixture, he will mix nuts that cost $3 per pound with pretzels that cost $1 per pound. How many pounds of each type of mix should he use?

50. **MODELING -** *Laboratory Research* Animals in an experiment are to be kept on a strict diet. Each animal is to receive, among other things, 20 g of protein and 6 g of carbohydrates. The scientist has only two food mixes of the following compositions available.

	Protein (%)	Carbohydrates (%)
Mix A	10	6
Mix *B*	20	2

How many grams of each mix should she use to obtain the right diet for a single animal?

51. **MODELING -** *School Play Tickets* Jefferson High School sold 250 tickets to its annual school play. Student tickets cost $2 per ticket and nonstudent tickets cost $5 per ticket. If $950 in ticket sales is collected, how many tickets of each type were sold?

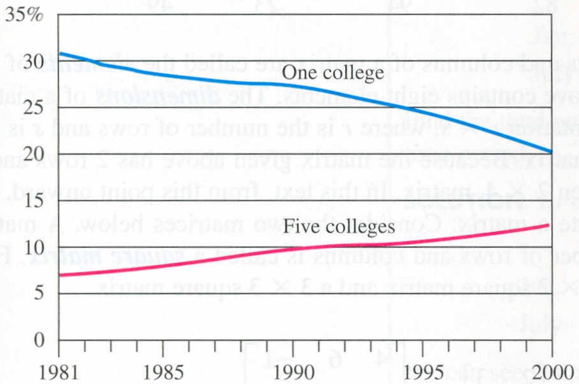
52. **MODELING -** *Golf Club Membership* Membership in Oakwood Country Club costs $3000 per year and entitles a member to play a round of golf for greens fee of $18. At Pinecrest Country Club, membership costs $2500 per year and the greens fee is $20.

a) How many rounds must a golfer play in a year for the costs at the two clubs to be the same?

b) If Sally Sestini planned to play 30 rounds of golf in a year, which club would be the least expensive?

53. **MODELING** - *College Applications* As the following graph shows, from 1981 through 2000 the percentage of high school students who applied to exactly one college decreased while the percentage of high school students who applied to exactly five colleges increased. The percentage of high school students who applied to one college (the blue curve) can be approximated by the linear equation $y = -0.58x + 31$, where x is the number of years since 1981. The percentage of high school students who applied to five colleges (the red curve) can be approximated by the linear equation $y = 0.32x + 7$. Assuming the present trend continues, use the substitution method to approximate when the percentage of high school students applying to one college will equal the percentage of high school students applying to five colleges.

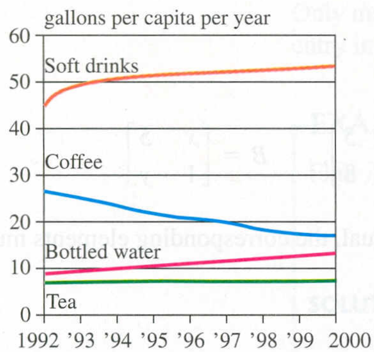
Percentage of Students Who Applied to:



Source: *U.S. News and World Report*

54. **MODELING** - *Drinking Coffee* The following graph shows how coffee consumption compared with consumption of other beverages from 1992 through 2000. The number of gallons of coffee consumed per capita per year (the blue curve) can be approximated by the linear equation $y = -1.13x + 27$, where x is the number of years since 1992. The number of gallons of bottled water consumed per capita per year (the red curve) can be approximated by the linear equation $y = 0.38x + 9$. Assuming the present trend continues, use the substitution method to approximate when the number of gallons per capita of coffee consumed will equal the number of gallons per capita of bottled water consumed.

Beverage Consumption in the U.S.



Source: U.S. Department of Agriculture

Challenge Problems/Group Activities

55. Solve the following system of equations for u and v by first substituting x for $\frac{1}{u}$ and y for $\frac{1}{v}$.

$$\frac{1}{u} + \frac{2}{v} = 8$$
$$\frac{3}{u} - \frac{1}{v} = 3$$

56. Develop a system of equations that has (6, 5) as its solution. Explain how you developed your system of equations.

57. The substitution or addition methods can also be used to solve a system of three equations in three variables. Consider the following system.

$$x + y + z = 7$$
$$x - y + 2z = 9$$
$$-x + 2y + z = 4$$

The *ordered triple* (x, y, z) is the solution to the system if it satisfies all three equations.

a) Show that the ordered triple (2, 1, 4) is a solution to the system.

b) Use the substitution or addition method to determine the solution to the system. (*Hint:* Eliminate one variable by using two equations. Then eliminate the same variable by using two different equations.)

58. Construct a system of two equations that has no solution. Explain how you know the system has no solution.

59. Construct a system of two equations that has an infinite number of solutions. Explain how you know the system has an infinite number of solutions.

Recreational Mathematics

60. In parts a)–d) make up a system of linear equations whose solution will be the ordered pair given. *Hint:* It may be helpful to visualize possible graphs that have the given solution. There are many possible answers for each part.

a) (0, 0) **b)** (1, 0) **c)** (0, 1) **d)** (1, 1)

7.3 MATRICES

DID YOU KNOW

Matrices Are Everywhere

You are already familiar with matrices, although you may not be aware of it. The matrix is a good way to display numerical data, as illustrated on this trail sign in Yosemite National Park.

We have discussed solving systems of equations by graphing, using substitution, and using the addition method. In Section 7.4, we will discuss solving systems of linear equations by using matrices. So that you will become familiar with matrices, in this section, we explain how to add, subtract, and multiply matrices. We also explain how to multiply a matrix by a real number. Matrix techniques are easily adapted to computers.

A ***matrix*** is a rectangular array of elements. An array is a systematic arrangement of numbers or symbols in rows and columns. Matrices (the plural of matrix) may be used to display information and to solve systems of linear equations. The following matrix displays the responses from a survey of 500 students at Morgan State University. The students were asked if they were in favor of an increase in their student fees to pay for building a new meeting room for student clubs.

$$\begin{array}{ll|cccc} & & \multicolumn{4}{c}{\text{Columns}} \\ & & \text{Freshmen} & \text{Sophomores} & \text{Juniors} & \text{Seniors} \\ \text{Rows} & \text{In favor} & 102 & 93 & 22 & 35 \\ & \text{Opposed} & 82 & 94 & 23 & 49 \end{array}$$

The numbers in the rows and columns of a matrix are called the ***elements*** of the matrix. The matrix given above contains eight elements. The ***dimensions*** of a matrix may be indicated with the notation $r \times s$, where r is the number of rows and s is the number of columns in the matrix. Because the matrix given above has 2 rows and 4 columns, it is a 2 by 4, written 2×4, matrix. In this text, from this point onward, we use brackets, [ ], to indicate a matrix. Consider the two matrices below. A matrix that contains the same number of rows and columns is called a ***square matrix***. Following is an example of a 2×2 square matrix and a 3×3 square matrix.

$$\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 6 & -1 \\ 2 & 3 & 0 \\ 5 & 2 & 1 \end{bmatrix}$$

Two matrices are equal if and only if they have the same elements in the same relative positions.

EXAMPLE 1 *Equal Matrices*

Given $A = B$, find x and y.

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 8 \end{bmatrix}, \qquad B = \begin{bmatrix} x & 5 \\ 1 & y \end{bmatrix}$$

SOLUTION: Since the matrices are equal, the corresponding elements must be the same, so $x = 2$ and $y = 8$. ▲

Addition of Matrices

Two matrices can be added only if they have the same dimensions (same number of rows and same number of columns). To obtain the sum of two matrices with the same dimensions, add the corresponding elements of the two matrices.

EXAMPLE 2 *Adding Matrices*

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 8 \\ 4 & 0 \end{bmatrix}. \quad \text{Find } A + B.$$

SOLUTION: $A + B = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ 4 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3+2 & 4+8 \\ -1+4 & 7+0 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 3 & 7 \end{bmatrix}$$

▲

EXAMPLE 3 **MODELING-** *Sales of Bicycles*

Peddler's Bicycle Corporation owns and operates two stores, one in Pennsylvania and one in New Jersey. The number of mountain bicycles, MB, and racing bicycles, RB, sold in each store during January through June and during July through December are indicated in the matrices that follow. We will call the matrices A and B.

$$\begin{array}{lcc} & \textit{Pennsylvania} & \textit{New Jersey} \\ & \begin{array}{cc} MB & RB \end{array} & \begin{array}{cc} MB & RB \end{array} \\ \begin{array}{l} \text{Jan.–June} \\ \text{July–Dec.} \end{array} & \begin{bmatrix} 515 & 425 \\ 290 & 250 \end{bmatrix} = A & \begin{bmatrix} 520 & 350 \\ 180 & 271 \end{bmatrix} = B \end{array}$$

Find the total number of each type of bicycle sold by the corporation during each time period.

SOLUTION: To solve the problem, we add matrices A and B.

$$\begin{array}{lcc} & \begin{array}{cc} MB & RB \end{array} & \begin{array}{cc} MB & RB \end{array} \\ \begin{array}{l} \text{Jan.–June} \\ \text{July–Dec.} \end{array} & \begin{bmatrix} 515+520 & 425+350 \\ 290+180 & 250+271 \end{bmatrix} = & \begin{bmatrix} 1035 & 775 \\ 470 & 521 \end{bmatrix} \end{array}$$

We can see from the sum matrix that during the period from January through June, a total of 1035 mountain bicycles and 775 racing bicycles were sold. During the period from July through December, a total of 470 mountain bicycles and 521 racing bicycles were sold. ▲

Subtraction of Matrices

Only matrices with the same dimension may be subtracted. To do so, we subtract each entry in one matrix from the corresponding entry in the other matrix.

EXAMPLE 4 *Subtracting Matrices*

Find $A - B$ if

$$A = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -4 \\ 7 & -3 \end{bmatrix}$$

SOLUTION:

$$A - B = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 3 & -4 \\ 7 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 6-(-4) \\ 3-7 & -1-(-3) \end{bmatrix} = \begin{bmatrix} -1 & 10 \\ -4 & 2 \end{bmatrix}$$

▲

Multiplying a Matrix by a Real Number

A matrix may be multiplied by a real number by multiplying each entry in the matrix by the real number. Sometimes when we multiply a matrix by a real number, we call that real number a ***scalar***.

EXAMPLE 5 *Multiplying a Matrix by a Scalar*

For matrices A and B, find (a) $3A$ and (b) $3A - 2B$.

$$A = \begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 6 \end{bmatrix}$$

SOLUTION:

a) $3A = 3\begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(4) \\ 3(-3) & 3(5) \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ -9 & 15 \end{bmatrix}$

b) We found $3A$ in part (a). Now we find $2B$.

$$2B = 2\begin{bmatrix} -1 & 3 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2(-1) & 2(3) \\ 2(5) & 2(6) \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 10 & 12 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 3 & 12 \\ -9 & 15 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 10 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3-(-2) & 12-6 \\ -9-10 & 15-12 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -19 & 3 \end{bmatrix}$$

▲

Multiplication of Matrices

Multiplication of matrices is slightly more difficult than addition of matrices. Multiplication of matrices is possible only when the number of *columns* of the first matrix, A, is the same as the number of *rows* of the second matrix, B. We use the notation

$$\underset{3 \times 4}{A}$$

to indicate that matrix A has three rows and four columns. Suppose matrix A is a 3×4 matrix and matrix B is a 4×5 matrix. Then

$$\underset{3 \times 4}{A} \qquad \underset{4 \times 5}{B}$$

Same

Product matrix 3×5

This notation indicates that matrix A has four columns and matrix B has four rows. Therefore, we can multiply these two matrices. The product matrix will have the same number of rows as matrix A and the same number of columns as matrix B. Thus, the dimensions of the product matrix are 3×5.

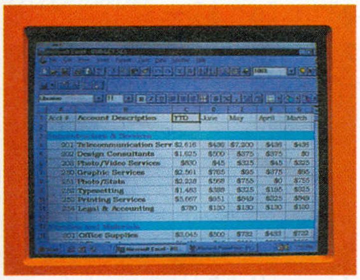
EXAMPLE 6 *Can These Matrices Be Multiplied?*

Determine which of the following pairs of matrices can be multiplied.

a) $A = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix}$

DID YOU KNOW

Spreadsheets



One of the most useful applications of computer software is the spreadsheet. On a computer screen, it looks very much like an accountant's ledger, but there the similarities end. A spreadsheet uses a matrix that assigns an identifying number and letter, in maplike fashion, to each "cell" in the matrix. The data may be dates, numbers, or sums of money. You instruct the computer what operation to perform in each column and row. You can change an entry to see what effect this has on the rest of the spreadsheet. The computer will automatically recompute all affected cells. The computer has the capacity to store information in thousands of cells and can reformat the data in graph form.

b) $A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & -1 \\ 6 & 8 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$

SOLUTION:

a)

$$\begin{array}{cc} A & B \\ 2 \times 2 & 2 \times 2 \end{array}$$

Same

Because matrix A has two columns and matrix B has two rows, the two matrices can be multiplied. The product is a 2×2 matrix.

b)

$$\begin{array}{cc} A & B \\ 2 \times 2 & 2 \times 3 \end{array}$$

Same

Because matrix A has two columns and matrix B has two rows, the two matrices can be multiplied. The product is a 2×3 matrix.

c)

$$\begin{array}{cc} A & B \\ 2 \times 3 & 2 \times 3 \end{array}$$

Not Same

Because matrix A has three columns and matrix B has two rows, the two matrices cannot be multiplied.

To explain matrix multiplication let's use matrices A and B that follow.

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix}$$

Since A contains two rows and B contains two columns, the product matrix will contain two rows and two columns. To multiply two matrices, we use a row–column scheme of multiplying. The numbers in the *first row* of matrix A are multiplied by the numbers in the *first column* of matrix B. These products are then added to determine the entry in the product matrix.

$$A \times B = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix}$$

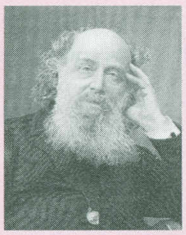
First row First column

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix}$$

$$(3 \times 0) + (2 \times 4) = 0 + 8 = 8$$

PROFILE IN MATHEMATICS

JAMES SYLVESTER, WILLIAM ROWAN HAMILTON, AND ARTHUR CAYLEY



James Sylvester

William Rowan Hamilton

Arthur Cayley

Three mathematicians played important roles in the development of matrix theory: James Sylvester (1814–1897), William Rowan Hamilton (1805–1865), and Arthur Cayley (1821–1895). Sylvester and Cayley cooperated in developing matrix theory. Sylvester was the first to use the term *matrix*. Hamilton, a noted physicist, astronomer, and mathematician, also used what was essentially the algebra of matrices under the name *linear and vector functions*. The mathematical concept of *vector space* grew out of Hamilton's work on the algebra of vectors.

The 8 is placed in the first-row, first-column position of the product matrix. The other numbers in the product matrix are obtained similarly, as illustrated in the matrix that follows.

First row First column

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix} \qquad (3 \times 0) + (2 \times 4) = 8$$

First row Second column

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix} \qquad (3 \times 6) + (2 \times 1) = 20$$

$$A \times B = \begin{bmatrix} 8 & 20 \\ 28 & 37 \end{bmatrix}$$

Second row First column

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix} \qquad (5 \times 0) + (7 \times 4) = 28$$

Second row Second column

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix} \qquad (5 \times 6) + (7 \times 1) = 37$$

We can shorten the procedure as follows.

$$\begin{aligned} A \times B &= \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3(0) + 2(4) & 3(6) + 2(1) \\ 5(0) + 7(4) & 5(6) + 7(1) \end{bmatrix} \\ &= \begin{bmatrix} 8 & 20 \\ 28 & 37 \end{bmatrix} \end{aligned}$$

In general, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

then

$$A \times B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

Let's do one more multiplication of matrices.

EXAMPLE 7 *Multiplying Matrices*

Find $A \times B$, given

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -1 & 3 \\ 2 & 8 & 0 \end{bmatrix}$$

SOLUTION: Matrix A contains two columns, and matrix B contains two rows. Thus, the matrices can be multiplied. Since A contains two rows and B contains three columns, the product matrix will contain two rows and three columns.

$$A \times B = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & -1 & 3 \\ 2 & 8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(6) + 1(2) & 2(-1) + 1(8) & 2(3) + 1(0) \\ 3(6) + 6(2) & 3(-1) + 6(8) & 3(3) + 6(0) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 6 & 6 \\ 30 & 45 & 9 \end{bmatrix}$$

It should be noted that multiplication of matrices *is not* commutative; that is, $A \times B \neq B \times A$, except in special instances.

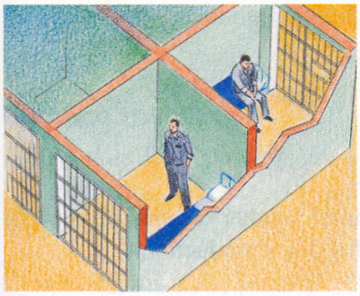
Square matrices have a ***multiplicative identity matrix***. The multiplicative identity matrices for a 2×2 and a 3×3 matrix, denoted I, follow. Note that in any multiplicative identity matrix, 1's go diagonally from top left to bottom right and all other elements in the matrix are 0's.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For any square matrix, A, $A \times I = I \times A = A$.

DID YOU KNOW

The Prisoner's Dilemma



When two parties pursue conflicting interests, the situation can sometimes be described and modeled in a matrix under a branch of mathematics known as game theory. Consider a famous problem called "the prisoner's dilemma." A pair of criminal suspects, A and B, are being held in separate jail cells and cannot communicate with each other. Each one is told that there are four possible outcomes: If both confess, each receives a 3-year sentence. If A confesses and B does not, A receives a 1-year sentence, whereas B receives a 10-year sentence. If B confesses and A does not, B receives a 1-year sentence and A receives a 10-year sentence. Finally, if neither confesses, each will be imprisoned for 2 years. (Try arranging this situation in a matrix.)

If neither prisoner knows whether the other will confess, what should each prisoner do? A study of game theory shows that it is in each prisoner's best interest to confess to the crime.

EXAMPLE 8 *Using the Identity Matrix in Multiplication*

Use the multiplicative identity matrix for a 2×2 matrix and matrix A to show that $A \times I = A$.

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

SOLUTION: The identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$A \times I = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4(1) + 3(0) & 4(0) + 3(1) \\ 2(1) + 1(0) & 2(0) + 1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = A$$

Example 9 illustrates an application of multiplication of matrices.

EXAMPLE 9 *A Manufacturing Application*

The Fancy Frock Company manufactures three types of women's outfits: a dress, a two-piece suit (skirt and jacket), and a three-piece suit (skirt, jacket, and a vest). On a particular day, the firm produces 20 dresses, 30 two-piece suits, and 50 three-piece suits. Each dress requires 4 units of material and 1 hour of work to produce, each two-piece suit requires 5 units of material and 2 hours of work to produce, and each three-piece suit requires 6 units of material and 3 hours to produce. Use matrix multiplication to determine the total number of units of material and the total number of hours needed for that day's production.

SOLUTION: Let matrix A represent the number of each type of women's outfits produced.

$$A = \begin{array}{ccc} \text{Dress} & \text{Two piece} & \text{Three piece} \\ \end{array}$$
$$A = [\,20 \quad 30 \quad 50\,]$$

The units of material and time requirements for each type are indicated in matrix B.

$$B = \begin{array}{c} \begin{array}{cc} \text{Material} & \text{Hours} \end{array} \\ \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Dress} \\ \text{Two piece} \\ \text{Three piece} \end{array}$$

The product of A and B, or $A \times B$, will give the total number of units of material and the total number of hours of work needed for that day's production.

$$\begin{aligned} A \times B &= [20 \quad 30 \quad 50] \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix} \\ &= [20(4) + 30(5) + 50(6) \qquad 20(1) + 30(2) + 50(3)] \\ &= [530 \quad 230] \end{aligned}$$

Thus, a total of 530 units of material and a total of 230 hours of work are needed that day.

TIMELY TIP Matrices can only be added or subtracted if they have the same dimensions.

Matrices can only be multiplied if the number of *columns* in the first matrix is the same as the number of *rows* in the second matrix.

SECTION 7.3 EXERCISES

Concept/Writing Exercises

1. What is a matrix?
2. Explain how to determine the dimensions of a matrix.
3. What is a square matrix?
4. How many rows does a 4×3 matrix have?
5. How many columns does a 3×2 matrix have?
6. To add or subtract two matrices, what must be true about the dimensions of those matrices?
7. a) In your own words, explain the procedure used to add matrices.
   b) Use the procedure given in part (a) to add

$$\begin{bmatrix} 1 & 4 & -1 \\ 3 & 2 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 5 & -6 \\ -1 & 2 & 4 \end{bmatrix}$$

8. a) In your own words, explain the procedure used to subtract matrices.
   b) Use the procedure given in part (a) to subtract

$$\begin{bmatrix} 8 & 4 & 2 \\ 0 & -2 & 4 \end{bmatrix} \text{ from } \begin{bmatrix} 3 & -5 & 6 \\ -2 & 3 & 4 \end{bmatrix}$$

9. a) To multiply two matrices, what must be true about the dimensions of those matrices?
   b) What will be the dimensions of the product matrix when multiplying a 2×2 matrix with a 2×3 matrix?
10. a) In your own words, explain the procedure used to multiply matrices.
    b) Use the procedure given in part (a) to multiply

$$\begin{bmatrix} 6 & -1 \\ 5 & 0 \end{bmatrix} \text{ by } \begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}$$

11. a) What is the multiplicative identity matrix for a 2×2 matrix?
    b) What is the multiplicative identity matrix for a 3×3 matrix?

12. A company has three offices: East, West, and Central. Each office has five divisions. The number of employees in each division of the three offices is as follows:
East: 110, 232, 103, 190, 212
West: 107, 250, 135, 203, 189
Central: 115, 218, 122, 192, 210
Express this information in the form of a 3×5 matrix.

Practice the Skills

In Exercises 13–16, determine A + B.

13. $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -1 \\ 7 & 2 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & 3 & -7 \\ 4 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & -3 & 8 \\ 6 & 5 & 0 \end{bmatrix}$

15. $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \\ 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 3 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$

16. $A = \begin{bmatrix} 2 & 6 & 3 \\ -1 & -6 & 4 \\ 3 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 3 & 1 \\ 7 & -2 & 1 \\ 2 & 3 & 8 \end{bmatrix}$

In Exercises 17–20, determine A − B.

17. $A = \begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 5 \\ 9 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 8 & 1 \\ 0 & 2 \\ -3 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 3 \\ -4 & 5 \\ -2 & 6 \end{bmatrix}$

19. $A = \begin{bmatrix} -4 & 3 \\ 6 & 2 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & -8 \\ -10 & -11 \\ 3 & -7 \end{bmatrix}$

20. $A = \begin{bmatrix} 5 & 3 & -1 \\ 7 & 4 & 2 \\ 6 & -1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & 6 \\ -2 & -4 & 9 \\ 0 & -2 & 4 \end{bmatrix}$

In Exercises 21–26,

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & 3 \\ 4 & 0 \end{bmatrix}.$$

Determine the following.

21. $2B$ **22.** $-3B$
23. $2B + 3C$ **24.** $2B + 3A$
25. $3B - 2C$ **26.** $4C - 2A$

In Exercises 27–32, determine A × B.

27. $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 \\ 8 & 4 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & -1 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 \\ -3 & -2 \end{bmatrix}$

29. $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

31. $A = \begin{bmatrix} 4 & 7 & 6 \\ -2 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

32. $A = \begin{bmatrix} -3 & 1 \\ 2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ 1 & 6 \end{bmatrix}$

In Exercises 33–38, determine A + B and A × B. If an operation cannot be performed, explain why.

33. $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 & 3 \\ 2 & -2 & 1 \end{bmatrix}$

34. $A = \begin{bmatrix} 6 & 4 & -1 \\ 2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$

35. $A = \begin{bmatrix} 4 & 5 & 3 \\ 6 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ -2 & 0 \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

37. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

38. $A = \begin{bmatrix} 5 & -1 \\ 6 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

In Exercises 39–41, show the commutative property of addition, A + B = B + A, holds for matrices A and B.

39. $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$

40. $A = \begin{bmatrix} 9 & 4 \\ 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix}$

41. $A = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 \\ 3 & -4 \end{bmatrix}$

42. Make up two matrices with the same dimensions, A and B, and show that $A + B = B + A$.

In Exercises 43–45, show that the associative property of addition, $(A + B) + C = A + (B + C)$, holds for the matrices given.

43. $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 4 \\ 5 & 0 \end{bmatrix}$

44. $A = \begin{bmatrix} 4 & 1 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -9 & 1 \\ -7 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -6 & -3 \\ 3 & 6 \end{bmatrix}$

45. $A = \begin{bmatrix} 7 & 4 \\ 9 & -36 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ -1 & -4 \end{bmatrix}$, $C = \begin{bmatrix} -7 & -5 \\ -1 & 3 \end{bmatrix}$

46. Make up three matrices with the same dimensions, A, B, and C, and show that $(A + B) + C = A + (B + C)$.

In Exercises 47–51, determine whether the commutative property of multiplication, $A \times B = B \times A$, holds for the matrices given.

47. $A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$

48. $A = \begin{bmatrix} 3 & 1 \\ 6 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

49. $A = \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$

50. $A = \begin{bmatrix} -3 & 2 \\ 6 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{5}{3} & -\frac{2}{3} \\ -2 & -1 \end{bmatrix}$

51. $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 0 \\ 0 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

52. Make up two square matrices A and B with the same dimensions, and determine whether $A \times B = B \times A$.

In Exercises 53–57, show that the associative property of multiplication, $(A \times B) \times C = A \times (B \times C)$, holds for the matrices given.

53. $A = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$

54. $A = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

55. $A = \begin{bmatrix} 4 & 3 \\ -6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 3 \\ 0 & -2 \end{bmatrix}$

56. $A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

57. $A = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$

58. Make up three matrices, A, B, and C, and show that $(A \times B) \times C = A \times (B \times C)$.

Problem Solving

59. MODELING - *Cookie Company Costs* The Original Cookie Factory bakes and sells four types of cookies: chocolate chip, sugar, molasses, and peanut butter. Matrix A shows the number of units of various ingredients used in baking a dozen of each type of cookie.

$$A = \begin{array}{c} \begin{array}{cccc} \text{Sugar} & \text{Flour} & \text{Milk} & \text{Eggs} \end{array} \\ \begin{bmatrix} 2 & 2 & \frac{1}{2} & 1 \\ 3 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \text{Chocolate chip} \\ \text{Sugar} \\ \text{Molasses} \\ \text{Peanut butter} \end{array}$$

The cost, in cents per cup or per egg, for each ingredient when purchased in small quantities and in large quantities is given in matrix B.

$$B = \begin{array}{c} \begin{array}{cc} \text{Large quantities} & \text{Small quantities} \end{array} \\ \begin{bmatrix} 10 & 12 \\ 5 & 8 \\ 8 & 8 \\ 4 & 6 \end{bmatrix} \end{array} \begin{array}{l} \text{Sugar} \\ \text{Flour} \\ \text{Milk} \\ \text{Eggs} \end{array}$$

Use matrix multiplication to find a matrix representing the comparative cost per item for small and large quantities purchased.

In Exercises 60 and 61, use the information given in Exercise 59. Suppose a typical day's order consists of 40 dozen chocolate chip cookies, 30 dozen sugar cookies, 12 dozen molasses cookies, and 20 dozen peanut butter cookies.

60. a) Express these orders as a 1×4 matrix.
b) Use matrix multiplication to determine the amount of each ingredient needed to fill the day's order.

61. Use matrix multiplication to determine the cost under the two purchase options (small and large quantities) to fill the day's order.

62. MODELING - *Food Prices* To raise money for a local charity, the Spanish Club at Montclair High School sold hot dogs, soft drinks, and candy bars for 3 days in the student lounge. The sales for the 3 days are summarized in matrix A.

$$A = \begin{array}{c} \begin{array}{ccc} \text{Hot dogs} & \text{Soft drinks} & \text{Candy bars} \end{array} \\ \begin{bmatrix} 52 & 50 & 75 \\ 48 & 43 & 60 \\ 62 & 57 & 81 \end{bmatrix} \end{array} \begin{array}{l} \text{Day 1} \\ \text{Day 2} \\ \text{Day 3} \end{array}$$

The cost and revenue (in dollars) for hot dogs, soft drinks, and candy are summarized in matrix B.

$$B = \begin{array}{c} \begin{array}{cc} \text{Cost} & \text{Revenue} \end{array} \\ \begin{bmatrix} 0.30 & 0.75 \\ 0.25 & 0.50 \\ 0.15 & 0.45 \end{bmatrix} \end{array} \begin{array}{l} \text{Hot dogs} \\ \text{Soft drinks} \\ \text{Candy bars} \end{array}$$

Multiply the two matrices to form a 3×2 matrix that shows the total cost and revenue for each item.

In Exercises 63 and 64, there are many acceptable answers.

63. a) Construct two matrices A and B whose product is a 3×1 matrix. Explain how you determined your answer.
b) For your matrices, determine $A \times B$.

64. a) Construct two matrices A and B whose product is a 4×1 matrix. Explain how you determined your answer.
b) For your matrices, determine $A \times B$.

Two matrices whose product is the multiplicative identity matrix are said to be **multiplicative inverses**. *That is, if $A \times B = B \times A = I$, where I is the multiplicative identity matrix, then A and B are multiplicative inverses. In Exercises 65 and 66, determine whether A and B are multiplicative inverses.*

65. $A = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

66. $A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$

Challenge Problems/Group Activities

In Exercises 67 and 68, determine whether the statement is true or false. Give an example to support your answer.

67. $A - B = B - A$, where A and B are any matrices.

68. For scalar a and matrices B and C,

$a(B + C) = aB + aC$.

69. MODELING - *Sofa Manufacturing Costs* The number of hours of labor required to manufacture one sofa of various sizes is summarized in matrix L.

$$L = \begin{array}{c} \overbrace{\begin{array}{ccc} \text{Cutting} & \text{Assembly} & \text{Packing} \end{array}}^{\text{Department}} \\ \begin{bmatrix} 1.4\text{ hr} & 0.7\text{ hr} & 0.3\text{ hr} \\ 1.8\text{ hr} & 1.4\text{ hr} & 0.3\text{ hr} \\ 2.7\text{ hr} & 2.8\text{ hr} & 0.5\text{ hr} \end{bmatrix} \end{array} \left.\begin{array}{l} \text{Small} \\ \text{Medium} \\ \text{Large} \end{array}\right\} \text{Sofa size}$$

The hourly labor rates for cutting, assembly, and packing at the Ames City Plant and at the Bay City Plant are given in matrix C.

$$C = \begin{array}{c} \overbrace{\begin{array}{cc} \text{Ames City} & \text{Bay City} \end{array}}^{\text{Plant}} \\ \begin{bmatrix} \$14 & \$12 \\ \$10 & \$9 \\ \$7 & \$5 \end{bmatrix} \end{array} \left.\begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array}\right\} \text{Department}$$

a) What is the total labor cost for manufacturing a small-sized sofa at the Ames City plant?
b) What is the total cost for manufacturing a large-sized sofa at the Bay City plant?
c) Determine the product $L \times C$ and explain the meaning of the results.

70. Is it possible that two matrices could be added but not multiplied? If so, give an example.

71. Is it possible that two matrices could be multiplied but not added? If so, give an example.

Recreational Mathematics

72. Make up two matrices A and B such that

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A \times B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Internet/Research Activities

73. Find an article that shows information illustrated in matrix form. Write a short paper explaining how to interpret the information provided by the matrix. Include the article with your report.

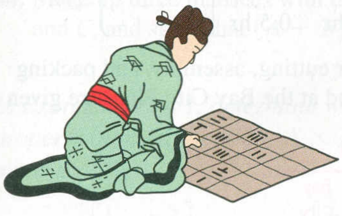
74. *Messages* The study of encoding and decoding messages is called *cryptography*. Do research on current real-life uses of cryptography and write a paper on how matrix multiplication is used to encode and decode messages. In your paper, include current real-life uses of cryptography.

75. ***Graphing Calculator*** Some graphing calculators are able to perform matrix operations such as addition, subtraction, and multiplication. Read the instruction manual for your graphing calculator and then use your graphing calculator to evaluate Exercises 13, 15, 17, and 19 and Exercises 33, 35, and 37. Verify that your answers are the same as the answers you obtained without the calculator.

7.4 SOLVING SYSTEMS OF EQUATIONS BY USING MATRICES

DID YOU KNOW

Early Matrices



The earliest known use of a matrix to solve linear equations appeared in the ancient Chinese mathematical classic *Jiuzhang Suanshu (Nine Chapters on the Mathematical Art)* in about 200 B.C. The use of matrices to solve problems did not appear in the West until the nineteenth century. Perhaps the fact that the Chinese used a counting board that took the form of a grid made it easier for them to make the leap to the development and use of matrices. The image above shows the same type of counting board as appeared in a book dated 1795.

In Section 7.3, we introduced matrices. Now we will discuss the procedure to solve a system of linear equations using matrices. We will illustrate how to solve a system of two equations and two unknowns. Systems of equations containing three equations and three unknowns (called third-order systems) and higher-order systems can also be solved by using matrices.

The first step in solving a system of equations using matrices is to represent the system of equations with an ***augmented matrix***. An augmented matrix consists of two smaller matrices, one for the coefficients of the variables in the equations and one for the constants in the equations. To determine the augmented matrix, first write each equation in standard form, $ax + by = c$. For the system of equations below, its augmented matrix is shown to its right.

System of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Augmented matrix

$$\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$$

Following is another example.

System of equations

$$x + 2y = 8$$
$$3x - y = 7$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -1 & 7 \end{array}\right]$$

Note that the bar in the augmented matrix separates the numerical coefficients from the constants. The matrix is just a shortened way of writing the system of equations. Thus, we can solve a system of equations by using matrices in a manner very similar to solving a system of equations with the addition method.

To solve a system of equations by using matrices, we use *row transformations* to obtain new matrices that have the same solution as the original system. We will discuss three row transformation procedures.

Procedures for Row Transformations

1. Any two rows of a matrix may be interchanged (which is the same as interchanging any two equations in the system of equations).
2. All the numbers in any row may be multiplied by any nonzero real number (which is the same as multiplying both sides of an equation by any nonzero real number).
3. All the numbers in any row may be multiplied by any nonzero real number, and these products may be added to the corresponding numbers in any other row of numbers.

We use row transformations to obtain an augmented matrix whose numbers to the left of the vertical bar are the same as in the *multiplicative identity matrix*. From this type of augmented matrix, we can determine the solution to the system of equations. For example, if we get

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

it tells us that $1x + 0y = 3$ or $x = 3$, and $0x + 1y = -2$ or $y = -2$. Thus, the solution to the system of equations that yielded this augmented matrix is $(3, -2)$. Now let's work an example.

EXAMPLE 1 *Using Row Transformations*

Solve the following system of equations by using matrices.

$$x + 2y = 5$$
$$3x - y = 8$$

SOLUTION: First we write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 8 \end{array}\right]$$

Our goal is to obtain a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{array}\right]$$

where c_1 and c_2 may represent any real numbers. It is generally easier to work by columns. Therefore, we will try to get the first column of the augmented matrix to be $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ and the second column to be $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$. Since the element in the top left position is already a 1, we must work to change the 3 in the first column, second row, into a 0. We use row transformation procedure 3 to change the 3 into a 0. If we multiply the top row of numbers by -3 and add these products to the second row of numbers, the element in the first column, second row will become a 0:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 8 \end{array}\right] \quad \text{Original augmented matrix}$$

The top row of numbers multiplied by -3 gives

$$1(-3), \quad 2(-3), \quad \text{and} \quad 5(-3)$$

Now add these products to their respective numbers in row 2.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 + 1(-3) & -1 + 2(-3) & 8 + 5(-3) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -7 \end{array}\right]$$

MATHEMATICS *Everywhere*

Matrices and Computer Graphics

Woody from *Toy Story*

Did you ever wonder how the characters in computer-generated movies such as *Shrek* and *Toy Story* are created? To create the graphics for computer-generated films, an object is first represented as a collection of geometric figures. Each geometric figure is then represented as points in the Cartesian coordinate plane. The coordinates of the points representing the object are written in columns in a matrix. Matrix operations are used if an object needs to be transformed by scaling, stretching, reflecting, or translating (see transformational geometry in Section 9.5). For example, matrix multiplication is used to translate the character Woody, from the movies *Toy Story* and *Toy Story 2*, from one position to another.

To add a shadow to an object, graphic artists perform several transformations on the matrix representing the object. Each transformation is created using matrix multiplication. The computer graphics used to create special effects in video games and used in areas such as medical imaging, architectural engineering, and weather forecasting are all based on a mathematical object called a matrix.

The next step is to obtain a 1 in the second column, second row. At present, -7 is in this position. To change the -7 to a 1, we use row transformation procedure 2. If we multiply -7 by $-\frac{1}{7}$, the product will be 1. Therefore, we multiply all the numbers in the second row by $-\frac{1}{7}$ to get

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0(-\frac{1}{7}) & -7(-\frac{1}{7}) & -7(-\frac{1}{7}) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

The next step is to obtain a 0 in the second column, first row. At present, a 2 is in this position. Multiplying the numbers in the second row by -2 and adding the products to the corresponding numbers in the first row gives a 0 in the desired position.

$$\left[\begin{array}{cc|c} 1+0(-2) & 2+1(-2) & 5+1(-2) \\ 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

We now have the desired augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

With this matrix, we see that $1x + 0y = 3$, or $x = 3$, and $0x + 1y = 1$, or $y = 1$. The solution to the system is $(3, 1)$.

Check:

$$\begin{aligned} x + 2y &= 5 \\ 3 + 2(1) &= 5 \\ 5 &= 5 \quad \text{True} \end{aligned} \qquad \begin{aligned} 3x - y &= 8 \\ 3(3) - 1 &= 8 \\ 8 &= 8 \quad \text{True} \end{aligned}$$

▲

Now we give a general procedure to change an augmented matrix to the desired form.

To Change an Augmented Matrix to the Form $\left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{array}\right]$

Use row transformations to:

1. Change the element in the first column, first row, to a 1.
2. Change the element in the first column, second row, to a 0.
3. Change the element in the second column, second row, to a 1.
4. Change the element in the second column, first row, to a 0.

Generally, when changing an element in the augmented matrix to a 1, *we use step* 2 *in the row transformation box* *on page 412*. *When changing an element to a* 0, *we use step* 3 *in the row transformation box.*

EXAMPLE 2 *Using Matrices to Solve a System of Equations*

Solve the following system of equations using matrices.

$$\begin{aligned} 2x + 4y &= 6 \\ 4x - 2y &= -8 \end{aligned}$$

SOLUTION: First write the augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 4 & -2 & -8 \end{array}\right]$$

To obtain a 1 in the first column, first row, multiply the numbers in the first row by $\frac{1}{2}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & -2 & -8 \end{array}\right]$$

To obtain a 0 in the first column, second row, multiply the numbers in the first row by -4 and add the products to the corresponding numbers in the second row.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4+1(-4) & -2+2(-4) & -8+3(-4) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -10 & -20 \end{array}\right]$$

To obtain a 1 in the second column, second row, multiply the numbers in the second row by $-\frac{1}{10}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0(-\frac{1}{10}) & -10(-\frac{1}{10}) & -20(-\frac{1}{10}) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

To obtain a 0 in the second column, first row, multiply the numbers in the second row by -2 and add the products to the corresponding numbers in the first row.

$$\left[\begin{array}{cc|c} 1+(0)(-2) & 2+(1)(-2) & 3+(2)(-2) \\ 0 & 1 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

The solution to the system of equations is $(-1, 2)$.

Inconsistent and Dependent Systems

Assume that you solve a system of two equations and obtain an augmented matrix in which one row of numbers on the left side of the vertical line are all zeroes but a zero does not appear in the same row on the right side of the vertical line. This situation indicates that the system is inconsistent and has no solution. For example, a system of equations that yields the following augmented matrix is an inconsistent system.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 4 \end{array}\right] \quad \text{Inconsistent system}$$

The second row of the matrix represents the equation

$$0x + 0y = 4 \qquad \text{or} \qquad 0 = 4$$

which is never true.

If you obtain a matrix in which a 0 appears across an entire row, the system of equations is dependent. For example, a system of equations that yields the following matrix is a dependent system.

$$\left[\begin{array}{cc|c} 1 & 5 & -6 \\ 0 & 0 & 0 \end{array}\right] \quad \text{Dependent system}$$

The second row of the matrix represents the equation

$$0x + 0y = 0 \qquad \text{or} \qquad 0 = 0$$

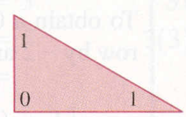
which is always true.

Triangularization Method

Another procedure to solve a system of two equations is to use row transformation procedures to obtain an augmented matrix of the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right]$$

where *a, b*, and *c* represent real numbers. This procedure is called the ***triangularization method***, because the ones and zeroes form a triangle.



When the matrix is in this form, we can write the following system of equations.

$$\begin{aligned} 1x + ay &= b \\ 0x + 1y &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + ay &= b \\ y &= c \end{aligned}$$

Using substitution, we can solve the system.

For example, in Example 2, in the process of solving the system we obtained the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

This matrix represents the following system of equations.

$$\begin{aligned} x + 2y &= 3 \\ y &= 2 \end{aligned}$$

To solve for x, we substitute 2 for y in the equation

$$\begin{aligned} x + 2y &= 3 \\ x + 2(2) &= 3 \\ x + 4 &= 3 \\ x &= -1 \end{aligned}$$

Thus, the solution to the system is $(-1, 2)$, as was obtained in Example 2. You may use either method when solving a system of equations with matrices unless your instructor specifies otherwise.

SECTION 7.4 EXERCISES

Concept/Writing Exercises

1. a) What is an augmented matrix?
   b) Determine the augmented matrix for the following system.

$$\begin{aligned} x + 3y &= 7 \\ 2x - y &= 4 \end{aligned}$$

2. In your own words, write the three row transformation procedures.
3. How will you know, when solving a system of equations by using matrices, whether the system is inconsistent?
4. How will you know, when solving a system of equations by using matrices, whether the system is dependent?
5. If you obtained the following augmented matrix when solving a system of equations, what would be your next step in completing the process? Explain your answer.

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2 & 1 \end{array}\right]$$

6. If you obtained the following augmented matrix when solving a system of equations, what would be your next step in completing the process? Explain your answer.

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

Practice the Skills

In Exercises 7–20, use matrices to solve the system of equations.

7. $x + 3y = 3$
$-x + y = -3$

8. $x - y = 5$
$2x - y = 6$

9. $x - 2y = -1$
$2x + y = 8$

10. $x + y = -1$
$2x + 3y = -5$

11. $2x - 5y = -6$
$-4x + 10y = 12$

12. $x + y = 5$
$3x - y = 3$

13. $2x - 3y = 10$
$2x + 2y = 5$

14. $x + 3y = 1$
$-2x + y = 5$

15. $4x + 2y = -10$
$-2x + y = -7$

16. $4x + 2y = 6$
$5x + 4y = 9$

17. $-3x + 6y = 5$
$2x - 4y = 8$

18. $2x - 5y = 10$
$3x + y = 15$

19. $2x + y = 11$
$x + 3y = 18$

20. $4x - 3y = 7$
$-2x + 5y = 14$

Problem Solving

In Exercises 21–24, use matrices to solve the problem.

21. **MODELING** - *Selling Flags* Michael's Arts and Crafts sells small flags for \$4 and large flags for \$6. Recently, they sold a total of 55 flags in a single day. If flag receipts for the day totaled \$290, how many of each type of flag were sold?

22. **MODELING** - *TV Dimensions* Vita Gunta just purchased a new high-definition television. She noticed that the perimeter of the screen is 124 in. The width of the screen is 8 in. greater than its height. Find the dimensions of the screen.

23. **MODELING** - *On the Job* Peoplepower, Inc., a daily employment agency, charges \$10 per hour for a truck driver and \$8 per hour for a laborer. On a certain job, the laborer worked two more hours than the truck driver, and together they cost \$144. How many hours did each work?

24. **MODELING** - *Sweets* If Meg Cohn buys 2 lb of chocolate-covered cherries and 3 lb of chocolate-covered mints, her total cost is \$23. If she buys 1 lb of chocolate-covered cherries and 2 lb of chocolate-covered mints, her total cost is \$14. Find the cost of 1 lb of chocolate-covered cherries and 1 lb of chocolate-covered mints.

Recreational Mathematics

25. **MODELING** - *Fill in the Missing Information* Pencil World sells two types of mechanical pencils to stationery stores. The nonrefillable pencil sells for \$1.50 each, and the refillable pencil sells for \$2.00 each. Pencil World received an order for 200 pencils and a check for \$337.50 for the pencils. When placing the order, the stationery store clerk failed to specify the number of each type of pencil being ordered. Can Pencil World fill the order with the information given? If so, determine the number of nonrefillable and the number of refillable pencils the clerk ordered.

Internet/Research Activities

26. Do research and write a paper on the development of matrices. In your paper, cover the contributions of James Joseph Sylvester, Arthur Cayley, and William Rowan Hamilton. (References include history of mathematics books, encyclopedias, and the Internet.)

7.5 SYSTEMS OF LINEAR INEQUALITIES

In earlier sections, we showed how to find the solution to a system of linear equations in two variables. Now we are going to explore the techniques of finding the solution set to a system of linear inequalities in two variables.

The solution set of a system of linear inequalities is the set of points that satisfy all inequalities in the system. The solution set of a system of linear inequalities may consist of infinitely many ordered pairs. To determine the solution set to a system of linear inequalities, graph each inequality on the same axes. The ordered pairs common to all the inequalities are the solution set to the system.

Procedure for Solving a System of Linear Inequalities

1. Select one of the inequalities. Replace the inequality symbol with an equal sign and draw the graph of the equation. Draw the graph with a dashed line if the inequality is $<$ or $>$ and with a solid line if the inequality is $\leq$ or $\geq$.
2. Select a test point on one side of the line and determine whether the point is a solution to the inequality. If so, shade the area on the side of the line containing the point. If the point is not a solution, shade the area on the other side of the line.
3. Repeat steps 1 and 2 for the other inequality.
4. The intersection of the two shaded areas and any solid line common to both inequalities form the solution set to the system of inequalities.

EXAMPLE 1 *Solving a System of Inequalities*

Graph the following system of inequalities and indicate the solution set.

$$x + y < 2$$
$$x - y < 4$$

SOLUTION: Graph both inequalities on the same axes. First draw the graph of $x + y < 2$. When drawing the graph, remember to use a dashed line, since the inequality is "less than" (see Fig. 7.10a on page 419). If you have forgotten how to graph inequalities, review Section 6.8.

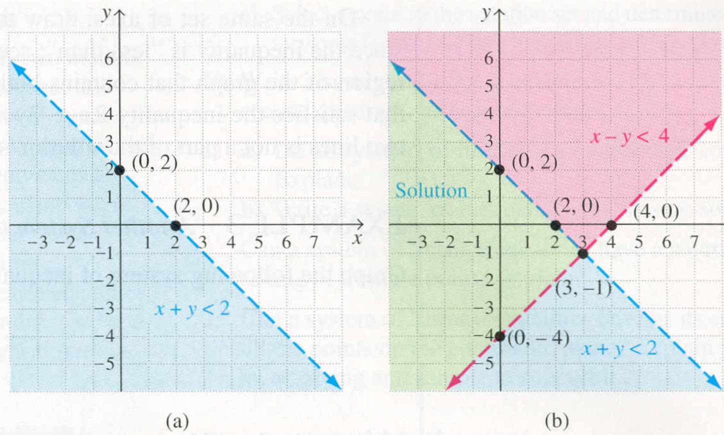


Figure 7.10

Now, on the same axes, shade the half-plane determined by the inequality $x - y < 4$ (see Fig. 7.10b). The solution set consists of all the points common to the two shaded half-planes. These are the points in the region on the graph containing both color shadings. Figure 7.10(b) shows that the two lines intersect at $(3, -1)$. This ordered pair can also be found by any of the algebraic methods discussed in Sections 7.2 and 7.3.

EXAMPLE 2 *Solving a System of Linear Inequalities*

Graph the following system of inequalities and indicate the solution set.

$$4x - 2y \geq 8$$
$$2x + 3y < 6$$

SOLUTION: Graph the inequality $4x - 2y \geq 8$. Remember to use a solid line, because the inequality is "greater than or equal to"; see Fig. 7.11(a).

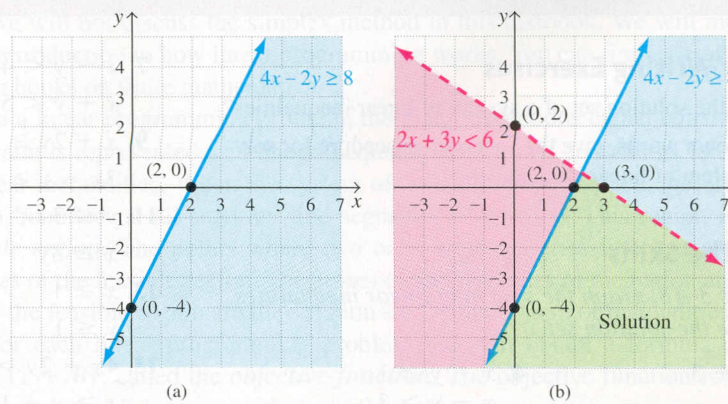


Figure 7.11

On the same set of axes, draw the graph of $2x + 3y < 6$. Use a dashed line, since the inequality is "less than"; see Fig. 7.11(b) on page 419. The solution is the region of the graph that contains both color shadings and the part of the solid line that satisfies the inequality $2x + 3y < 6$. Note that the point of intersection of the two lines is not a part of the solution set.

EXAMPLE 3 ***Another System of Inequalities***

Graph the following system of inequalities and indicate the solution set.

$$x \geq -2$$
$$y < 3$$

SOLUTION: Graph the inequality $x \geq -2$; see Fig. 7.12(a). On the same axes, graph the inequality $y < 3$; see Fig. 7.12(b). The solution set is that region of the graph that is shaded in both colors and the part of the solid line that satisfies the inequality $y < 3$. The point of intersection of the two lines, $(-2, 3)$, is not part of the solution because it does not satisfy the inequality $y < 3$.

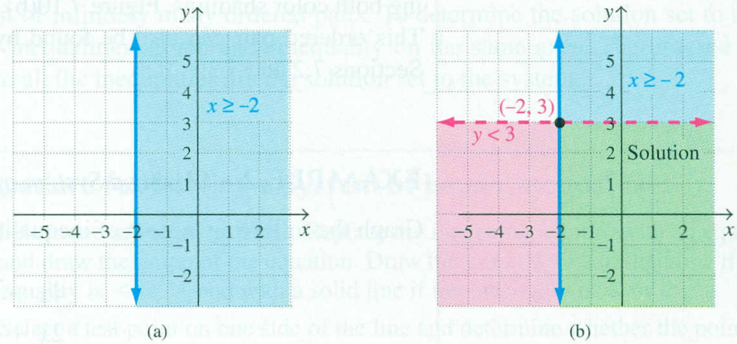


Figure 7.12

SECTION 7.5 EXERCISES

Concept/Writing Exercises

1. What is the solution set of a system of linear inequalities?
2. In your own words, give the four-step procedure for solving a system of linear inequalities.

Practice the Skills

In Exercises 3–18, graph the system of linear inequalities and indicate the solution set.

3. $y > x + 3$
   $y > 2x$
4. $x + y \geq 1$
   $x - y > 3$
5. $y \leq x - 4$
   $y < -2x + 4$
6. $x + y < 4$
   $3x + 2y \geq 6$
7. $x - y < 4$
   $x + y < 5$
8. $3x - y \leq 6$
   $x - y > 4$
9. $x + 2y \geq 4$
   $3x - y \geq -6$
10. $x - 3y \leq 3$
    $x + 2y \geq 4$
11. $y \leq 3x$
    $x \geq 3y$
12. $y \leq 4$
    $x - y < 1$
13. $x \geq 1$
    $y \leq 1$
14. $x \leq 0$
    $y \leq 0$
15. $4x + 2y > 8$
    $x \geq y - 1$
16. $5y > 3x + 10$
    $3y < -2x - 3$
17. $3x \geq 2y + 10$
    $x \leq y + 8$
18. $3x - 5y < 7$
    $x > 2y + 1$

Challenge Problems/Group Activities

19. **MODELING** - *Camcorder Sales* Best Buy sells two models of a certain brand of camcorder. Based on demand, it is necessary to stock at least twice as many units of Panasonic as Sony. The costs to the store for the two models are \$600 and \$900, respectively. The management wants at least 10 Panasonic camcorders and five Sony camcorders in inventory at all times and does not want more than \$18,000 in camcorder inventory at any one time.
   a) Translate the problem into a system of linear inequalities.
   b) Solve the system graphically. Graph inventory for Panasonic on the horizontal axis and inventory for Sony on the vertical axis.
   c) Select a point in the solution set and determine the inventory cost for the two models that corresponds to that point.

20. Write a system of linear inequalities whose solution is the second quadrant, including the axes.
21. a) Do all systems of linear inequalities have solutions? Explain.
   b) Write a system of inequalities that has no solution.
22. Can a system of linear inequalities have a solution set consisting of a single point? Explain.
23. Can a system of linear inequalities have as its solution set all the points on the coordinate plane? Explain your answer, giving an example to support it.

Recreational Mathematics

24. Write a system of linear inequalities that has the ordered pair (0, 0) as its only solution. There are many possible answers.
25. Write a system of linear inequalities that has the following ordered pairs as some of its solutions. There are many possible answers.

$$\ldots(-3, -3), (-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), (3, 3), \ldots$$

7.6 LINEAR PROGRAMMING

Government, business, and industry often require decision makers to find cost-effective solutions to a variety of problems. Linear programming often serves as a method of expressing the relationships in many of these problems and uses systems of linear inequalities.

The typical linear programming problem has many variables and is generally so lengthy that it is solved on a computer by a technique called the ***simplex method***. The simplex method was developed in the 1940s by George B. Dantzig. Linear programming is used to solve problems in the social sciences, health care, land development, nutrition, military, and many other fields.

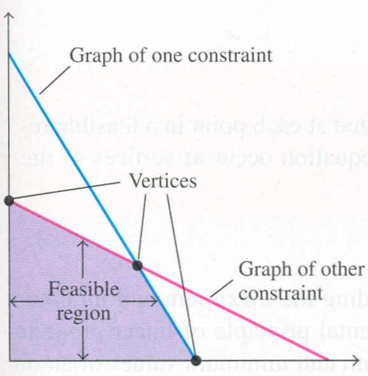


Figure 7.13

We will not discuss the simplex method in this textbook. We will merely give a brief introduction to how linear programming works. You can find a detailed explanation in books on finite mathematics.

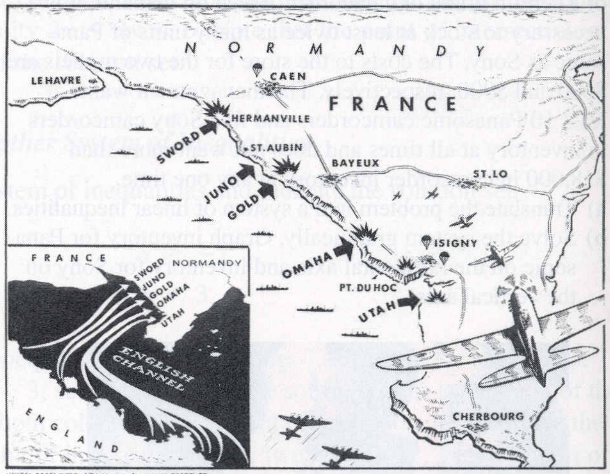
In a linear programming problem, there are restrictions called ***constraints***. Each constraint is represented as a linear inequality. The list of constraints forms a system of linear inequalities. When the system of inequalities is graphed, we often obtain a region bounded on all sides by line segments (Fig. 7.13). This region is called the ***feasible region***. The points where two or more boundaries intersect are called the ***vertices*** of the feasible region. The points on the boundary of the region and the points inside the feasible region are the solution set for the system of inequalities.

For each linear programming problem, we will obtain a formula of the form $K = Ax + By$, called the ***objective function***. The objective function is the formula for the quantity K (or some other variable) that we want to maximize or minimize. The values we substitute for x and y determine the value of K. From the information given in the problem, we determine the real number constants A and B. In a particular

DID YOU KNOW

The Logistics of D-Day

Linear programming was first used to deal with the age-old military problem of logistics: obtaining, maintaining, and transporting military equipment and personnel. George Dantzig developed the simplex method for the Allies of World War II to do just that. Consider the logistics of the Allied invasion of Normandy. Meteorologic experts had settled on three possible dates in June 1944. It had to be a day when low tide and first light would coincide, when the winds should not exceed 8 to 13 mph, and when visibility was not less than 3 miles. A force of 170,000 assault troops was to be assembled and moved to 22 airfields in England where 1200 air transports and 700 gliders would then take them to the coast of France to converge with 5000 ships of the D-day armada. The code name for the invasion was Operation Overlord, but it is known to most as D-day.



Winston Churchill called the invasion of Normandy "the most difficult and complicated operation that has ever taken place."

linear programming problem, a typical equation that might be used to find the maximum profit, P, is $P = 3x + 7y$. We would find the maximum profit by substituting the ordered pairs (x, y) of the vertices of the feasible region into the formula $P = 3x + 7y$ to see which ordered pair yields the greatest value of P and therefore the maximum profit. The ordered pair that yields the smallest value of P determines the minimum profit.

Linear programming is used to determine which ordered pair will yield the maximum (or minimum) value of the variable that is being maximized (or minimized). The fundamental principle of linear programming provides a rule for finding those maximum and minimum values.

Fundamental Principle of Linear Programming

If the objective function, $K = Ax + By$, is evaluated at each point in a feasible region, the maximum and minimum values of the equation occur at vertices of the region.

Linear programming is a powerful tool for finding the maximum and minimum values of an objective function. Using the fundamental principle of linear programming, we are quickly able to determine the maximum and minimum values of an objective function by using just a few of many points in the feasible region.

Example 1 illustrates how the fundamental principle is used to solve a linear programming problem.

EXAMPLE 1 **MODELING** - *Using the Fundamental Principle of Linear Programming*

The Ric Shaw Chair company makes two types of rocking chairs, a plain chair and a fancy chair. Each rocking chair must be assembled and then finished. The plain chair takes 4 hours to assemble and 4 hours to finish. The fancy chair takes 8 hours to assemble and 12 hours to finish. The company can provide at most 160 worker-hours of assembling and 180 worker-hours of finishing a day. If the profit on a plain chair is \$25 and the profit on a fancy chair is \$40, how many rocking chairs of each type should the company make per day to maximize profits? What is the maximum profit?

SOLUTION: From the information given, we know the following facts.

	Assembly Time (hr)	Finishing Time (hr)	Profit (\$)
Plain chair	4	4	25.00
Fancy chair	8	12	40.00

Let

$$x = \text{the number of plain chairs per day}$$
$$y = \text{the number of fancy chairs per day}$$
$$25x = \text{profit on the plain chairs}$$
$$40y = \text{profit on the fancy chairs}$$
$$P = \text{the total profit}$$

The total profit is the sum of the profit on the plain chairs and the profit on the fancy chairs. Since $25x$ is the profit on the plain chairs and $40y$ is the profit on the fancy chairs, the profit formula is $P = 25x + 40y$.

The maximum profit, P, is dependent on several conditions, called *constraints*. The number of chairs manufactured each day cannot be a negative amount. This condition gives us the constraints $x \geq 0$ and $y \geq 0$. Another constraint is determined by the total number of hours allocated for assembling. Four hours are needed to assemble the plain chair, so the total number of hours per day to assemble x plain chairs is $4x$. Eight hours are required to assemble a fancy chair, so the total number of hours needed to assemble y fancy chairs is $8y$. The maximum number of hours allocated for assembling is 160 per day. Thus, the third constraint is $4x + 8y \leq 160$. The final constraint is determined by the number of hours allotted for finishing. Finishing a plain chair takes 4 hours, or $4x$ hours to finish x plain chairs. Finishing a fancy chair takes 12 hours, or $12y$ hours to finish y fancy chairs. The total number of hours allotted for finishing is 180 per day. Therefore, the fourth constraint is $4x + 12y \leq 180$. Thus, the four constraints are

$$x \geq 0$$
$$y \geq 0$$
$$4x + 8y \leq 160$$
$$4x + 12y \leq 180$$

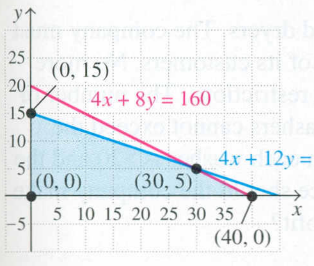


Figure 7.14

The list of constraints is a system of linear inequalities in two variables. The solution to the system of inequalities is the set of ordered pairs that satisfies all the constraints. These points are plotted in Fig. 7.14. Note that the solution to the system consists of the colored region and the solid boundaries. The points (0, 0), (0, 15), (30, 5), and (40, 0) are the points at which the boundaries intersect. These points can also be found by the addition or substitution method described in Section 7.2.

The goal in this example is to maximize the profit. The objective function is given by the profit formula $P = 25x + 40y$. According to the fundamental principle, the maximum profit will be found at one of the vertices of the feasible region. Calculate P for each one of the vertices.

$$\begin{array}{ll} & P = 25x + 40y \\ \text{At } (0, 0), & P = 25(0) + 40(0) = 0 \\ \text{At } (0, 15), & P = 25(0) + 40(15) = 600 \\ \text{At } (30, 5), & P = 25(30) + 40(5) = 950 \\ \text{At } (40, 0), & P = 25(40) + 40(0) = 1000 \end{array}$$

The maximum profit is at (40, 0), which means that the company should manufacture 40 plain rocking chairs and no fancy rocking chairs. The maximum profit would be \$1000. The minimum profit would be at (0, 0), when no rocking chairs of either style were manufactured. ▲

A variation of the problem in Example 1 could be that the company knows that it cannot sell more than 15 plain rocking chairs per day. With this additional constraint, we now have the following set of constraints.

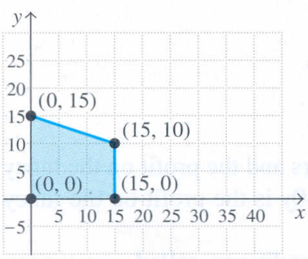


Figure 7.15

$$\begin{aligned} x &\geq 0 \\ x &\leq 15 \\ y &\geq 0 \\ 4x + 8y &\leq 160 \\ 4x + 12y &\leq 180 \end{aligned}$$

The graph of these constraints is shown in Fig. 7.15.

The vertices of the feasible region are (0, 0), (0, 15), (15, 10), and (15, 0). To determine the maximum profit, we calculate P for each of these vertices:

$$\begin{array}{ll} & P = 25x + 40y \\ \text{At } (0, 0), & P = 25(0) + 40(0) = 0 \\ \text{At } (0, 15), & P = 25(0) + 40(15) = 600 \\ \text{At } (15, 10), & P = 25(15) + 40(10) = 775 \\ \text{At } (15, 0), & P = 25(15) + 40(0) = 375 \end{array}$$

This set of constraints gives the maximum profit of \$775 when the company manufactures 15 plain rocking chairs and 10 fancy rocking chairs.

EXAMPLE 2 **MODELING** - *Washers and Dryers, Maximizing Profit*

The Alexander Appliance Company makes washers and dryers. The company must manufacture at least one washer per day to ship to one of its customers. No more than 6 washers can be manufactured due to production restrictions. The number of dryers cannot exceed 7 per day. Also, the number of washers cannot exceed the number of dryers manufactured per day. If the profit on each washer is \$20 and the profit on each dryer is \$30, how many of each appliance should the company make per day to maximize profits? What is the maximum profit?

SOLUTION: Let

$$x = \text{the number of washers manufactured per day}$$
$$y = \text{the number of dryers manufactured per day}$$
$$20x = \text{the profit on washers}$$
$$30x = \text{the profit on dryers}$$
$$P = \text{the total profit}$$

The maximum profit is dependent on several constraints. The number of appliances manufactured each day cannot be a negative amount. This condition gives us the constraints $x \geq 0$ and $y \geq 0$. The company must manufacture at least one washer per day; therefore, $x \geq 1$. No more than 6 washers can be manufactured per day; therefore, $x \leq 6$. No more than 7 dryers can be manufactured per day; therefore, $y \leq 7$. The number of washers cannot exceed the number of dryers manufactured per day; therefore, $x \leq y$. Thus, the six constraints are

$$x \geq 0, y \geq 0, x \geq 1, x \leq 6, y \leq 7, x \leq y$$

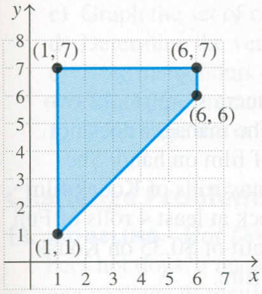


Figure 7.16

Since $20x$ is the profit on x washers and $30y$ is the profit on y dryers, the objective function, the profit formula, is $P = 20x + 30y$. Figure 7.16 shows the feasible region. The feasible region consists of the shaded region and the boundaries. The vertices of the feasible region are the points (1, 1), (1, 7), (6, 7), and (6, 6).

Next we calculate the value of the objective function, P, at each one of the vertices.

$$P = 20x + 30y$$

At (1, 1),	$P = 20(1) + 30(1) = 50$
At (1, 7),	$P = 20(1) + 30(7) = 230$
At (6, 7),	$P = 20(6) + 30(7) = 330$
At (6, 6),	$P = 20(6) + 30(6) = 300$

The maximum profit is at (6, 7). This means the company should manufacture 6 washers and 7 dryers to maximize their profit. The maximum profit is \$330. ▲

Use the following steps to solve a linear programming problem.

Solving a Linear Programming Problem

1. Determine all necessary constraints.
2. Determine the objective function.
3. Graph the constraints and determine the feasible region.
4. Determine the vertices of the feasible region.
5. Determine the value of the objective function at each vertex.

The solution is determined by the vertex that yields the maximum or minimum value of the objective function.

SECTION 7.6 EXERCISES

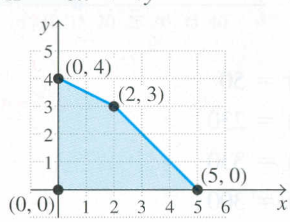
Concept/Writing Exercises

1. What are constraints in a linear programming problem? How are they represented?
2. In a linear programming problem, how is a feasible region formed?
3. What are the points of intersection of the boundaries of the feasible region called?
4. What is the general form of the objective function?
5. In your own words, state the fundamental principle of linear programming.
6. A profit function is $P = 4x + 6y$ and the vertices of the feasible region are (1, 1), (1, 4), (5, 1), and (7, 1). Determine the maximum profit. Explain how you determined your answer.

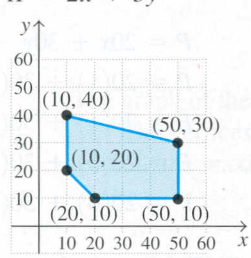
Practice the Skills

Exercises 7 and 8 show a feasible region and its vertices. Find the maximum and minimum values of the given objective function.

7. $K = 6x + 4y$



8. $K = 2x + 3y$



In Exercises 9–14, a set of constraints and a profit formula are given.

a) *Draw the graph of the constraints and find the vertices of the feasible region.*

b) *Use the vertices as obtained in part (a) to determine the maximum and minimum profit.*

9. $x + y \le 5$
$2x + y \le 8$
$x \ge 0$
$y \ge 0$
$P = 5x + 4y$

10. $2x + 3y \le 12$
$2x + y \le 8$
$x \ge 0$
$y \ge 0$
$P = 2x + 4y$

11. $x + y \le 4$
$x + 3y \le 6$
$x \ge 0$
$y \ge 0$
$P = 7x + 6y$

12. $x + y \le 50$
$x + 3y \le 90$
$x \ge 0$
$y \ge 0$
$P = 20x + 40y$

13. $4x + 3y \ge 12$
$3x + 4y \le 36$
$x \ge 2$
$y \le 5$
$y \ge 1$
$P = 2.20x + 1.65y$

14. $x + 2y \le 14$
$7x + 4y \ge 28$
$x \ge 2$
$x \le 10$
$y \ge 1$
$P = 15.13x + 9.35y$

Problem Solving

15. MODELING - *Stocking Film* A camera store stocks two brands of film, Kodak and Fuji. The manager does not want to keep more than 24 rolls of film on hand. She wants to stock at least twice as many rolls of Kodak film as Fuji film. She also wants to stock at least 4 rolls of Fuji film. Assume that she makes a profit of \$0.35 on Kodak film and a profit of \$0.50 on Fuji film.

a) List the constraints.
b) Determine the objective function.
c) Graph the set of constraints.
d) Determine the vertices of the feasible region.
e) How many rolls of each brand of film should she stock to maximize her profit?
f) Determine the maximum profit.

16. MODELING - *On Wheels* The Boards and Blades Company manufactures skateboards and in-line skates. The company can produce a maximum of 20 skateboards and pairs of in-line skates per day. It makes a profit of \$25 on a skateboard and a profit of \$20 on a pair of in-line skates. The company's planners want to make at least 3 skateboards but not more than 6 skateboards per day. To keep customers happy, they must make at least 2 pairs of in-line skates per day.

a) List the constraints.
b) Determine the objective function.
c) Graph the set of constraints.
d) Determine the vertices of the feasible region.
e) How many skateboards and pairs of in-line skates should be made to maximize the profit?
f) Find the maximum profit.

Central Park, New York City

17. **MODELING** - *Paint Production* A paint supplier has two machines that produce both indoor paint and outdoor paint. To meet one of its contractual obligations, the company must produce at least 60 gal of indoor paint and 100 gal of outdoor paint. Machine I makes 3 gal of indoor paint and 10 gal of outdoor paint per hour. Machine II makes 4 gal of indoor paint and 5 gal of outdoor paint per hour. It costs $28 per hour to run machine I and $33 per hour to run machine II.
    - **a)** List the constraints.
    - **b)** Determine the objective function.
    - **c)** Graph the set of constraints.
    - **d)** Determine the vertices of the feasible region.
    - **e)** How many hours should each machine be operated to fulfill the contract at a minimum cost?
    - **f)** Determine the minimum cost.

Challenge Problems/Group Activities

18. **MODELING** - *Hot Dog Profits* To make one package of all-beef hot dogs, a manufacturer uses 1 lb of beef; to make one package of regular hot dogs, the manufacturer uses $\frac{1}{2}$ lb each of beef and pork. The profit on the all-beef hot dogs is 40 cents per pack and the profit on regular hot dogs is 30 cents per pack. If there are 200 lb of beef and 150 lb of pork available, how many packs of all-beef and regular hot dogs should the manufacturer make to maximize the profit? What is the profit?

19. **MODELING** - *Car Seats and Strollers* A company makes car seats and strollers. Each car seat and stroller passes through three processes: assembly, safety testing, and packaging. A car seat requires 1 hr in assembly, 2 hr in safety testing, and 1 hr in packaging. A stroller requires 3 hr in assembly, 1 hr in safety testing, and 1 hr in packaging. Employee work schedules allow for 24 hr per day for assembly, 16 hr per day for safety testing, and 10 hr per day for packaging. The profit for each car seat is $25 and the profit for each stroller is $35. How many units of each type should the company make per day to maximize the profit? What is the maximum profit?

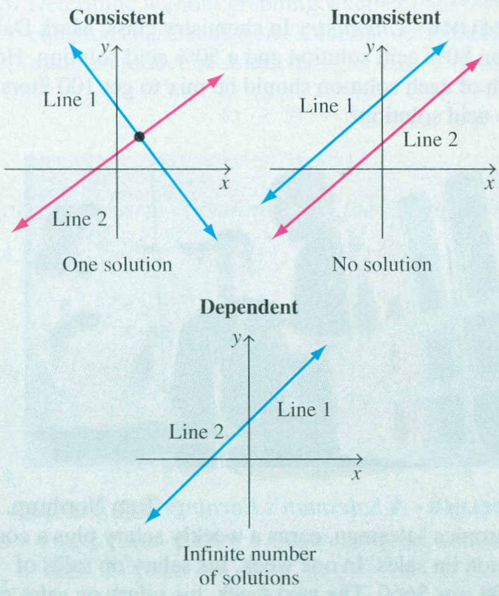
Internet/Research Activity

20. *Operations research* draws on several disciplines, including mathematics, probability theory, statistics, and economics. George Dantzig was one of the key people in developing operations research. Write a paper on Dantzig and his contributions to operations research and linear programming.

CHAPTER 7 SUMMARY

IMPORTANT FACTS

Systems of equations



Methods of solving systems of equations

1. Graphing
2. Substitution
3. Addition (or elimination) method

Multiplicative identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Fundamental principle of linear programming

If the objective function $K = Ax + By$ is evaluated at each point in a feasible region, the maximum and minimum values of the equation occur at vertices of the region.

CHAPTER 7 REVIEW EXERCISES

7.1

In Exercises 1–4, solve the system of equations graphically. If the system does not have a single ordered pair as a solution, state whether the system is inconsistent or dependent.

1. $x = 3$
$y = 4$

2. $2x + y = 5$
$-3x + y = 5$

3. $x = 3$
$x + y = 5$

4. $x + 2y = 5$
$2x + 4y = 4$

In Exercises 5–8, determine without graphing whether the system of equations has exactly one solution, no solution, or an infinite number of solutions.

5. $y = \frac{2}{3}x + 5$
$3y - 2x = 15$

6. $2y - 4x = 12$
$2y = 4x + 15$

7. $6y - 2x = 20$
$4y + 2x = 10$

8. $2x - 4y = 8$
$-2x + y = 6$

7.2

In Exercises 9–12, solve the system of equations by the substitution method. If the system does not have a single ordered pair as a solution, state whether the system is inconsistent or dependent.

9. $-x + y = 12$
$x + 2y = -3$

10. $x - 2y = 9$
$y = 2x - 3$

11. $2x - y = 4$
$3x - y = 2$

12. $3x + y = 1$
$3y = -9x - 4$

In Exercises 13–18, solve the system of equations by the addition method. If the system does not have a single ordered pair as a solution, state whether the system is inconsistent or dependent.

13. $x - 2y = 1$
$2x + y = 7$

14. $2x + y = 2$
$-3x - y = 5$

15. $x + y = 2$
$x + 3y = -2$

16. $4x - 8y = 16$
$x - 2y = 4$

17. $3x + 5y = 15$
$2x + 4y = 0$

18. $3x + 4y = 6$
$2x - 3y = 4$

7.3

Given $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & -5 \\ 6 & 3 \end{bmatrix}$, *determine the following.*

19. $A + B$

20. $A - B$

21. $2A$

22. $2A - 3B$

23. $A \times B$

24. $B \times A$

7.4

In Exercises 25–30, use matrices to solve the system of equations.

25. $x + 2y = 6$
$x + y = 4$

26. $-x + y = 4$
$x + 2y = 2$

27. $2x + y = 3$
$3x - y = 12$

28. $2x + 3y = 2$
$4x - 9y = 4$

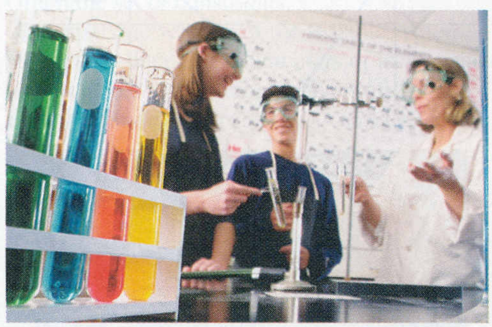
29. $x + 3y = 3$
$3x - 2y = 2$

30. $3x - 6y = -9$
$4x + 5y = 14$

7.1–7.4

31. **MODELING - *Borrowing Money*** A company borrows \$600,000 for 1 year to expand its product line. Some of the money was borrowed at an 8% simple interest rate and the rest of the money was borrowed at a 10% simple interest rate. How much money was borrowed at each rate if the annual interest was \$53,000?

32. **MODELING - *Chemistry*** In chemistry class, Mark Damon has an 80% acid solution and a 50% acid solution. How much of each solution should he mix to get 100 liters of a 75% acid solution?



33. **MODELING - *A Salesman's Earnings*** Tom Northrup, an electronics salesman, earns a weekly salary plus a commission on sales. In one week, his salary on sales of \$4000 was \$660. The next week, his salary on sales of

$6000 was $740. Determine his weekly salary and his commission rate.

34. **MODELING** - *Cool Air* Emily Richelieu needs to purchase a new air conditioner for the office. Model 1600A costs $950 to purchase and $32 per month to operate. Model 6070B, a more efficient unit, costs $1275 to purchase and $22 per month to operate.

a) After how many months will the total cost of both units be equal?

b) Which model will be the more cost effective if the life of both units is guaranteed for 10 years?

35. **MODELING** - *Minimizing Parking Costs* The cost of parking in All-Day parking lot is $5 for the first hour and $0.50 for each additional hour. Sav-a-Lot parking lot costs $4.25 for the first hour and $0.75 for each additional hour.

a) In how many hours would the total cost of parking at All-Day and Sav-a-Lot be the same?

b) If Mark McMahon needed to park his car for 5 hr, which parking lot would be less expensive?

7.5

In Exercises 36–39, graph the system of linear inequalities and indicate the solution set.

36. $y \le 3x - 1$
$y > -2x + 1$

37. $2x + y < 8$
$y \ge 2x - 1$

38. $x + 3y \le 6$
$2x - 7y \ge 14$

39. $x - y > 5$
$6x + 5y \le 30$

7.6

40. For the following set of constraints and profit formula, graph the constraints and find the vertices. Use the vertices to determine the maximum profit.

$$\begin{aligned} x + y &\le 10 \\ 2x + 1.8y &\le 18 \\ x &\ge 0 \\ y &\ge 0 \\ P &= 6x + 3y \end{aligned}$$

CHAPTER 7 TEST

1. From a graph, explain how you would identify a consistent system of equations, an inconsistent system of equations, and a dependent system of equations.

2. Solve the system of equations graphically.

$$\begin{aligned} y &= -2x - 3 \\ -2x + y &= -11 \end{aligned}$$

3. Determine without graphing whether the system of equations has exactly one solution, no solution, or an infinite number of solutions.

$$\begin{aligned} 4x + 5y &= 6 \\ -3x + 5y &= 13 \end{aligned}$$

Solve the system of equations by the method indicated.

4. $x - y = 5$
$2x + 3y = -5$
(substitution)

5. $y = 5x + 7$
$y = 2x + 1$
(substitution)

6. $x - y = 4$
$2x + y = 5$
(addition)

7. $4x + 3y = 5$
$2x + 4y = 10$
(addition)

8. $3x + 4y = 6$
$2x - 3y = 4$
(addition)

9. $x + 3y = 4$
$5x + 7y = 4$
(matrices)

In Exercises 10–12, for $A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, *determine the following.*

10. $A + B$ **11.** $3A - B$ **12.** $A \times B$

13. Graph the system of linear inequalities and indicate the solution set.

$$\begin{aligned} y &< -2x + 2 \\ y &> 3x + 2 \end{aligned}$$

Solve Exercises 14 and 15 by using a system of equations.

14. **MODELING** - *A Coffee Blend* Louis DiMento plans to mix coffee that sells for $6 per pound with coffee that sells for $7.50 per pound to get a 30 lb blend that sells for $7 per pound. How many pounds of each type should Louis use?

15. **MODELING -** *Checking Accounts* The charge for maintaining a checking account at Union Bank is \$6 per month plus 10 cents for each check that is written. The charge at Citrus Bank is \$2 per month and 20 cents per check.
    a) How many checks would a customer have to write in a month for the total charges to be the same at both banks?
    b) If Brent Pickett planned to write 14 checks per month, which bank would be the least expensive?

16. The set of constraints and profit formula for a linear programming problem are

$$x + 3y \le 6$$
$$4x + 3y \le 15$$
$$x \ge 0$$
$$y \ge 0$$
$$P = 5x + 3y$$

    a) Draw the graph of the constraints and determine the vertices of the feasible region.
    b) Use the vertices to determine the maximum and minimum profit.

GROUP PROJECTS

1. Make up three different systems of equations that have (1, 4) as a solution. Explain how you determined your systems.

Linear Programming

2. **MODELING -** *Profit from Bookcases* The Bookholder Company manufactures two types of bookcases out of oak and walnut. Model 01 requires 5 board feet of oak and 2 board feet of walnut. Model 02 requires 4 board feet of oak and 3 board feet of walnut. A profit of \$75 is made on each Model 01 bookcase and a profit of \$125 is made on each Model 02 bookcase. The company has a supply of 1000 board feet of oak and 600 board feet of walnut. The company has orders for 40 Model 01 bookcases and 50 Model 02 bookcases. These orders indicate the minimum number the company must manufacture of each model.
    a) Write the set of constraints.
    b) Write the objective function.
    c) Graph the set of constraints.
    d) Determine the number of bookcases of each type the company should manufacture in order to maximize profits.
    e) Determine the maximum profit.

Create Your Own Word Problem

3. **a)** Write a word problem that can be solved by using a system of two equations with two unknowns.
    b) For the problem in part (a), write the system of equations and find the answer.
    c) Explain how you developed the problem in part (a).