CHAPTER 9

The geometry of fractals provides mathematicians with a means for describing objects in nature in which a pattern endlessly repeats itself in smaller and smaller versions. This mountain scene is a computer-generated fractal.

GEOMETRY

Many objects in our lives can be described in terms of geometry. The spherical basketball you dribble down a rectangular court, the cylindrical can of soda you drink, and even the rectangular solid shape of this book are all examples of geometric objects that affect our lives. Throughout human history, geometry has played an important role in education, technology, and commerce. This role continues in modern times.

Albert Einstein's use of non-Euclidean geometry in his theory of relativity has enabled mathematicians and scientists to model the universe more accurately. Benoit Mandelbrot's work in fractal geometry has led scientists to discover ways to describe such intricate and detailed objects as weather systems, air passages in our lungs, and earthquake frequency patterns. A recently discovered form of pure carbon naturally forms molecules whose structure involves hexagons and pentagons in a pattern similar to those found on a soccer ball. As the human mind continues to uncover and interact with nature's secrets, geometry undoubtedly will continue to play a vital role.

9.1 POINTS, LINES, PLANES, AND ANGLES

PROFILE IN MATHEMATICS

EUCLID

Euclid (320–275 B.C.) lived in Alexandria, Egypt, and was a teacher and scholar at Alexandria's school called the *Museum*. It was here that Euclid collected and arranged many of the mathematical results known at the time. This collection of works became his 13-volume masterpiece known as *Elements*. Beginning with a list of definitions, postulates, and axioms, Euclid proved one theorem after another, using only previously proven results. This method of proof became a model of mathematical and scientific investigation that survives today. Remarkably, the geometry in *Elements* does not rely on making exact geometric measurements using a ruler or protractor. Rather, the work is developed using only an unmarked straightedge and a drawing compass. Next to the Bible, Euclid's *Elements* may be the most translated, published, and studied of all the books produced in the Western world.

Human beings recognized shapes, sizes, and physical forms long before geometry was developed. Geometry as a science is said to have begun in the Nile Valley of ancient Egypt. The Egyptians used geometry to measure land and to build pyramids and other structures.

The word *geometry* is derived from two Greek words, *ge*, meaning earth, and *metron*, meaning measure. Thus geometry means "earth measure" or "measurement of the earth."

Unlike the Egyptians, the Greeks were interested in more than just the applied aspects of geometry. The Greeks attempted to apply their knowledge of logic to geometry. In about 600 B.C., Thales of Miletus was the first to be credited with using deductive methods to develop geometric concepts. Another outstanding Greek geometer, Pythagoras, continued the systematic development of geometry that Thales had begun.

In about 300 B.C., Euclid collected and summarized much of the Greek mathematics of his time. In a set of 13 books called *Elements*, Euclid laid the foundation for plane geometry, which is also called ***Euclidean geometry***.

Euclid is credited with being the first mathematician to use the ***axiomatic method*** in developing a branch of mathematics. First, Euclid introduced ***undefined terms*** such as point, line, plane, and angle. He related these to physical space by such statements as "A line is length without breadth" so that we may intuitively understand them. Because such statements play no further role in his system, they constitute primitive or undefined terms.

Second, Euclid introduced certain ***definitions***. The definitions are introduced when needed and are often based on the undefined terms. Some terms that Euclid introduced and defined include triangle, right angle, and hypotenuse.

Third, Euclid stated certain primitive propositions called ***postulates*** (now called ***axioms****) about the undefined terms and definitions. The reader is asked to accept these statements as true on the basis of their "obviousness" and their relationship with the physical world. For example, the Greeks accepted all right angles as being equal, which is Euclid's fourth postulate.

Fourth, Euclid proved, using deductive reasoning (see Section 1.1), other propositions called ***theorems***. One theorem that Euclid proved is known as the Pythagorean theorem: "The sum of the areas of the squares constructed on the arms of a right triangle is equal to the area of the square constructed on the hypotenuse." He also proved that the sum of the angles of a triangle is 180°.

Using only 10 axioms, Euclid deduced 465 propositions (or theorems) in plane and solid geometry, number theory, and Greek geometric algebra.

Point and Line

Three basic terms in geometry are ***point***, ***line***, and ***plane***. These three terms are not given a formal definition, but we recognize points, lines, and planes when we see them.

*The concept of the axiom has changed significantly since Euclid's time. Now any statement may be designated as an axiom, whether it is self-evident or not. All axioms are *accepted* as true. A set of axioms forms the foundation for a mathematical system.

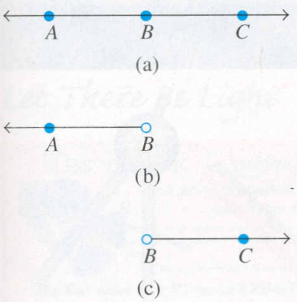


Figure 9.1

Let's consider some properties of a line. Assume that a line means a straight line unless otherwise stated.

1. A line is a set of points. Each point is on the line and the line passes through each point. When we wish to refer to a specific point, we will label it with a single capital letter. For example, in Figure 9.1(a) three points are labled *A, B,* and *C*, respectively.
2. Any two distinct points determine a unique line. Figure 9.1(a) illustrates a line. The arrows at both ends of the line indicate that the line continues in each direction. The line in Fig. 9.1(a) may be symbolized with any two points on the line by placing a line with a double-sided arrow above the letters that correspond to the points—for example, $\overleftrightarrow{AB}$, $\overleftrightarrow{BA}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{BC}$, or $\overleftrightarrow{CB}$.
3. Any point on a line separates the line into three parts: the point itself and two ***half lines*** (neither of which includes the point). For example, in Fig. 9.1(a) point *B* separates the line into the point *B* and two half lines. Half line *BA*, symbolized $\overset{\circ\!\rightarrow}{BA}$, is illustrated in Fig. 9.1(b). The open circle above the *B* indicates that point *B* is not included in the half line. Figure 9.1(c) illustrates half line *BC*, symbolized $\overset{\circ\!\rightarrow}{BC}$.

Look at the half line $\overset{\circ\!\rightarrow}{AB}$ in Fig. 9.2(b). If the ***end point***, *A*, is included with the set of points on the half line, the result is called a ***ray***. Ray *AB*, symbolized $\overrightarrow{AB}$, is illustrated in Fig. 9.2(c). Ray *BA*, symbolized $\overrightarrow{BA}$, is illustrated in Fig. 9.2(d).

A ***line segment*** is that part of a line between two points, including the end points. Line segment *AB*, symbolized $\overline{AB}$, is illustrated in Fig. 9.2(e).

Description	Diagram	Symbol
(a) Line *AB*		$\overleftrightarrow{AB}$
(b) Half line *AB*		$\overset{\circ\!\rightarrow}{AB}$
(c) Ray *AB*		$\overrightarrow{AB}$
(d) Ray *BA*		$\overrightarrow{BA}$
(e) Line segment *AB*		$\overline{AB}$
(f) Open line segment *AB*		$\overset{\circ\!-\!\circ}{AB}$
(g) Half open line segments *AB*		$\overset{-\circ}{AB}$, $\overset{\circ-}{AB}$

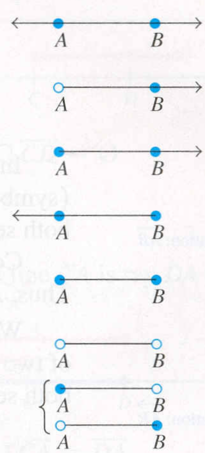
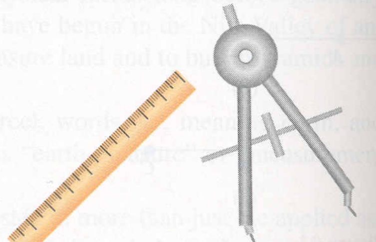


Figure 9.2

An open line segment is the set of points on a line between two points, excluding the end points. Open line segment *AB*, symbolized $\overset{\circ\!-\!\circ}{AB}$, is illustrated in Fig. 9.2(f). Figure 9.2(g) illustrates two half open line segments, symbolized $\overset{-\circ}{AB}$ and $\overset{\circ-}{AB}$.

DID YOU KNOW

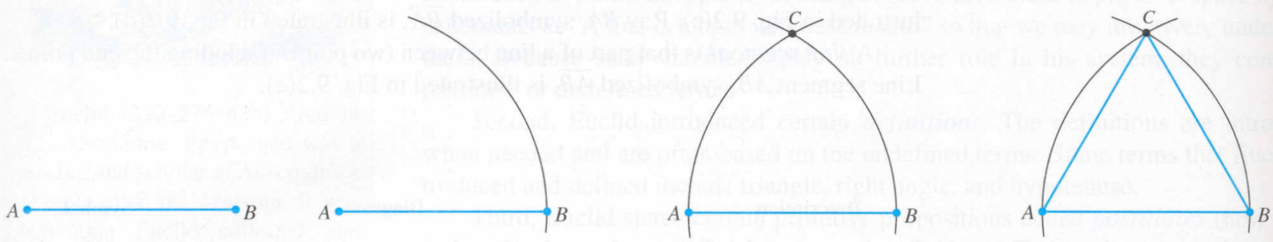
Compass and Straightedge Constructions



Geometric constructions were central to ancient Greek mathematics. Although these constructions are often referred to as *Euclidean constructions*, they were used centuries before Euclid wrote his classic work, *Elements*. The tools *allowed* in geometric constructions are a pencil, an unmarked straightedge, and a drawing compass. The straightedge is used to draw line segments, and the compass is used to draw circles and arcs. One example of a construction using these tools is shown below. The Internet has many sites devoted to classic geometric constructions.

To construct a triangle with sides of equal length (i.e., an equilateral triangle) do the following:

1. Use the straightedge to draw a line segment of any length and label the end points A and B.
2. Place one end of the compass at point A and the other end on point B and draw an arc as shown.
3. Now turn the compass around and draw another arc as shown. Label the point of intersection of the two arcs C.
4. Draw line segments AC and BC. This completes the construction of equilateral triangle ABC.



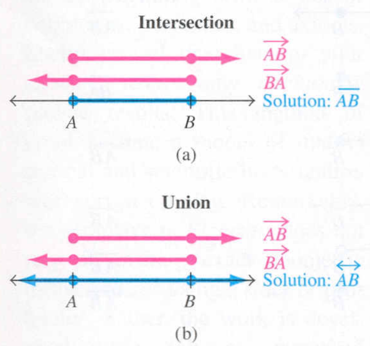


Figure 9.3

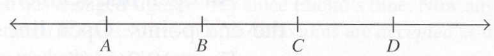
In Chapter 2 we discussed intersection of sets. Recall that the intersection (symbolized $\cap$) of two sets is the set of elements (points in this case) common to both sets.

Consider the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ in Fig. 9.3(a). The intersection of $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is $\overline{AB}$. Thus, $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

We also discussed the union of two sets in Chapter 2. The union (symbolized $\cup$) of two sets is the set of elements (points in this case) that belong to either of the sets or both sets. The union of $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is $\overleftrightarrow{AB}$ (Fig. 9.3b). Thus, $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

EXAMPLE 1 *Unions and Intersections of Parts of a Line*

Using line AD, determine the solution to each part.



a) $\overrightarrow{AB} \cap \overrightarrow{DC}$ b) $\overrightarrow{AB} \cup \overrightarrow{DC}$ c) $\overrightarrow{AB} \cap \overrightarrow{CD}$ d) $\overline{AD} \cup \overset{\circ}{\overrightarrow{CA}}$

DID YOU KNOW

Let There Be Light

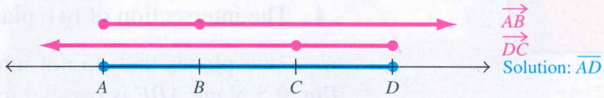
A Kodak Camera advertisement appeared in the first issue of the *Photographic Herald and Amateur Sportsman,* in November 1889. The slogan "You press the button, we do the rest" summed up George Eastman's groundbreaking snapshot camera system.

Sunlight travels in straight lines, or rays. When these rays pass through a small hole of a darkened room, they converge and then spread out to form an inverted image of the sun on the opposite wall. In the sixteenth century, artists used lenses to focus images on paper, which could then be traced to create a reproduction. In the eighteenth century, chemists discovered that certain salts of silver would darken when exposed to light, effectively creating a negative image. For the population at large, it all came together in 1888 when 23-year-old George Eastman, using basic geometric concepts, developed the first modern camera intended for the "casual millions," which sold for $25 apiece.

SOLUTION:

a) $\overrightarrow{AB} \cap \overrightarrow{DC}$

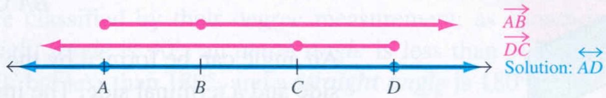
Ray AB and ray DC are shown below. The intersection of these two rays is that part of line AD that is a part of *both* ray AB *and* ray DC. The intersection of ray AB and ray DC is line segment AD.



$$\overrightarrow{AB} \cap \overrightarrow{DC} = \overline{AD}$$

b) $\overrightarrow{AB} \cup \overrightarrow{DC}$

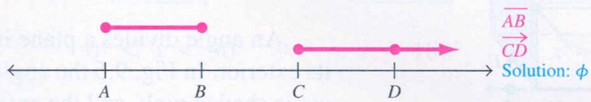
Once again ray AB and ray DC are shown below. The union of these two rays is that part of line AD that is part of *either* ray AB *or* ray DC. The union of ray AB and ray DC is the entire line AD.



$$\overrightarrow{AB} \cup \overrightarrow{DC} = \overleftrightarrow{AD}$$

c) $\overline{AB} \cap \overrightarrow{CD}$

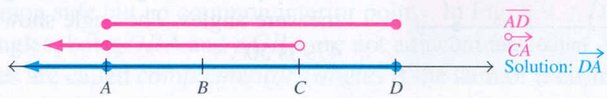
Line segment AB and ray CD have no points in common, so their intersection is empty.



$$\overline{AB} \cap \overrightarrow{CD} = \emptyset$$

d) $\overline{AD} \cup \overset{\circ\!\!\rightarrow}{CA}$

The union of line segment AD and half line CA is ray DA (or $\overrightarrow{DB}$ or $\overrightarrow{DC}$).



$$\overline{AD} \cup \overset{\circ\!\!\rightarrow}{CA} = \overrightarrow{DA}$$

Plane

The term *plane* is one of Euclid's undefined terms. For our purposes, we can think of a plane as a two-dimensional surface that extends infinitely in both directions, like an infinitely large blackboard. Euclidean geometry is called ***plane geometry*** because it is the study of two-dimensional figures in a plane.

Two lines in the same plane that do not intersect are called ***parallel lines***. Figure 9.4(a) on page 478 illustrates two parallel lines in a plane ($\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$).

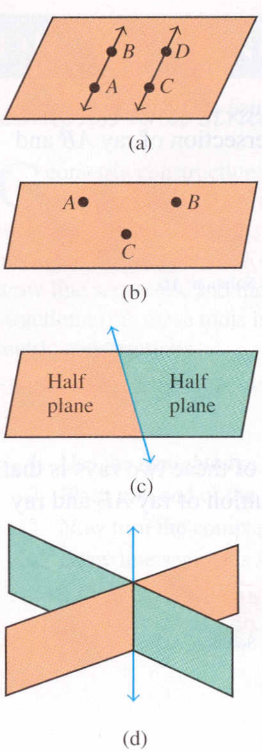


Figure 9.4

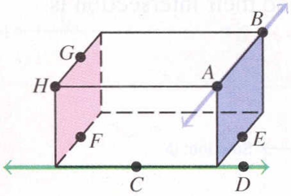


Figure 9.5

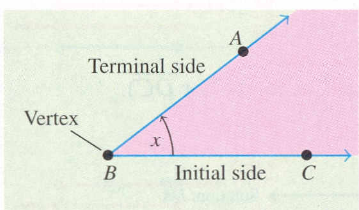


Figure 9.6

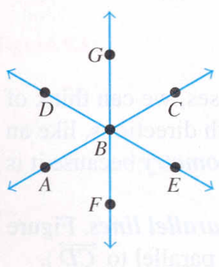


Figure 9.7

Properties of planes include the following:

1. Any three points that are not on the same line (noncollinear points) determine a unique plane (Fig. 9.4b).
2. A line in a plane divides the plane into three parts, the line and two half planes (Fig. 9.4c).
3. Any line and a point not on the line determine a unique plane.
4. The intersection of two planes is a line (Fig. 9.4d).

Two planes that do not intersect are said to be ***parallel planes***. For example, in Fig. 9.5 plane ABE is parallel to plane GHF.

Two lines that do not lie in the same plane and do not intersect are called ***skewed lines***. Figure 9.5 illustrates many skewed lines (for example, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$).

Angles

An ***angle***, denoted $\measuredangle$, is the union of two rays with a common end point (Fig. 9.6):

$$\overrightarrow{BA} \cup \overrightarrow{BC} = \measuredangle ABC \text{ (or } \measuredangle CBA)$$

An angle can be formed by the rotation of a ray about a point. An angle has an initial side and a terminal side. The initial side indicates the position of the ray prior to rotation; the terminal side indicates the position of the ray after rotation. The point common to both rays is called the ***vertex*** of the angle. The letter designating the vertex is always the middle one of the three letters designating an angle. The rays that make up the angle are called its ***sides***.

There are several ways to name an angle. The angle in Fig. 9.6 may be denoted

$$\measuredangle ABC, \qquad \measuredangle CBA, \qquad \text{or} \qquad \measuredangle B$$

An angle divides a plane into three distinct parts: the angle itself, its interior, and its exterior. In Fig. 9.6 the angle is represented by the blue lines, the interior of the angle is shaded pink, and the exterior is shaded green.

The ***measure of an angle***, symbolized m, is the amount of rotation from its initial side to its terminal side. In Fig. 9.6, the letter x represents the measure of $\measuredangle ABC$; therefore, we may write $m\measuredangle ABC = x$.

Angles can be measured in ***degrees***, radians, or gradients. In this text we will discuss only the degree unit of measurement. The symbol for degrees is the same as the symbol for temperature degrees. An angle of 45 degrees is written 45°. A *protractor* is used to measure angles. The angle shown being measured by the protractor in Fig. 9.8 on page 479 is 50°.

EXAMPLE 2 *Union and Intersection*

Refer to Fig. 9.7. Determine the following.

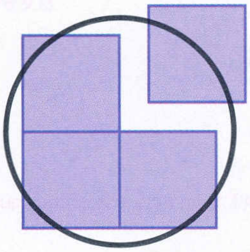
a) $\overrightarrow{BG} \cup \overrightarrow{BF}$ b) $\measuredangle ABG \cap \measuredangle DBC$ c) $\overleftrightarrow{DE} \cap \measuredangle CBE$ d) $\overrightarrow{BD} \cup \overrightarrow{BC}$

SOLUTION:

a) $\overrightarrow{BG} \cup \overrightarrow{BF} = \overleftrightarrow{GF}$
b) $\measuredangle ABG \cap \measuredangle DBC = \{B\}$
c) $\overleftrightarrow{DE} \cap \measuredangle CBE = \overrightarrow{BE}$
d) $\overrightarrow{BD} \cup \overrightarrow{BC} = \measuredangle DBC$ (or $\measuredangle CBD$)

DID YOU KNOW

Classical Geometry Problems



Have you ever spent several hours trying to solve a difficult homework problem? Can you imagine studying the same problem for your entire life? Entire generations of mathematicians throughout history spent their entire lives studying three geometry construction problems that originated in ancient Greece. The three problems are (1) *trisecting an angle*, (2) *squaring the circle*, and (3) *doubling the cube*. All three problems involve using only an unmarked straightedge and a compass to construct geometric figures. Trisecting an angle refers to dividing a given angle into three equal angles. Squaring the circle refers to constructing a square that has the exact same area as the area of a given circle. Doubling the cube refers to constructing a cube that has exactly double the volume of a given cube. Although these problems are easily stated, they puzzled mathematicians for thousands of years. The solutions were never found, but in the process of studying these problems, mathematicians were able to expand their knowledge about mathematics. Eventually, mathematicians came to realize that if one is limited to using only an unmarked straightedge and a compass, then these constructions are impossible. Finally, in 1837 Pierre Wantzel proved the impossibility of the first two constructions, and in 1882 Carl Lindemann proved the impossibility of the third construction.

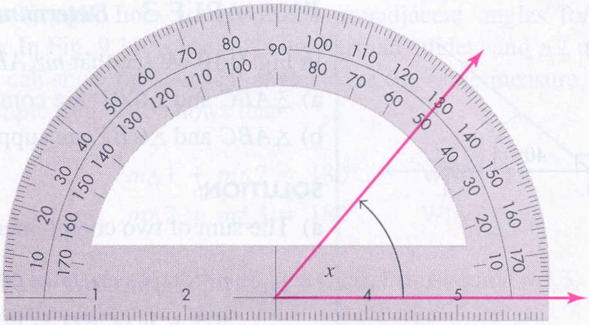
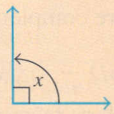
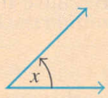
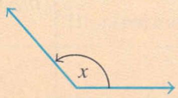
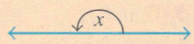


Figure 9.8

Consider a circle whose circumference is divided into 360 equal parts. If we draw a line from each mark on the circumference to the center of the circle, we get 360 wedge-shaped pieces. The measure of an angle formed by the straight sides of each wedge-shaped piece is defined to be 1°.

Angles are classified by their degree measurement, as shown in the following summary. A ***right angle*** is 90°, an ***acute angle*** is less than 90°, an ***obtuse angle*** is greater than 90° but less than 180°, and a ***straight angle*** is 180°.

Right Angle	**Acute Angle**	**Obtuse Angle**	**Straight Angle**
			
$x = 90°$ The symbol ∟ is used to indicate right angles.	$0° < x < 90°$	$90° < x < 180°$	$x = 180°$

Two angles in the same plane are ***adjacent angles*** when they have a common vertex and a common side but no common interior points. In Fig. 9.9, $\measuredangle DBC$ and $\measuredangle CBA$ are adjacent angles, but $\measuredangle DBA$ and $\measuredangle CBA$ are not adjacent angles.

Two angles are called ***complementary angles*** if the sum of their measures is 90°. Two angles are called ***supplementary angles*** if the sum of their measures is 180°.

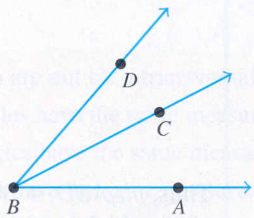


Figure 9.9

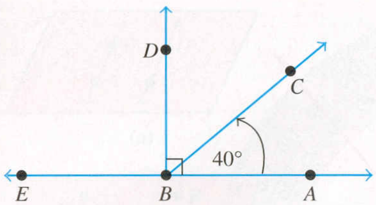


Figure 9.10

EXAMPLE 3 *Determining Complementary and Supplementary Angles*

In Fig. 9.10 we see that $m\measuredangle ABC = 40°$.

a) $\measuredangle ABC$ and $\measuredangle CBD$ are complementary angles. Determine $m\measuredangle CBD$.

b) $\measuredangle ABC$ and $\measuredangle CBE$ are supplementary angles. Determine $m\measuredangle CBE$.

SOLUTION:

a) The sum of two complementary angles must be 90°, so

$$\begin{aligned} m\measuredangle ABC + m\measuredangle CBD &= 90° \\ 40° + m\measuredangle CBD &= 90° \\ m\measuredangle CBD &= 90° - 40° = 50° \end{aligned}$$

Subtract 40° from each side of the equation.

b) The sum of two supplementary angles must be 180°, so

$$\begin{aligned} m\measuredangle ABC + m\measuredangle CBE &= 180° \\ 40° + m\measuredangle CBE &= 180° \\ m\measuredangle CBE &= 180° - 40° = 140° \end{aligned}$$

Subtract 40° from each side of the equation.

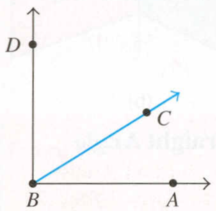


Figure 9.11

EXAMPLE 4 *Determining Complementary Angles*

If $\measuredangle ABC$ and $\measuredangle CBD$ are complementary angles and $m\measuredangle ABC$ is 26° less than $m\measuredangle CBD$, determine the measure of each angle (Fig. 9.11).

SOLUTION: Let $m\measuredangle CBD = x$. Then $m\measuredangle ABC = x - 26$ since it is 26° less than $m\measuredangle CBD$. Because these angles are complementary, we have

$$\begin{aligned} m\measuredangle CBD + m\measuredangle ABC &= 90 \\ x + (x - 26) &= 90 \\ 2x - 26 &= 90 \\ 2x &= 116 \\ x &= 58 \end{aligned}$$

Therefore, $m\measuredangle CBD = 58°$ and $m\measuredangle ABC = 58° - 26°$, or 32°. Note that $58° + 32° = 90°$, which is what we expected.

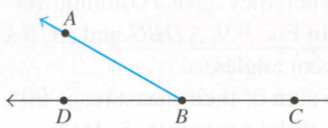


Figure 9.12

EXAMPLE 5 *Determining Supplementary Angles*

If $\measuredangle ABC$ and $\measuredangle ABD$ are supplementary and $m\measuredangle ABC$ is five times larger than $m\measuredangle ABD$, determine $m\measuredangle ABC$ and $m\measuredangle ABD$ (Fig. 9.12).

SOLUTION: Let $m\measuredangle ABD = x$, then $m\measuredangle ABC = 5x$. Since these angles are supplementary, we have

$$\begin{aligned} m\measuredangle ABC + m\measuredangle ABD &= 180° \\ 5x + x &= 180° \\ 6x &= 180° \\ x &= 30° \end{aligned}$$

Thus, $m\measuredangle ABD = 30°$ and $m\measuredangle ABC = 5(30°) = 150°$. Note that $30° + 150° = 180°$, which is what we expected.

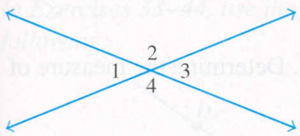


Figure 9.13

When two straight lines intersect, the nonadjacent angles formed are called ***vertical angles***. In Fig. 9.13, $\measuredangle 1$ and $\measuredangle 3$ are vertical angles, and $\measuredangle 2$ and $\measuredangle 4$ are vertical angles. We can show that vertical angles have the same measure, that is, they are equal. For example, Fig. 9.13 shows that

$$m\measuredangle 1 + m\measuredangle 2 = 180^\circ. \qquad \text{Why?}$$
$$m\measuredangle 2 + m\measuredangle 3 = 180^\circ. \qquad \text{Why?}$$

Since $\measuredangle 2$ has the same measure in both cases, $m\measuredangle 1$ must equal $m\measuredangle 3$.

> Vertical angles have the same measure.

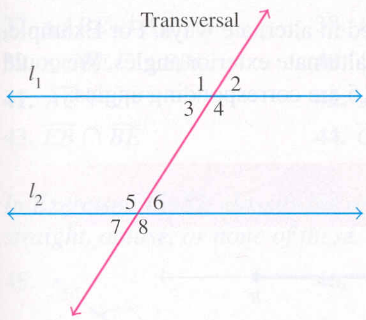
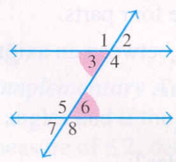
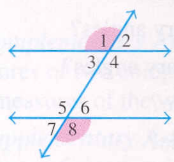
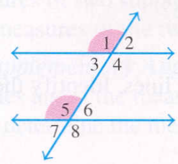


Figure 9.14

A line that intersects two different lines, l_1 and l_2, at two different points is called a ***transversal***. Figure 9.14 illustrates that when two parallel lines are cut by a transversal, eight angles are formed. Angles 3, 4, 5, and 6 are called ***interior angles***, and angles 1, 2, 7, and 8 are called ***exterior angles***. Eight pairs of supplementary angles are formed. Can you list them?

Special names are given to the angles formed by a transversal crossing two parallel lines.

Name	Description	Illustration	Pairs of Angles Meeting Criteria
Alternate interior angles	Interior angles on opposite sides of the transversal		$\measuredangle 3$ and $\measuredangle 6$ $\measuredangle 4$ and $\measuredangle 5$
Alternate exterior angles	Exterior angles on opposite sides of the transversal		$\measuredangle 1$ and $\measuredangle 8$ $\measuredangle 2$ and $\measuredangle 7$
Corresponding angles	One interior and one exterior angle on the same side of the transversal		$\measuredangle 1$ and $\measuredangle 5$ $\measuredangle 2$ and $\measuredangle 6$ $\measuredangle 3$ and $\measuredangle 7$ $\measuredangle 4$ and $\measuredangle 8$

> When two parallel lines are cut by a transversal
> 1. alternate interior angles have the same measure.
> 2. alternate exterior angles have the same measure.
> 3. corresponding angles have the same measure.

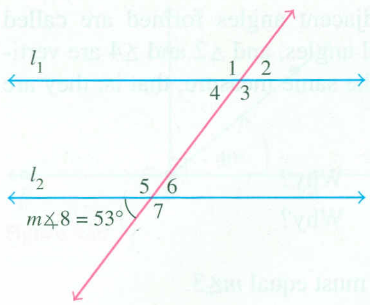


Figure 9.15

EXAMPLE 6 *Finding Angle Measures*

Figure 9.15 shows two parallel lines cut by a transversal. Determine the measure of $\measuredangle 1$ through $\measuredangle 7$.

SOLUTION:

$m\measuredangle 6 = 53°$	$\measuredangle 8$ and $\measuredangle 6$ are vertical angles.
$m\measuredangle 5 = 127°$	$\measuredangle 8$ and $\measuredangle 5$ are supplementary angles.
$m\measuredangle 7 = 127°$	$\measuredangle 5$ and $\measuredangle 7$ are vertical angles.
$m\measuredangle 1 = 127°$	$\measuredangle 1$ and $\measuredangle 7$ are alternate exterior angles.
$m\measuredangle 4 = 53°$	$\measuredangle 4$ and $\measuredangle 6$ are alternate interior angles.
$m\measuredangle 2 = 53°$	$\measuredangle 6$ and $\measuredangle 2$ are corresponding angles.
$m\measuredangle 3 = 127°$	$\measuredangle 3$ and $\measuredangle 1$ are vertical angles.

In Example 6, the angles could have been determined in alternate ways. For Example, we mentioned $m\measuredangle 1 = 127°$ because $\measuredangle 1$ and $\measuredangle 7$ are alternate exterior angles. We could have also stated that $m\measuredangle 1 = 127°$ because $\measuredangle 1$ and $\measuredangle 5$ are corresponding angles.

SECTION 9.1 EXERCISES

Concept/Writing Exercises

1. a) What are the four key parts in the axiomatic method used by Euclid?
   b) Discuss each of the four parts.
2. What is the difference between an axiom and a theorem?
3. What are parallel lines?
4. What are skewed lines?
5. What are adjacent angles?
6. What are supplementary angles?
7. What are complementary angles?
8. What is a straight angle?
9. What is an obtuse angle?
10. What is an acute angle?
11. What is a right angle?
12. Draw two intersecting lines. Identify the two pairs of vertical angles.

Practice the Skills

In Exercises 13–20, identify the figure as a line, half line, ray, line segment, open line segment, or half open line segment. Denote it by its appropriate symbol.

13. 

14. 

15. 

16. 

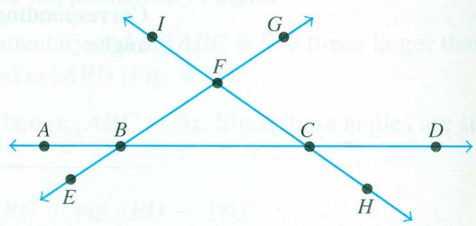
17. 

18. 

19. 

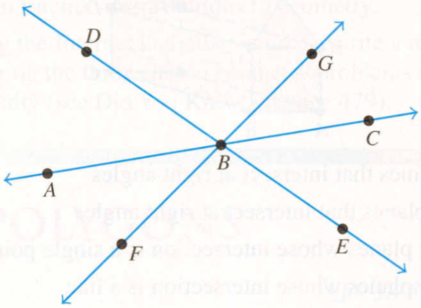
20. 

In Exercises 21–32, use the figure to find the following:



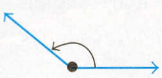
21. $\measuredangle GBD \cap \overrightarrow{AD}$
22. $\overrightarrow{FE} \cup \overrightarrow{FG}$
23. $\overset{\circ\!\!-\!\!\circ}{BC} \cup \overset{\circ\!\!-\!\!\circ}{CD}$
24. $\overleftrightarrow{AD} \cup \overline{BC}$
25. $\measuredangle ICA \cap \overleftrightarrow{EG}$
26. $\measuredangle HCD \cap \measuredangle ACF$
27. $\overrightarrow{AB} \cap \overline{HC}$
28. $\overset{\circ\!\!\rightarrow}{BD} \cap \overset{\circ\!\!\rightarrow}{CB}$
29. $\overleftrightarrow{AD} \cap \overline{BC}$
30. $\overline{BC} \cup \overline{CF} \cup \overline{FB}$
31. $\overset{\circ\!\!\rightarrow}{BD} \cup \overset{\circ\!\!\rightarrow}{CB}$
32. $\{C\} \cap \overset{\circ\!\!\rightarrow}{CH}$

In Exercises 33–44, use the figure to find each of the following:



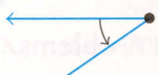
33. $\overrightarrow{BF} \cap \overrightarrow{BE}$
34. $\overrightarrow{BD} \cup \overrightarrow{BE}$
35. $\measuredangle GBC \cap \measuredangle CBE$
36. $\overleftrightarrow{DE} \cup \overrightarrow{BE}$
37. $\measuredangle ABE \cup \overline{AB}$
38. $\overrightarrow{BF} \cup \overrightarrow{BE}$
39. $\measuredangle CBE \cap \measuredangle EBC$
40. $\{B\} \cap \overrightarrow{BA}$
41. $\overleftrightarrow{AC} \cap \overline{AC}$
42. $\overleftrightarrow{AC} \cap \overrightarrow{BE}$
43. $\overrightarrow{EB} \cap \overrightarrow{BE}$
44. $\overline{GF} \cap \overline{AB}$

In Exercises 45–52, classify the angle as acute, right, straight, obtuse, or none of these.

45. 

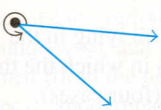
46. 

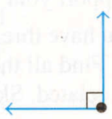
47. 

48. 

49. 

50. 

51. 

52. 

In Exercises 53–58, find the complementary angle of the given angle.

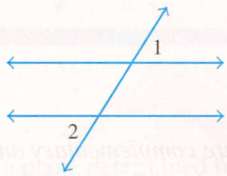
53. 19°
54. 89°
55. $32\frac{3}{4}°$
56. $43\frac{1}{3}°$
57. 64.7°
58. 0.01°

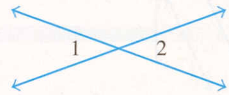
In Exercises 59–64, find the supplementary angle of the given angle.

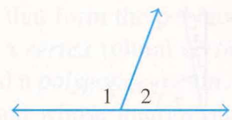
59. 91°
60. 8°
61. 20.5°
62. 179.99°
63. $43\frac{5}{7}°$
64. $64\frac{7}{16}°$

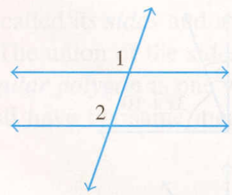
In Exercises 65–70, match the names of the angles with the corresponding figure in parts (a)–(f).

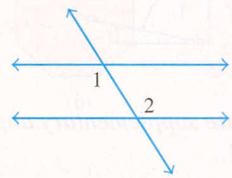
65. Corresponding angles
66. Vertical angles
67. Supplementary angles
68. Complementary angles
69. Alternate interior angles
70. Alternate exterior angles

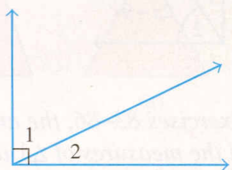
a) 

b) 

c) 

d) 

e) 

f) 

Problem Solving

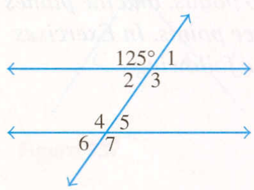
71. **MODELING** - *Complementary Angles* If $\measuredangle 1$ and $\measuredangle 2$ are complementary angles and if the measure of $\measuredangle 1$ is four more than the measure of $\measuredangle 2$, determine the measures of $\measuredangle 1$ and $\measuredangle 2$.

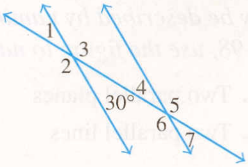
72. **MODELING** - *Complementary Angles* The difference between the measures of two complementary angles is 62°. Determine the measures of the two angles.

73. **MODELING** - *Supplementary Angles* The difference between the measures of two supplementary angles is 88°. Determine the measures of the two angles.

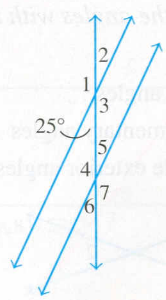
74. **MODELING** - *Supplementary Angles* If $\measuredangle 1$ and $\measuredangle 2$ are supplementary angles and if the measure of $\measuredangle 2$ is 17 times the measure of $\measuredangle 1$, determine the measures of the two angles.

In Exercises 75–78, parallel lines are cut by the transversal shown. Determine the measures of $\measuredangle 1$ through $\measuredangle 7$.

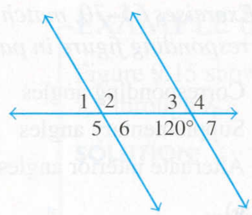
75. 

76. 

77.

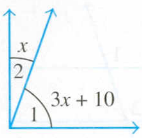


78.

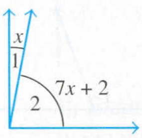


In Exercises 79–82, the angles are complementary angles. Find the measures of ∡1 and ∡2.

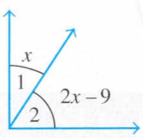
79.



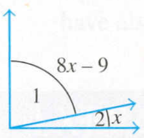
80.



81.

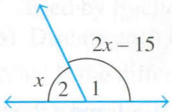


82.

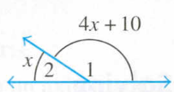


In Exercises 83–86, the angles are supplementary angles. Find the measures of ∡1 and ∡2.

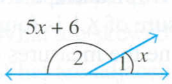
83.



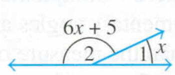
84.



85.



86.



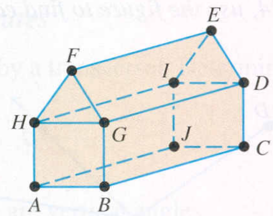
87. a) How many lines can be drawn through a given point?
b) How many planes can be drawn through a given point?

88. What is the intersection of two distinct nonparallel planes?

89. How many planes can be drawn through a given line?

90. a) Will three noncollinear points A, B, and C always determine a plane? Explain.
b) Is it possible to determine more than one plane with three noncollinear points? Explain.
c) How many planes can be constructed through three collinear points?

The figure suggests a number of lines and planes. The lines may be described by naming two points, and the planes may be described by naming three points. In Exercises 91–98, use the figure to name the following:



91. Two parallel planes

92. Two parallel lines

93. Two lines that intersect at right angles

94. Two planes that intersect at right angles

95. Three planes whose intersection is a single point

96. Three planes whose intersection is a line

97. A line and a plane whose intersection is a point

98. A line and a plane whose intersection is a line

In Exercises 99–104, determine whether the statement is always true, sometimes true, or never true. Explain your answer.

99. Two lines that are both parallel to a third line must be parallel to each other.

100. A triangle contains two acute angles.

101. Vertical angles are complementary angles.

102. Alternate exterior angles are supplementary angles.

103. Alternate interior angles are complementary angles.

104. A triangle contains two obtuse angles.

Challenge Problems/Group Activities

105. If lines l and m are parallel lines and if lines l and n are skewed lines, is it true that lines m and n must also be skewed? (*Hint:* Look at Fig. 9.5 on page 478.) Explain your answer and include a sketch to support your answer.

106. Two lines are *perpendicular* if they intersect at right angles. If lines l and m are perpendicular and if lines m and n are perpendicular, is it true that lines l and n must also be perpendicular? Explain your answer and include a sketch to support your answer.

107. Suppose you have three distinct lines, all lying in the same plane. Find all the possible ways in which the three lines can be related. Sketch each case (four cases).

Recreational Mathematics

108. If two straight lines intersect at a point, determine the sum of the measures of the 4 angles formed.

109. ∡ABC and ∡CBD are complementary and m∡CBD is twice the m∡ABC. ∡ABD and ∡DBE are supplementary angles.
a) Draw a sketch illustrating ∡ABC, ∡CBD, and ∡DBE.
b) Determine m∡ABC.
c) Determine m∡CBD.
d) Determine m∡DBE.

Internet/Research Activities

110. Using the Internet and other sources, write a research paper on Euclid's contributions to geometry.

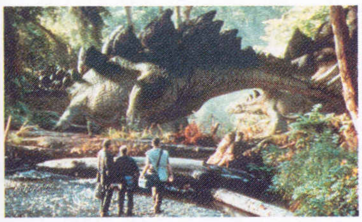
111. Using the Internet and other sources, write a research paper on the three classic geometry problems of Greek antiquity (see Did You Know on page 479).

112. Search the Internet or other sources such as a geometry textbook to study the geometric constructions that use a straightedge and a compass only. Prepare a poster demonstrating five of these basic constructions.

9.2 POLYGONS

MATHEMATICS *Everywhere*

Making Movies Come Alive



Mathematics plays a key role in the animation you see in movies such as those in the *Jurassic Park* series. The images you see on the movie screen are created using software that combines pixels (the smallest piece of a screen image) into geometric shapes including polygons. These shapes are then stored in a computer and manipulated using various mathematical techniques so that the new shapes formed (from the original geometric shapes) approximate curves. Each movie frame has over 2 million pixels and can have over 40 million polygons. With such a huge amount of data, computers are used to carry out the mathematics needed to create animation. One computer animation specialist stated that "it's all controlled by math. . . . All those little *X*'s, *Y*'s, and *Z*'s that you had in school—oh my gosh, suddenly they all apply."

A ***polygon*** is a closed figure in a plane determined by three or more straight line segments. Examples of polygons are given in Fig. 9.16.

The straight line segments that form the polygon are called its ***sides*** and a point where two sides meet is called a ***vertex*** (plural ***vertices***). The union of the sides of a polygon and its interior is called a ***polygonal region***. A ***regular polygon*** is one whose sides are all the same length and whose interior angles all have the same measure. Figures 9.16(b) and (d) are regular polygons.

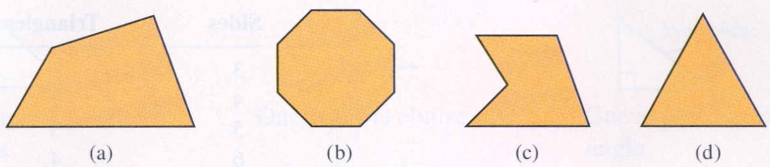


Figure 9.16

Polygons are named according to their number of sides. The names of some polygons are given in Table 9.1.

TABLE 9.1

Number of Sides	Name	Number of Sides	Name
3	Triangle	8	Octagon
4	Quadrilateral	9	Nonagon
5	Pentagon	10	Decagon
6	Hexagon	12	Dodecagon
7	Heptagon	20	Icosagon

One of the most important polygons is the triangle. The sum of the measures of the interior angles of a triangle is 180°. To illustrate, consider triangle ABC given in Fig. 9.17. The triangle is formed by drawing two transversals through two parallel lines l_1 and l_2 with the two transversals intersecting at a point on l_1.

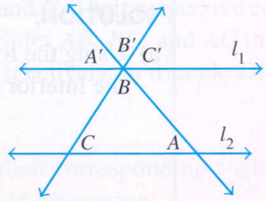


Figure 9.17

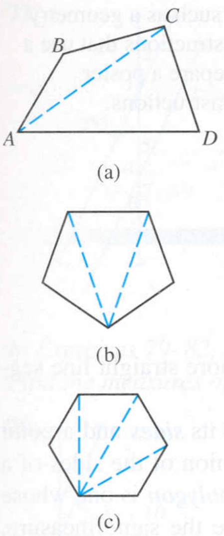


Figure 9.18

In Fig. 9.17, notice that $\measuredangle A$ and $\measuredangle A'$ are corresponding angles. Recall from Section 9.1 that corresponding angles are equal, so $m\measuredangle A = m\measuredangle A'$. Also, $\measuredangle C$ and $\measuredangle C'$ are corresponding angles; therefore, $m\measuredangle C = m\measuredangle C'$. Next, we notice that $\measuredangle B$ and $\measuredangle B'$ are vertical angles. In Section 9.1 we learned that vertical angles are equal; therefore, $m\measuredangle B = m\measuredangle B'$. Figure 9.17 shows that $\measuredangle A'$, $\measuredangle B'$, and $\measuredangle C'$ form a straight angle; therefore, $m\measuredangle A' + m\measuredangle B' + m\measuredangle C' = 180°$. Since $m\measuredangle A = m\measuredangle A'$, $m\measuredangle B = m\measuredangle B'$, and $m\measuredangle C = m\measuredangle C'$, we can reason that $m\measuredangle A + m\measuredangle B + m\measuredangle C = 180°$. This illustrates that the sum of the interior angles of a triangle is 180°.

Consider the quadrilateral *ABCD* (Fig. 9.18a). Drawing a straight line segment between any two vertices forms two triangles. Since the sum of the measures of the angles of a triangle is 180°, the sum of the measures of the interior angles of a quadrilateral is $2 \cdot 180°$, or 360°.

Now let's examine a pentagon (Fig. 9.18b). We can draw two straight line segments to form three triangles. Thus, the sum of the measures of the interior angles of a five-sided figure is $3 \cdot 180°$, or 540°. Figure 9.18(c) shows that four triangles can be drawn in a six-sided figure. Table 9.2 summarizes this information.

TABLE 9.2

Sides	Triangles	Sum of the Measures of the Interior Angles
3	1	$1(180°) = 180°$
4	2	$2(180°) = 360°$
5	3	$3(180°) = 540°$
6	4	$4(180°) = 720°$

If we continue this procedure, we can see that for an *n*-sided polygon the sum of the measures of the interior angles is $(n - 2)180°$.

The **sum** of the measures of the interior angles of an *n*-sided polygon is $(n - 2)180°$.

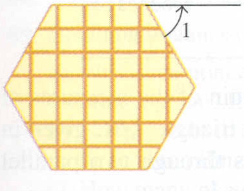


Figure 9.19

EXAMPLE 1 *Angles of a Hexagon*

Crispy Hexagons is a breakfast cereal whose pieces are in the shape of regular hexagons. A regular hexagon is a six-sided figure with all the sides the same length and all interior angles with the same measure. See Fig. 9.19. Determine

a) the measure of an interior angle.

b) the measure of exterior $\measuredangle 1$.

SOLUTION:

a) Using the formula $(n - 2)180°$, we can determine the sum of the measures of the interior angles of a hexagon as follows.

$$\begin{aligned} \text{Sum} &= (6 - 2)180° \\ &= 4(180°) \\ &= 720° \end{aligned}$$

The measure of an interior angle of a regular polygon can be determined by dividing the sum of the interior angles by the number of angles.

The measure of an interior angle of a regular hexagon is determined as follows:

$$\text{Measure} = \frac{720°}{6} = 120°$$

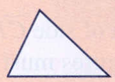
b) Since $\measuredangle 1$ is the supplement of an interior angle,

$$m\measuredangle 1 = 180° - 120° = 60°$$ ▲

In order to discuss area in the next section, we must be able to identify various types of triangles and quadrilaterals. The following is a summary of certain types of triangles and their characteristics.

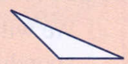
Triangles

Acute Triangle



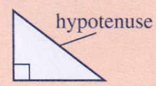
All angles are acute angles.

Obtuse Triangle



One angle is obtuse.

Right Triangle

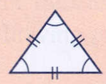


One angle is a right angle.

Isosceles Triangle

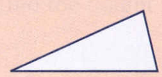
Two equal sides
Two equal angles

Equilateral Triangle



Three equal sides
Three equal angles
(60° each)

Scalene Triangle



No two sides are equal in length.

Similar Figures

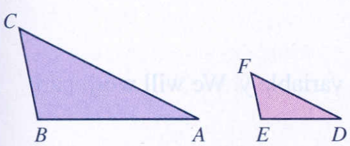


Figure 9.20

In everyday living we often have to deal with geometric figures that have the "same shape" but are of different sizes. For example, an architect will make a small-scale drawing of a floor plan or a photographer will make an enlargement of a photograph. Figures that have the same shape but may be of different sizes are called ***similar figures***. Two similar figures are illustrated in Fig. 9.20.

Similar figures have *corresponding angles* and *corresponding sides*. In Fig. 9.20 triangle *ABC* has angles *A, B*, and *C*. Their respective corresponding angles in triangle *DEF* are angles *D, E*, and *F*. Sides *AB, BC*, and *AC* in triangle *ABC* have corresponding sides *DE, EF*, and *DF*, respectively, in triangle *DEF*.

Two polygons are **similar** if their corresponding angles have the same measure and their corresponding sides are in proportion.

In Figure 9.20, $\measuredangle A$ and $\measuredangle D$ have the same measure, $\measuredangle B$ and $\measuredangle E$ have the same measure, and $\measuredangle C$ and $\measuredangle F$ have the same measure. Also, the corresponding sides of similar triangles are in proportion:

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}}.$$

EXAMPLE 2 *Similar Figures*

Consider the similar figures in Fig. 9.21.

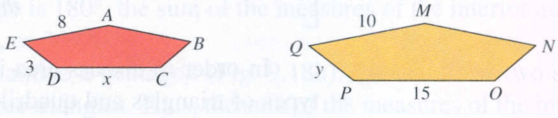


Figure 9.21

Determine

a) the length of side $\overline{CD}$. b) the length of side $\overline{PQ}$.

SOLUTION:

a) We will represent the length of side $\overline{CD}$ with the variable x. Because the corresponding sides of similar figures must be in proportion, we can write a proportion (as explained in Section 6.2) to find the length of side $\overline{CD}$. Corresponding sides $\overline{AE}$ and $\overline{MQ}$ are known, so we use them as one ratio in the proportion. The corresponding side of $\overline{CD}$ is $\overline{OP}$.

$$\frac{\overline{AE}}{\overline{MQ}} = \frac{\overline{CD}}{\overline{OP}}$$

$$\frac{8}{10} = \frac{x}{15}$$

Now we solve for x.

$$8 \cdot 15 = 10 \cdot x$$
$$120 = 10x$$
$$12 = x$$

Thus, the length of side $\overline{CD}$ is 12 units.

b) We will represent the length of side $\overline{PQ}$ with the variable y. We will work part (b) in a similar manner to part (a).

$$\frac{\overline{AE}}{\overline{MQ}} = \frac{\overline{DE}}{\overline{PQ}}$$

$$\frac{8}{10} = \frac{3}{y}$$

$$8 \cdot y = 10 \cdot 3$$
$$8y = 30$$
$$y = \frac{30}{8} = \frac{15}{4} = 3.75$$

Thus, the length of side $\overline{PQ}$ is $\frac{15}{4}$ or 3.75.

EXAMPLE 3 *Using Similar Triangles to Find the Height of a Tree*

Saraniti Walker plans to remove a tree from her back yard. She needs to know the height of the tree. Saraniti is 5 ft tall and determines that when her shadow is 8 ft long, the shadow of the tree is 50 ft long (see Fig. 9.22). How tall is the tree?

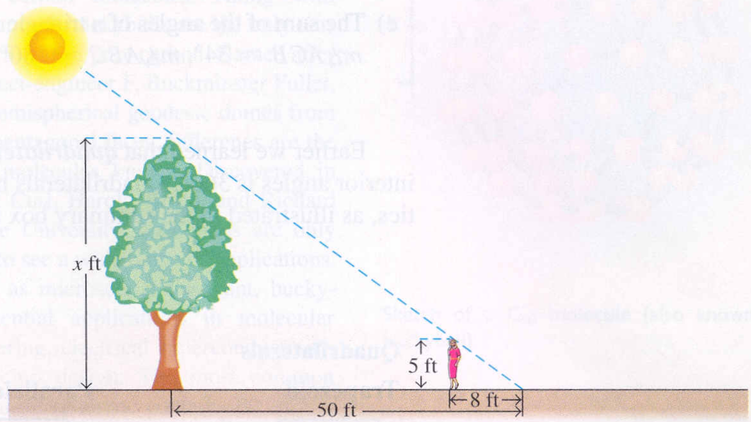


Figure 9.22

SOLUTION: We will let x represent the height of the tree. From Fig. 9.22 we can see that the triangle formed by the sun's rays, Saraniti, and her shadow is similar to the triangle formed by the sun's rays, the tree, and its shadow. To find the height of the tree we will set up and solve the following proportion:

$$\frac{\text{Height of the tree}}{\text{Height of Saraniti}} = \frac{\text{length of tree's shadow}}{\text{length of Saraniti's shadow}}$$

$$\frac{x}{5} = \frac{50}{8}$$

$$8x = 250$$

$$x = 31.25$$

Therefore, the tree is 31.25 ft tall.

Congruent Figures

If the corresponding sides of two similar figures are the same length, the figures are called *congruent figures*. Corresponding angles of congruent figures have the same measure, and the corresponding sides are equal in length. Two congruent figures coincide when placed one upon the other.

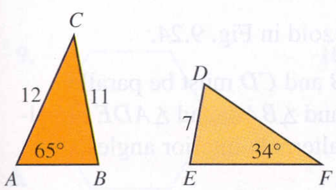


Figure 9.23

EXAMPLE 4 *Congruent Triangles*

Triangles ABC and DEF in Fig. 9.23 are congruent. Determine

a) the length of side $\overline{DF}$. b) the length of side $\overline{AB}$.

c) $m\measuredangle FDE$. d) $m\measuredangle ACB$.

e) $m\measuredangle ABC$.

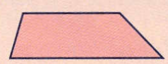
SOLUTION: Because ΔABC is congruent to ΔDEF, we know that the corresponding sides and angles are equal.

a) $\overline{DF} = \overline{AC} = 12$
b) $\overline{AB} = \overline{DE} = 7$
c) $m\measuredangle FDE = m\measuredangle CAB = 65°$
d) $m\measuredangle ACB = m\measuredangle DFE = 34°$
e) The sum of the angles of a triangle is 180°. Since $m\measuredangle BAC = 65°$ and $m\measuredangle ACB = 34°$, $m\measuredangle ABC = 180° - 65° - 34° = 81°$.

Earlier we learned that ***quadrilaterals*** are four-sided polygons, the sum of whose interior angles is 360°. Quadrilaterals may be classified according to their characteristics, as illustrated in the summary box below.

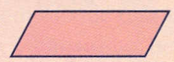
Quadrilaterals

Trapezoid



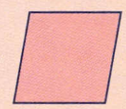
Two sides are parallel.

Parallelogram



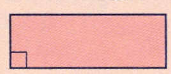
Both pairs of opposite sides are parallel. Both pairs of opposite sides are equal in length.

Rhombus



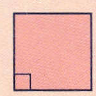
Both pairs of opposite sides are parallel. The four sides are equal in length.

Rectangle



Both pairs of opposite sides are parallel. Both pairs of opposite sides are equal in length. The angles are right angles.

Square



Both pairs of opposite sides are parallel. The four sides are equal in length. The angles are right angles.

EXAMPLE 5 *Angles of a Trapezoid*

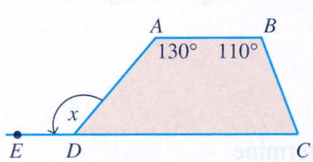


Figure 9.24

Find the measure of the exterior angle, x, of the trapezoid in Fig. 9.24.

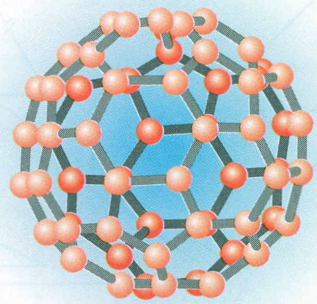
SOLUTION: By the definition of a trapezoid, sides AB and CD must be parallel. Therefore, side AD may be considered a transversal and $\measuredangle BAD$ and $\measuredangle ADE$ are alternate interior angles. Recall from Section 9.1, that alternate interior angles are equal. Thus, $m\measuredangle BAD = m\measuredangle ADE$ and $m\measuredangle x = 130°$.

DID YOU KNOW

Buckyballs

The molecular structure of C_{60} resembles the patterns found on a soccer ball.

Buckminsterfullerenes, also known as fullerenes and affectionately known as buckyballs, are a class of pure carbon molecules. Along with graphite and diamond, buckyballs are the only naturally occurring forms of pure carbon. Named after American architect-engineer F. Buckminster Fuller, who designed hemispherical geodesic domes from hexagonal and pentagonal faces, fullerenes are the most spherical molecules known. Discovered in 1985 by Robert Curl, Harold Kroto, and Richard Smalley at Rice University, buckyballs are only now beginning to see a wide range of applications. Used primarily as microscopic lubricant, buckyballs have potential applications in molecular medical engineering, electrical superconductivity, and computer chip design. The most common form of buckminsterfullerene contains 60 carbon atoms and has the chemical symbol C_{60}. The molecular structure of C_{60} contains 12 pentagons and 20 hexagons arranged in a pattern similar to that found on a soccer ball.



Sketch of a C_{60} molecule (also known as a buckyball)

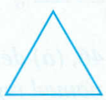
SECTION 9.2 EXERCISES

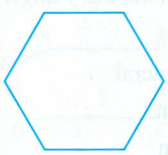
Concept/Writing Exercises

1. What is a polygon?
2. What distinguishes regular polygons from other polygons?
3. List six different types of triangles and in your own words describe the characteristics of each.
4. List five different types of quadrilaterals and in your own words describe the characteristics of each.
5. What are congruent figures?
6. What are similar figures?

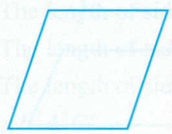
In Exercises 7–14, (a) name the polygon. If the polygon is a quadrilateral, give its specific name. (b) State whether or not the polygon is a regular polygon.

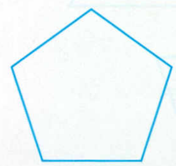
7. 

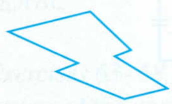
8. 

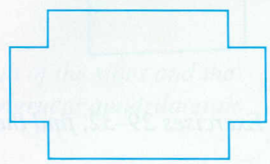
9. 

10. 

11. 

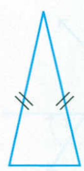
12. 

13. 

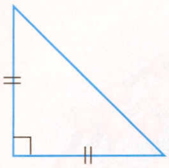
14. 

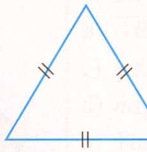
In Exercises 15–22, identify the triangle as (a) scalene, isosceles, or equilateral and as (b) acute, obtuse, or right. The parallel markings (the two small parallel lines) on two or more sides indicate that the marked sides are of equal length.

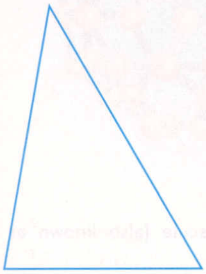
15. 

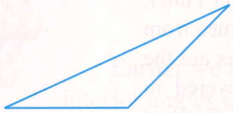
16. 

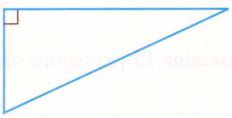
17. 

18. 

19. 

20. 

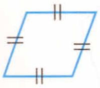
21. 

22. 

In Exercises 23–28, identify the quadrilateral.

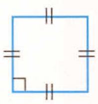
23. 

24. 

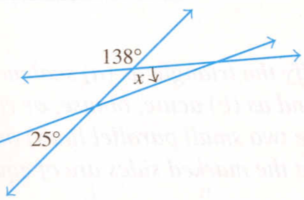
25. 

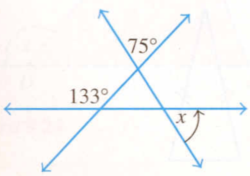
26. 

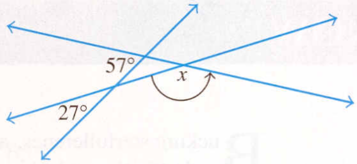
27. 

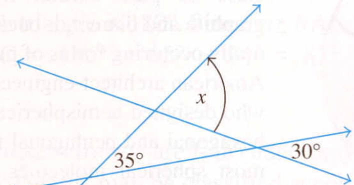
28. 

In Exercises 29–32, find the measure of ∡x.

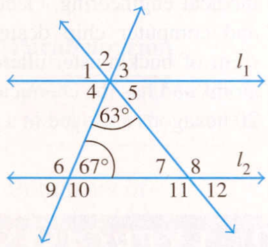
29. 

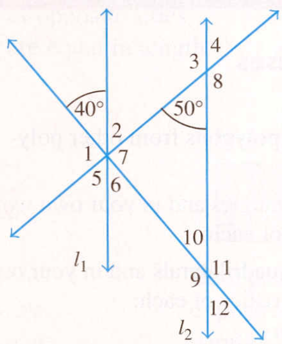
30. 

31. 

32. 

In Exercises 33–34, lines l_1 and l_2 are parallel. Determine the measures of ∡1 through ∡12.

33. 

34. 

In Exercises 35–40, determine the sum of the measures of the interior angles of the indicated polygon.

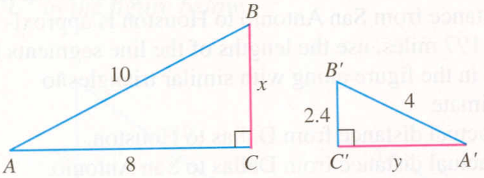
35. Pentagon
36. Nonagon
37. Hexagon
38. Decagon
39. Icosagon
40. Dodecagon

In Exercises 41–46, (a) determine the measure of an interior angle of the named regular polygon. (b) If a side of the polygon is extended, determine the supplementary angle of an interior angle. See Example 1.

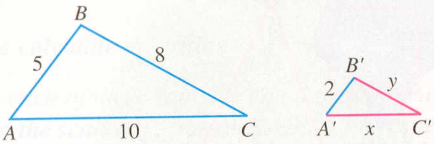
41. Triangle
42. Quadrilateral
43. Octagon
44. Nonagon
45. Dodecagon
46. Icosagon

In Exercises 47–52, the figures are similar. Find the length of side x and side y.

47.



48.



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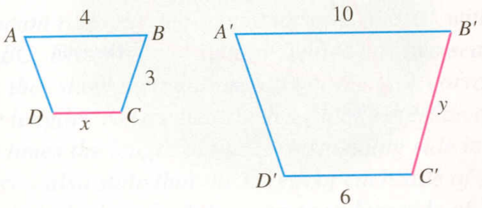
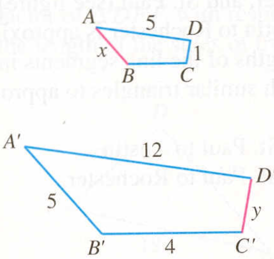


50.



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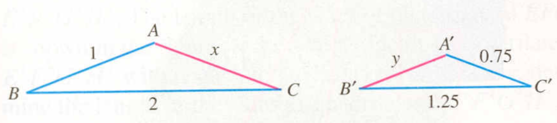
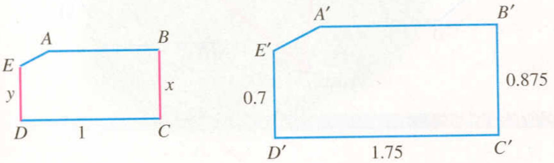


52.



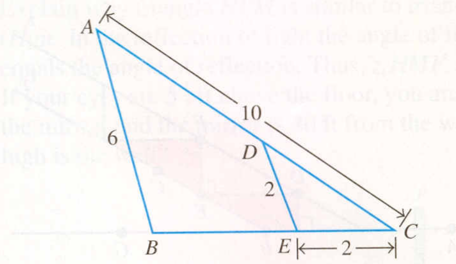
In Exercises 53–56, triangles ABC and DEC are similar figures. Find the length of

53. side $\overline{BC}$.

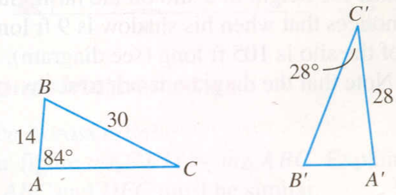
54. side $\overline{DC}$.

55. side $\overline{AD}$.

56. side $\overline{BE}$.



In Exercises 57–62, find the length of the sides and the measures of the angles for the congruent triangles ABC and A′B′C′.



57. The length of side $\overline{A'B'}$

58. The length of side $\overline{B'C'}$

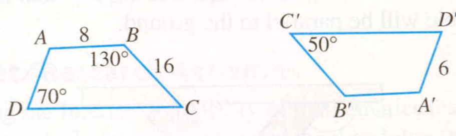
59. The length of side $\overline{AC}$

60. $\measuredangle B'A'C'$

61. $\measuredangle ACB$

62. $\measuredangle ABC$

In Exercises 63–68, find the length of the sides and the measures of the angles for the congruent quadrilaterals ABCD and A′B′C′D′.



63. The length of side $\overline{A'B'}$

64. The length of side $\overline{AD}$

65. The length of side $\overline{B'C'}$

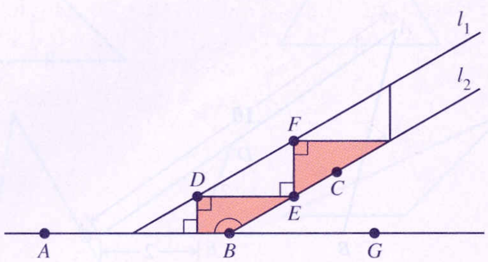
66. $\measuredangle BCD$

67. $\measuredangle A'D'C'$

68. $\measuredangle DAB$

Problem Solving

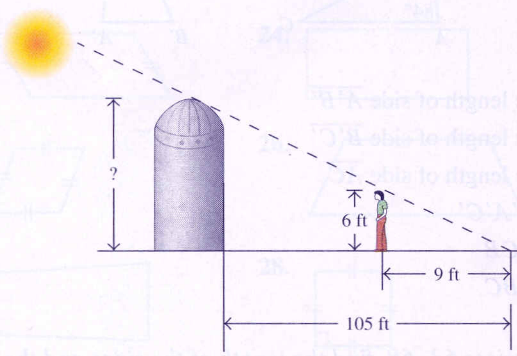
In Exercises 69–72, determine the measure of the angle. In the figure, $\measuredangle ABC$ makes an angle of 125° with the floor and l_1 and l_2 are parallel.



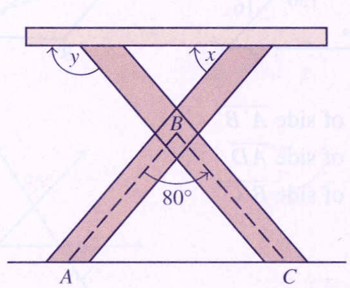
69. $\measuredangle GBC$ **70.** $\measuredangle EDF$

71. $\measuredangle DFE$ **72.** $\measuredangle DEC$

73. ***Height of a Silo*** Steve Runde is buying a farm and needs to determine the height of a silo on the farm. Steve, who is 6 ft tall, notices that when his shadow is 9 ft long, the shadow of the silo is 105 ft long (see diagram). How tall is the silo? Note that the diagram is not to scale.



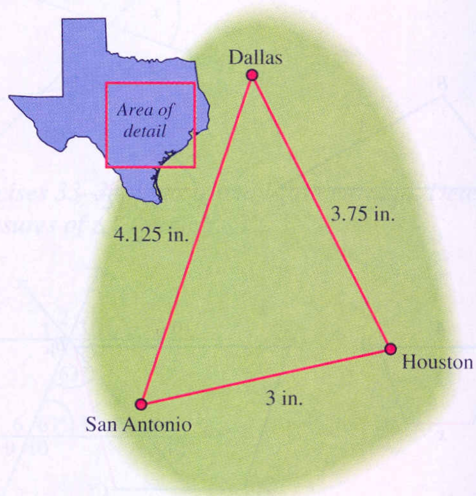
74. ***Angles on a Picnic Table*** The legs of a picnic table form an isosceles triangle as indicated in the figure. If $\measuredangle ABC = 80°$, determine $m\measuredangle x$ and $m\measuredangle y$ so that the top of the table will be parallel to the ground.



75. ***Distances in Texas*** A triangle can be formed by drawing line segments on a map of Texas connecting the cities of Dallas, Houston, and San Antonio (see figure). If the actual distance from San Antonio to Houston is approximately 197 miles, use the lengths of the line segments indicated in the figure along with similar triangles to approximate

a) the actual distance from Dallas to Houston.

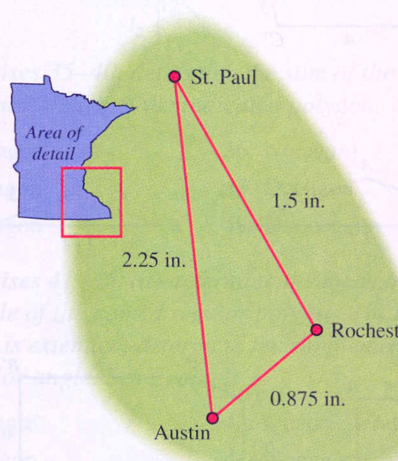
b) the actual distance from Dallas to San Antonio.



76. ***Distances in Minnesota*** A triangle can be formed by drawing line segments on a map of Minnesota connecting the cities of Austin, Rochester, and St. Paul (see figure). If the actual distance from Austin to Rochester is approximately 44 miles, use the lengths of the line segments indicated in the figure along with similar triangles to approximate

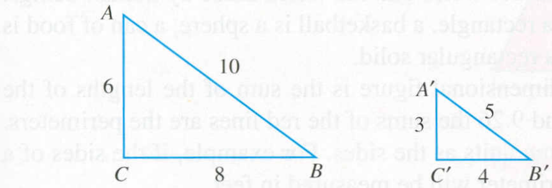
a) the actual distance from St. Paul to Austin.

b) the actual distance from St. Paul to Rochester.



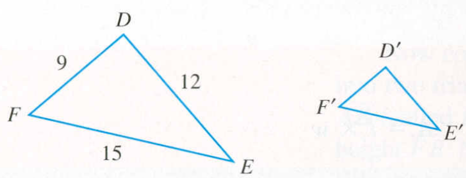
Challenge Problems/Group Activities

Scaling Factor *Examine the similar triangles ABC and A′B′C′ in the figure below.*

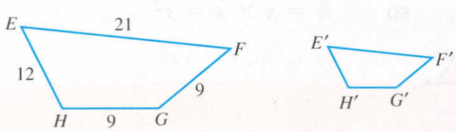


If we calculate the ratios $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, *and* $\frac{CA}{C'A'}$, *we see that each of these ratios is equal to 2. We call this common ratio the* ***scaling factor*** *of* ΔABC *with respect to* $\Delta A'B'C'$. *If we calculate the reciprocal ratios* $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, *and* $\frac{C'A'}{CA}$, *we see that each of these ratios is equal to* $\frac{1}{2}$. *We call this common ratio the scaling factor of* $\Delta A'B'C'$ *with respect to* ΔABC. *Every pair of similar figures has two scaling factors that show the relationship between the corresponding side lengths. Notice that the length of each side of* ΔABC *is two times the length of the corresponding side in* $\Delta A'B'C'$. *We can also state that the length of each side of* $\Delta A'B'C'$ *is one-half the length of the corresponding side of* ΔABC.

77. In the figure, ΔDEF is similar to $\Delta D'E'F'$. The length of the sides of ΔDEF is shown in the figure. If the scaling factor of ΔDEF with respect to $\Delta D'E'F'$ is 3, determine the length of the sides of triangle $\Delta D'E'F'$.



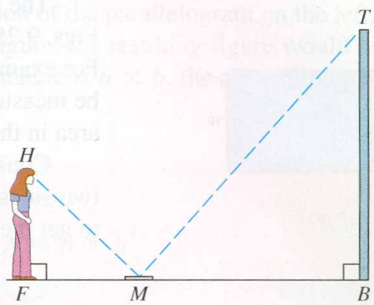
78. In the figure, quadrilateral $EFGH$ is similar to quadrilateral $E'F'G'H'$. The length of the sides of quadrilateral $EFGH$ is shown in the figure. If the scaling factor of quadrilateral $E'F'G'H'$ with respect to quadrilateral $EFGH$ is $\frac{1}{3}$, determine the length of the sides of quadrilateral $E'F'G'H'$.



79. *Height of a Wall* You are asked to measure the height of an inside wall of a warehouse. No ladder tall enough to measure the height is available. You borrow a mirror from a salesclerk and place it on the floor. You then move away from the mirror until you can see the reflection of the top of the wall in it, as shown in the figure.

a) Explain why triangle HFM is similar to triangle TBM. (*Hint:* In the reflection of light the angle of incidence equals the angle of reflection. Thus, $\measuredangle HMF = \measuredangle TMB$.)

b) If your eyes are $5\frac{1}{2}$ ft above the floor, you are $2\frac{1}{2}$ ft from the mirror, and the mirror is 20 ft from the wall, how high is the wall?

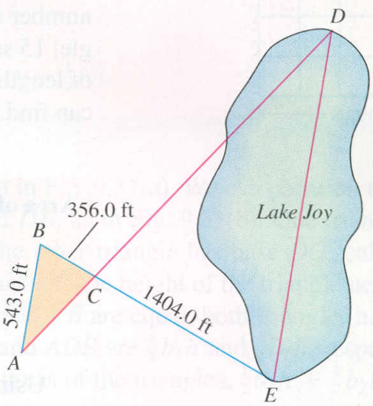


Recreational Mathematics

80. *Distance Across a Lake*

a) In the figure $m\measuredangle CED = m\measuredangle ABC$. Explain why triangles ABC and DEC must be similar.

b) Determine the distance across the lake, $\overline{DE}$.



Internet/Research Activities

81. Using the Internet and history of mathematics texts, write a paper on the history and use of the theodolite, a surveying instrument.

82. Using the Internet and other sources, write a paper on the use of geometry in the photographic process. Include discussions on the use of similar figures.

9.3 PERIMETER AND AREA

Perimeter and Area

Figure 9.25

Geometric shapes abound in the natural world and the world made by human beings. For example, a basketball court is a rectangle, a basketball is a sphere, a can of food is a cylinder, and a ream of paper is a rectangular solid.

The ***perimeter***, P, of a two-dimensional figure is the sum of the lengths of the sides of the figure. In Figs. 9.25 and 9.26 the sums of the red lines are the perimeters. Perimeters are measured in the same units as the sides. For example, if the sides of a figure are measured in feet, the perimeter will be measured in feet.

The ***area***, A, is the region within the boundaries of the figure. The blue color in Figs. 9.25 and 9.26 indicates the areas of the figures. Area is measured in square units. For example, if the sides of a figure are measured in inches, the area of the figure will be measured in square inches (in.^{2}). (See Table 8.7 on page 461 for common units of area in the U.S. customary and metric systems.)

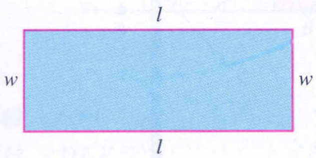


Figure 9.26

Consider the rectangle in Fig. 9.26. Two sides of the rectangle have length l, and two sides of the rectangle have width w. Thus, if we add the lengths of the four sides to get the perimeter, we find $P = l + w + l + w = 2l + 2w$.

Perimeter of a Rectangle

$$P = 2l + 2w$$

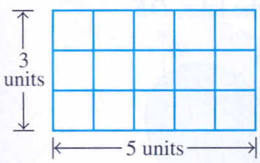


Figure 9.27

Consider a rectangle of length 5 units and width 3 units (Fig. 9.27). Counting the number of 1-unit by 1-unit squares within the figure we obtain the area of the rectangle, 15 square units. The area can also be obtained by multiplying the number of units of length by the number of units of width, or 5 units $\times$ 3 units $=$ 15 square units. We can find the area of a rectangle by the formula area $=$ length $\times$ width.

Area of a Rectangle

$$A = l \times w$$

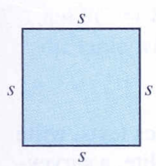


Figure 9.28

Using the formula for the area of a rectangle, we can determine the formulas for the areas of other figures.

A square (Fig. 9.28) is a rectangle that contains four equal sides. Therefore, the length equals the width. If we call both the length and the width of the square s, then

$$A = l \times w, \qquad \text{so} \qquad A = s \times s = s^2$$

Area of a Square

$$A = s^2$$

A parallelogram with height h and base b is shown in Fig. 9.29(a).

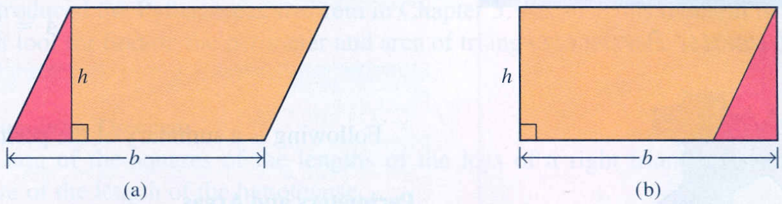


Figure 9.29

If we were to cut off the red portion of the parallelogram on the left, Fig. 9.29(a), and attach it to the right side of the figure, the resulting figure would be a rectangle, Fig. 9.29(b). Since the area of the rectangle is $b \times h$, the area of the parallelogram is also $b \times h$.

Area of a Parallelogram

$$A = b \times h$$

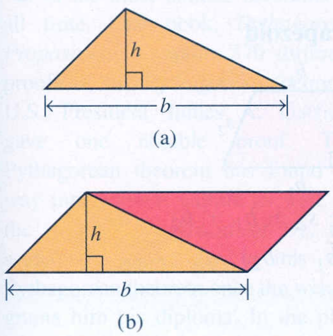


Figure 9.30

Consider the triangle with height, h, and base, b, shown in Fig. 9.30(a). Using this triangle and a second identical triangle, we can construct a parallelogram, Fig. 9.30(b). The area of the parallelogram is bh. The area of the triangle is one-half that of the parallelogram. Therefore, the area of the triangle is $\frac{1}{2}(\text{base})(\text{height})$.

Area of a Triangle

$$A = \frac{1}{2}bh$$

Now consider the trapezoid shown in Fig. 9.31(a). We can partition the trapezoid into two triangles by drawing diagonal $\overline{DB}$, as in Fig. 9.31(b). One triangle has base $\overline{AB}$ (called b_2) with height $\overline{DE}$, and the other triangle has base $\overline{DC}$ (called b_1) with height $\overline{FB}$. Note that the line used to measure the height of the triangle need not be inside the triangle. Because heights $\overline{DE}$ and $\overline{FB}$ are equal, both triangles have the same height, h. The area of triangles DCB and ADB are $\frac{1}{2}b_1h$ and $\frac{1}{2}b_2h$, respectively. The area of the trapezoid is the sum of the areas of the triangles, $\frac{1}{2}b_1h + \frac{1}{2}b_2h$, which can be written $\frac{1}{2}h(b_1 + b_2)$.

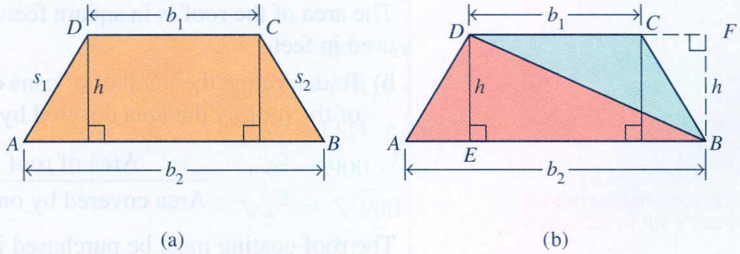
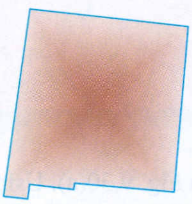


Figure 9.31

DID YOU KNOW

Area versus Perimeter



New Mexico

Which geographic area, Italy or New Mexico, do you think has the larger land area? As a matter of fact, the land areas of Italy and New Mexico are very similar: Italy's area is 116,304 mi^2 and New Mexico's area is 121,593 mi^2. Italy's perimeter, 5812 mi, is almost five times greater than New Mexico's 1200 mi, however.

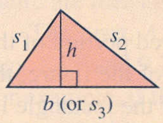
Area of a Trapezoid

$$A = \tfrac{1}{2}h(b_1 + b_2)$$

Following is a summary of the perimeters and areas of selected figures.

Perimeters and Areas

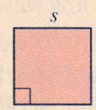
Triangle



$P = s_1 + s_2 + s_3\ (s_3 = b)$

$A = \tfrac{1}{2}bh$

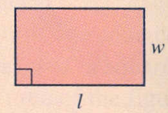
Square



$P = 4s$

$A = s^2$

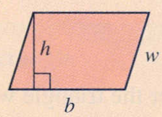
Rectangle



$P = 2l + 2w$

$A = lw$

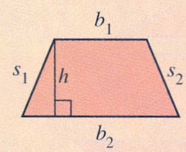
Parallelogram



$P = 2b + 2w$

$A = bh$

Trapezoid



$P = s_1 + s_2 + b_1 + b_2$

$A = \tfrac{1}{2}h(b_1 + b_2)$

EXAMPLE 1 *Resurfacing a Restaurant Roof*

Richard McMenamy recently purchased the Lazy Lobster Restaurant and needs to resurface the roof of the sunporch with aluminum roof coating. One can of Kopper's Aluminum roof coating costs \$10.99 and covers 330 ft^2. If the roof of the sunporch is 40 ft long and 16 ft wide, determine

a) the area of the roof.

b) how many cans of roof coating he needs.

c) the cost of resurfacing the roof of the sunporch.

SOLUTION:

a) The area of the roof is

$$A = l \cdot w = 40 \cdot 16 = 640 \text{ ft}^2$$

The area of the roof is in square feet because both the length and width are measured in feet.

b) To determine the number of cans of roof coating Richard needs, divide the area of the roof by the area covered by one can of roof coating.

$$\frac{\text{Area of roof}}{\text{Area covered by one can}} = \frac{640}{330} = 1\frac{31}{33} \approx 1.94$$

The roof coating must be purchased in whole cans, so Richard needs to buy two cans of roof coating.

c) The cost of two cans of the roof coating is $2 \times \$10.99$, or \$21.98. ▲

MATHEMATICS *Everywhere*

Pythagorean Theorem

The Pythagorean theorem is one of the most famous theorems of all time. One book, *Pythagorean Propositions*, contains 370 different proofs of the Pythagorean theorem. U.S. President James A. Garfield gave one notable proof. The Pythagorean theorem has found its way into popular culture as well. In the movie *The Wizard of Oz*, the scarecrow incorrectly recites the Pythagorean theorem once the wizard grants him his diploma. In the play *The Pirates of Penzance*, the Major General refers to the Pythagorean theorem when he sings "I'm teeming with a lot o' news, with many cheerful facts about the square of the hypotenuse." Lewis Carroll, author of *Through the Looking Glass* and *Alice's Adventures in Wonderland*, stated that the Pythagorean theorem "is as dazzlingly beautiful now as it was in the day when Pythagoras discovered it." Actually, the Pythagorean theorem was known to the ancient Babylonians in about 1600 B.C., 1000 years before Pythagoras, and it continues to play a huge role in mathematics.

Pythagorean Theorem

We introduced the Pythagorean theorem in Chapter 5. Because this theorem is an important tool for finding the perimeter and area of triangles, we restate it here.

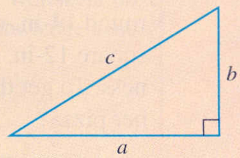
Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse.

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$

Symbolically, if a and b represent the lengths of the legs and c represents the length of the hypotenuse (the side opposite the right angle), then

$$a^2 + b^2 = c^2$$



EXAMPLE 2 *Crossing a Moat*

The moat surrounding a castle is 18 ft wide and the wall by the moat of the castle is 24 ft high (see Figure 9.32). If an invading army wishes to use a ladder to cross the moat and reach the top of the wall, how long must the ladder be?

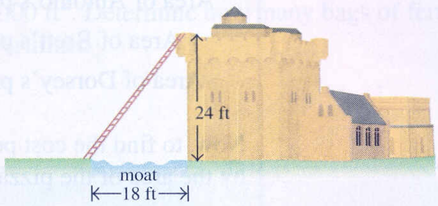


Figure 9.32

SOLUTION: The moat, the castle wall, and the ladder form a right triangle. The moat and the castle wall form the legs of the triangle (sides a and b), and the ladder forms the hypotenuse (side c). By the Pythagorean theorem,

$$c^2 = a^2 + b^2$$
$$c^2 = (18)^2 + (24)^2$$
$$c^2 = 324 + 576$$
$$c^2 = 900$$
$$\sqrt{c^2} = \sqrt{900} \quad \text{Take the square root of both sides of the equation.}$$
$$c = 30$$

Therefore, the ladder would need to be at least 30 ft long.

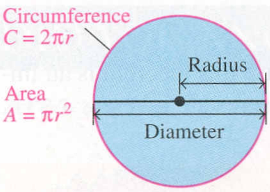


Figure 9.33

Circles

A commonly used plane figure that is not a polygon is a *circle*. A ***circle*** is a set of points equidistant from a fixed point called the center. A ***radius***, r, of a circle is a line segment from the center of the circle to any point on the circle (Fig. 9.33). A ***diameter***, d, of a circle is a line segment through the center of a circle with both end points on the circle. Note that the diameter of the circle is twice its radius. The ***circumference*** is the length of the simple closed curve that forms the circle. The formulas for the area and circumference of a circle are given in Fig. 9.33. The symbol pi, π, was introduced in Chapter 5. Recall that π is approximately 3.14. If your calculator contains a $\boxed{\pi}$ key, you should use that key when working calculations involving pi.

EXAMPLE 3 *Comparing Pizzas*

Victoria Montoya wishes to order a large cheese pizza. She can choose among three pizza parlors in town: Antonio's, Brett's, and Dorsey's. Antonio's large cheese pizza is a round 16-in.-diameter pizza that sells for \$15. Brett's large cheese pizza is a round 14-in.-diameter pizza that sells for \$12. Dorsey's large cheese pizza is a square 12-in. by 12-in. pizza that sells for \$10. All three pizzas have the same thickness. To get the most for her money, from which pizza parlor should Victoria order her pizza?

SOLUTION: To determine the best value, we will calculate the cost per square inch of pizza for each of the three pizzas. To do so we will divide the cost of each pizza by its area. The areas of the two round pizzas can be determined using the formula for the area of a circle, $A = \pi r^2$. Since the radius is half the diameter, we will use $r = 8$ and $r = 7$ for Antonio's and Brett's large pizzas, respectively. The area for the square pizza can be determined using the formula for the area of a square, $A = s^2$. We will use $s = 12$.

$$\text{Area of Antonio's pizza} = \pi r^2 \approx (3.14)(8)^2 \approx 3.14(64) \approx 200.96 \text{ in.}^2*$$
$$\text{Area of Brett's pizza} = \pi r^2 \approx (3.14)(7)^2 \approx 3.14(49) \approx 153.86 \text{ in.}^2*$$
$$\text{Area of Dorsey's pizza} = s^2 = (12)^2 = 144 \text{ in.}^2$$

Now, to find the cost per square inch of pizza, we will divide the cost of the pizza by the area of the pizza.

$$\text{Cost per square inch of Antonio's pizza} \approx \frac{\$15}{200.96 \text{ in.}^2} \approx \$0.0746$$

Thus, Antonio's pizza costs about \$0.0746, or about 7.5 cents, per square inch.

$$\text{Cost per square inch of Brett's pizza} \approx \frac{\$12}{153.86 \text{ in.}^2} \approx \$0.0780$$

Thus, Brett's pizza costs about \$0.0780, or about 7.8 cents, per square inch.

$$\text{Cost per square inch of Dorsey's pizza} = \frac{\$10}{144 \text{ in.}^2} \approx \$0.0694$$

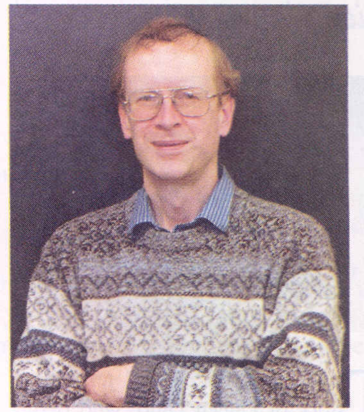
*If you use the $\boxed{\pi}$ key on your calculator, your answers will be slightly more accurate.

DID YOU KNOW

Fermat's Last Theorem

Pierre de Fermat

In 1637 Pierre de Fermat, an amateur French mathematician, scribbled a note in the margin of the book *Arithmetica* by Diophantus. The note would haunt mathematicians for centuries. Fermat stated that the generalized form of the Pythagorean theorem, $a^n + b^n = c^n$, has no positive integer solutions where $n > 2$. Fermat's note concluded, "I have a truly marvelous demonstration of this proposition, which this margin is too narrow to contain." This conjecture became known as Fermat's last theorem. A formal proof of this conjecture escaped mathematicians until on September 19, 1994, Andrew J. Wiles of Princeton University announced he had found a proof. It took Wiles over 8 years of work—including fixing a flaw in an earlier announced solution—to accomplish the task. Wiles was awarded the Wolfskehl prize at Göttingen University in Germany in acknowledgement of his achievement.



Andrew J. Wiles

Thus, Dorsey's pizza costs about \$0.0694, or about 6.9 cents, per square inch.

Since the cost per square inch of pizza is the lowest for Dorsey's pizza, Victoria would get the most pizza for her money by ordering her pizza from Dorsey's. ▲

EXAMPLE 4 *Applying Lawn Fertilizer*

Steve May plans to fertilize his lawn. The shapes and dimensions of his lot, house, driveway, pool, and rose garden are shown in Fig. 9.34. One bag of fertilizer costs \$29.95 and covers 5000 ft^2. Determine how many bags of fertilizer Steve needs and the total cost of the fertilizer.

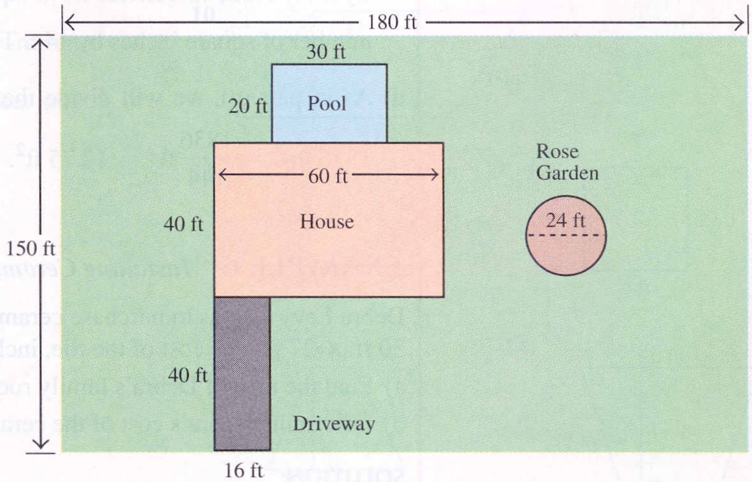
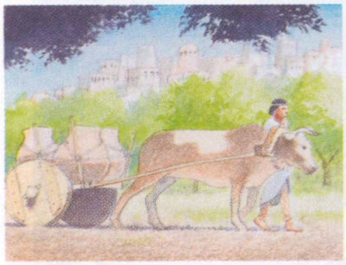


Figure 9.34

DID YOU KNOW

City Planning



The Roman poet Virgil tells the story of Queen Dido, who fled to Africa after her brother murdered her husband. There, she begged for some land from King Iarbus, telling him she only needed as much land as the hide of an ox would enclose. Being very clever, she decided that the greatest area would be enclosed if she tore the hide into thin strips and formed the strips into a circle. On this land she founded the city of Byrsa (the Greek word for "hide"), later known as Carthage in present-day Tunisia.

SOLUTION: The total area of the lot is $150 \cdot 180$, or 27,000 ft^2. To determine the area to be fertilized, subtract the area of house, driveway, pool, and rose garden from the total area.

$$\text{Area of house} = 60 \cdot 40 = 2400 \text{ ft}^2$$
$$\text{Area of driveway} = 40 \cdot 16 = 640 \text{ ft}^2$$
$$\text{Area of pool} = 20 \cdot 30 = 600 \text{ ft}^2$$

The diameter of the rose garden is 24 ft, so its radius is 12 ft.

$$\text{Area of rose garden} = \pi r^2 = \pi(12)^2 \approx 3.14(144) \approx 452.16 \text{ ft}^2$$

The total area of the house, driveway, pool, and rose garden is approximately $2400 + 640 + 600 + 452.16$, or 4092.16 ft^2. The area to be fertilized is $27{,}000 - 4092.16$ ft^2, or 22,907.84 ft^2. The number of bags of fertilizer is found by dividing the total area to be fertilized by the number of square feet covered per bag. The number of bags of fertilizer is $\frac{22{,}907.84}{5000}$, or about 4.58 bags. Therefore, Steve needs five bags. At \$29.95 per bag, the total cost is $5 \times \$29.95$, or \$149.75. ▲

EXAMPLE 5 *Converting between Square Feet and Square Inches*

a) Convert 1 ft^2 to square inches.

b) Convert 86 ft^2 to square inches.

c) Convert 288 in.^{2} to square feet.

d) Convert 1836 in.^{2} to square feet.

SOLUTION:

a) 1 ft = 12 in. Therefore, 1 ft^2 = 12 in. × 12 in. = 144 in.^{2}

b) From part (a) we know that 1 ft^2 = 144 in.^{2} Therefore, 86 ft^2 = 86 × 144 in.^{2} = 12,384 in.^{2}

c) In part (b) we converted from ft^2 to in.^{2} by *multiplying* the number of square feet by 144. Now, to convert from square inches to square feet we will *divide* the number of square inches by 144. Therefore, 288 in.^{2} $= \frac{288}{144}$ ft^2 = 2 ft^2.

d) As in part (c), we will divide the number of square inches by 144. Therefore, 1836 in.^{2} $= \frac{1836}{144}$ ft^2 = 12.75 ft^2. ▲

EXAMPLE 6 *Installing Ceramic Tile*

Debra Levy wishes to purchase ceramic tile for her family room, which measures 30 ft × 27 ft. The cost of the tile, including installation, is \$21 per square yard.

a) Find the area of Debra's family room in square *yards*.

b) Determine Debra's cost of the ceramic tile for her family room.

SOLUTION:

a) The area of the family room in square feet is $30 \cdot 27 = 810$ ft^2.

Since 1 yd $= 3$ ft, 1 yd$^2 = 3$ ft $\times$ 3 ft $= 9$ ft^2. To find the area of the family room in square yards, divide the area in square feet by 9 ft^2.

$$\text{Area in square yards} = \frac{810}{9} = 90$$

Therefore, the area is 90 yd^2.

b) The cost of 90 yd^2 of ceramic tile, including installation, is $90 \cdot \$21 = \$1890$.

When multiplying units of length, be sure that the units are the same. You can multiply feet by feet to get square feet or yards by yards to get square yards. However, you cannot get a valid answer if you multiply numbers expressed in feet by numbers expressed in yards.

SECTION 9.3 EXERCISES

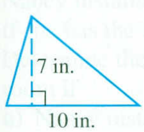
Concept/Writing Exercises

1. a) Describe in your own words how to determine the *perimeter* of a two-dimensional figure.
   b) Describe in your own words how to determine the *area* of a two-dimensional figure.
   c) Draw a rectangle with a length of 6 units and a width of 2 units. Determine the area and perimeter of this rectangle.
2. What is the relationship between the *radius* and the *diameter* of a circle?
3. a) How do you convert an area from square feet into square inches?
   b) How do you convert an area from square inches into square feet?
4. a) How do you convert an area from square yards into square feet?
   b) How do you convert an area from square feet into square yards?

Practice the Skills

In Exercises 5–8, find the area of the triangle.

5.



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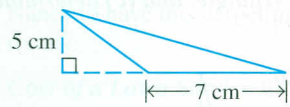

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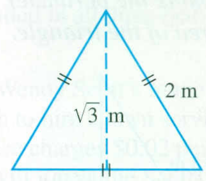
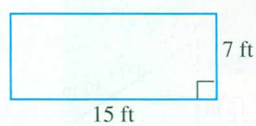


In Exercises 9–14, find the area and perimeter of the quadrilateral.

9.



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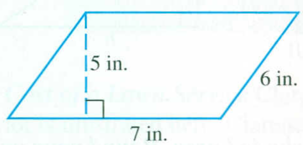

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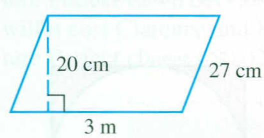

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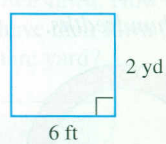

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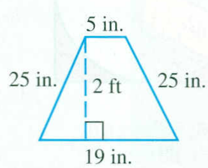

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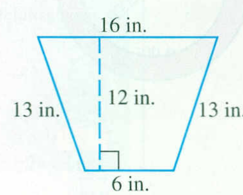
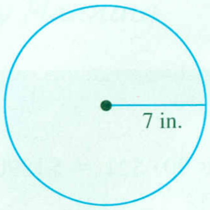


In Exercises 15–18, find the area and circumference of the circle.

15.



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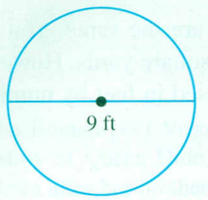

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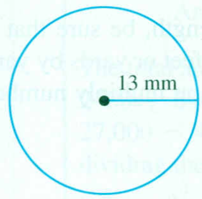

In Exercises 19–22, (a) use the Pythagorean theorem to determine the length of the unknown side of the triangle, (b) determine the perimeter of the triangle, and (c) determine the area of the triangle.

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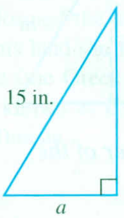

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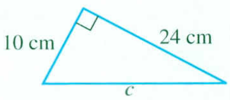

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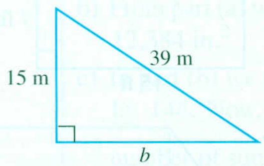
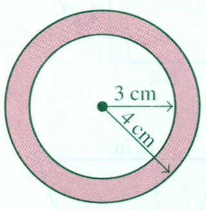


Problem Solving

In Exercises 23–32, find the shaded area. Round your answers to hundredths.

23.



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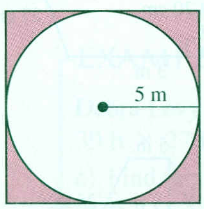

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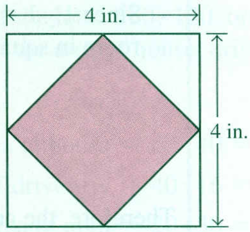

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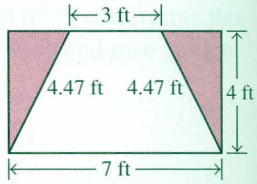

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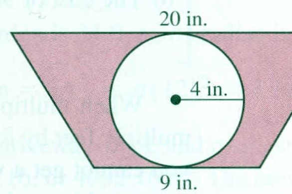

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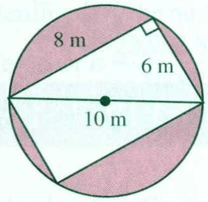

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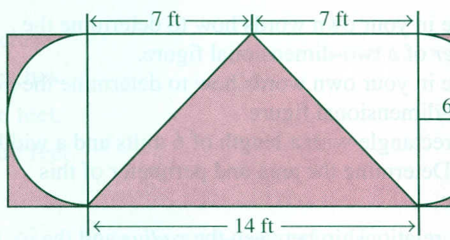

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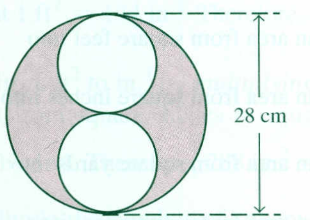

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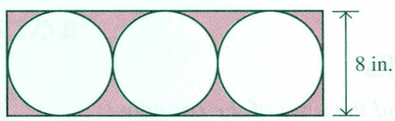

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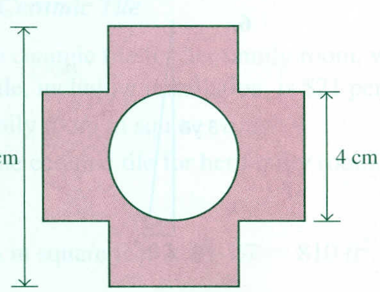


One square yard equals 9 ft^2. Use this information to convert the following.

33. 107 ft^2 to square yards **34.** 15.2 ft^2 to square yards

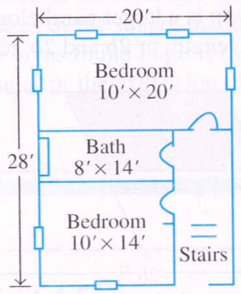
35. 14.7 yd^2 to square feet **36.** 18.3 yd^2 to square feet

One square meter equals 10,000 cm^2. Use this information to convert the following.

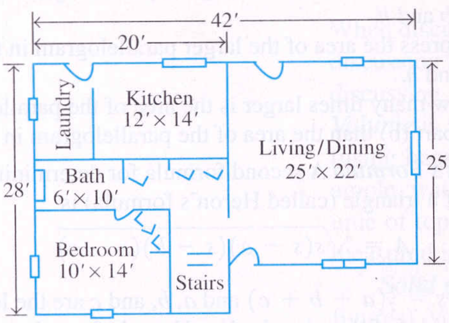
37. 23.4 m^2 to square centimeters **38.** 14.7 m^2 to square centimeters

39. 1075 cm^2 to square meters **40.** 608 cm^2 to square meters

Nancy Wallin has just purchased a new house that is in need of new flooring. In Exercises 41–46, use the measurements given on the floor plans of Nancy's house to obtain the answer.



Second floor



First floor

41. ***Cost of Laminate Flooring*** The cost of Pergo Select Helsinki Birch laminate flooring is \$5.89 per square foot if Nancy installs the flooring herself or \$8.89 per square foot if she has the flooring installed by the flooring company. Determine the cost for the flooring in the living/dining room if

a) Nancy installs it herself.

b) Nancy has it installed by the flooring company.

42. ***Cost of Hardwood Flooring*** The cost of Mannington Chestnut hardwood flooring is \$10.86 per square foot if Nancy installs the flooring herself or \$13.86 per square foot if she has the flooring installed by the flooring company. Determine the cost for hardwood flooring in the living/dining room if

a) Nancy installs it herself.

b) Nancy has it installed by the flooring company.

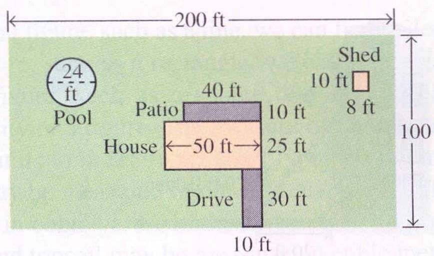
43. ***Cost of Linoleum*** The cost of Armstrong Solarian Woodcut linoleum is \$5.00 per square foot. This price includes the cost of installation. Determine the cost for Nancy to have this linoleum installed in the kitchen and in both bathrooms.

44. ***Cost of Ceramic Tile*** The cost of Mohawk Porcelain ceramic tile is \$8.50 per square foot. This price includes the cost of installation. Determine the cost for Nancy to have this ceramic tile installed in the kitchen and in both bathrooms.

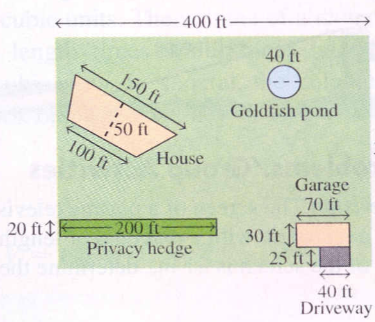
45. ***Cost of Berber Carpeting*** The cost of Bigelow Commodore Berber carpeting is \$6.06 per square foot. This price includes the cost of installation. Determine the cost for Nancy to have this carpeting installed in all three bedrooms.

46. ***Cost of Saxony Carpeting*** The cost of DuPont Stainmaster Saxony carpeting is \$5.56 per square foot. This price includes the cost of installation. Determine the cost for Nancy to have this carpeting installed in all three bedrooms.

47. ***Cost of a Lawn Service*** Jim and Wendy Scott's home lot is illustrated here. The Scotts wish to hire a lawn service to cut their lawn. M&M Lawn Service charges \$0.02 per square yard of lawn. How much will it cost the Scotts to have their lawn cut?



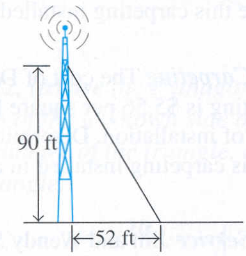
48. ***Cost of a Lawn Service*** Clarence and Rose Cohen's home lot is illustrated here. Clarence and Rose wish to hire Picture Perfect Lawn Service to cut their lawn. How much will it cost Clarence and Rose to have their lawn cut if Picture Perfect charges \$0.02 per square yard?



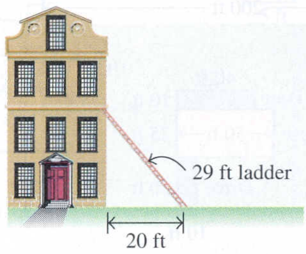
49. *Area of a Garden* Gaetano Cannata's rectangular garden is 11.5 m by 15.4 m.
  a) How large is his garden in square meters?
  b) If 1 hectare (a measurement of area in the metric system) equals 10,000 m^2, how large is his garden in hectares?

50. *Hamburger Comparison* Which hamburger has the larger surface area: a square hamburger 3 in. on a side from Wendy's or a $3\frac{1}{2}$-in.-diameter round hamburger from Burger King? Explain your answer and give the difference in their surface areas.

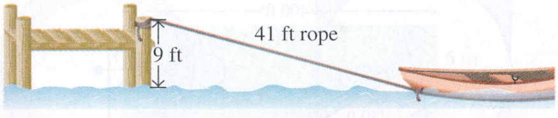
51. *Anchoring a Radio Signal Tower* A 100 ft radio signal tower is being constructed. To steady the tower, guy wires are attached to the tower. One end of the highest guy wire is attached to the tower at a point 90 ft above the ground (see figure). The other end is anchored into the ground at a point 52 ft from the base of the tower. How long is this guy wire?



52. *Ladder on a Wall* Lorrie Morgan places a 29 ft ladder against the side of a building with the bottom of the ladder 20 ft away from the building (see figure). How high up on the wall does the ladder reach?



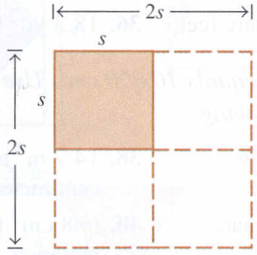
53. *Docking a Boat* Brian Murphy is bringing his boat into a dock that is 9 ft above the water level (see figure). If a 41 ft rope is attached to the dock on one side and to the boat on the other side, determine the horizontal distance from the dock to the boat.



Challenge Problems/Group Activities

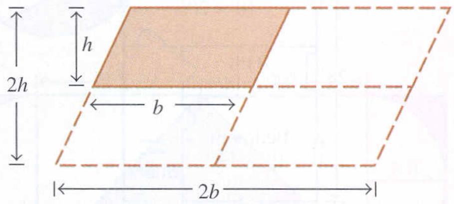
54. *Plasma Television* The screen of a plasma television is in the shape of a rectangle with a diagonal of length 43 in. If the height of the screen is 21 in., determine the width of the screen.

55. *Doubling the Sides of a Square* In the figure below, an original square with sides of length s is shown. Also shown is a larger square with sides double in length, or $2s$.



  a) Express the area of the original square in terms of s.
  b) Express the area of the larger square in terms of s.
  c) How many times larger is the area of the square in part (b) than the area of the square in part (a)?

56. *Doubling the Sides of a Parallelogram* In the figure below, an original parallelogram with base b and height h is shown. Also shown is a larger parallelogram with base and height double in length, or $2b$ and $2h$, respectively.

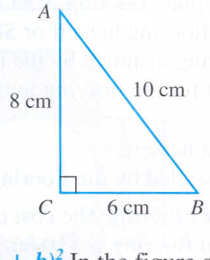


  a) Express the area of the original parallelogram in terms of b and h.
  b) Express the area of the larger parallelogram in terms of b and h.
  c) How many times larger is the area of the parallelogram in part (b) than the area of the parallelogram in part (a)?

57. *Heron's Formula* A second formula for determining the area of a triangle (called Heron's formula) is

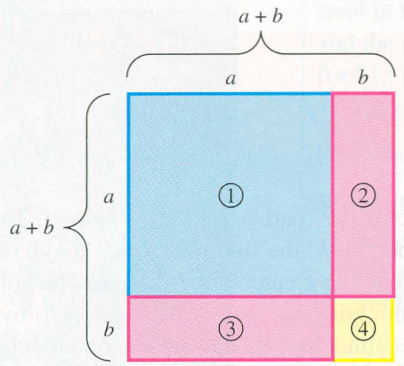
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ and a, b, and c are the lengths of the sides of the triangle. Use Heron's formula to determine the area of right triangle ABC and check your answer using the formula $A = \frac{1}{2}ab$.



58. *Expansion of $(a + b)^2$* In the figure on page 507, one side of the largest square has length $a + b$. Therefore, the area

of the largest square is $(a+b)^2$. Answer the following questions to find a formula for the expansion of $(a+b)^2$.



a) What is the area of the square marked 1?
b) What is the area of the rectangle marked 2?
c) What is the area of the rectangle marked 3?
d) What is the area of the square marked 4?
e) Add the four areas found in parts (a) through (d) to write a formula for the expansion of $(a+b)^2$.

Recreational Mathematics

59. *Scarecrow's Error* In the movie *The Wizard of Oz,* once the scarecrow gets his diploma he states the following: "In an isosceles triangle, the sum of the square roots of the two equal sides is equal to the square root of the third side." Discuss why this statement is incorrect.

Internet/Research Activities

For Exercises 60–62, references include the Internet, history of mathematics textbooks, and encyclopedias.

60. Research the proof of the Pythagorean theorem provided by President James Garfield. Write a brief paper and make a poster of this proof and the associated diagrams.
61. The early Babylonians and Egyptians did not know about π and had to devise techniques to approximate the area of a circle. Do research and write a paper on the techniques these societies used to approximate the area of a circle.
62. Write a paper on the contributions of Heron of Alexandria to geometry.

9.4 VOLUME

When discussing a one-dimensional figure, such as a line, we can find its length. When discussing a two-dimensional figure, such as a rectangle, we can find its area. When discussing a three-dimensional figure, such as a sphere, we can find its volume. ***Volume*** is a measure of the capacity of a figure. The measure of volume may be confusing because we use different units to measure different types of volumes. For example, water and other liquids may be measured in ounces, quarts, or gallons. A volume of topsoil may be measured in cubic yards. In the metric system, liquid may be measured in liters or milliliters, and topsoil may be measured in cubic meters.

Solid geometry is the study of three-dimensional solid figures (also called space figures). Volumes of three-dimensional figures are measured in cubic units such as cubic feet or cubic meters.

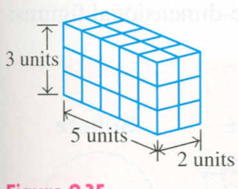


Figure 9.35

We will begin our discussion with the *rectangular solid.* If the length of the solid is 5 units, the width is 2 units, and the height is 3 units, the total number of cubes is 30 (Fig. 9.35). Thus, the volume is 30 cubic units. The volume of a rectangular solid can also be found by multiplying its length times width times height; in this case, 5 units $\times$ 2 units $\times$ 3 units $=$ 30 cubic units. In general, the volume of any rectangular solid (shown in part a in the box on page 508), is $V = l \times w \times h$.

Volume of a Rectangular Solid

$$V = l \times w \times h$$

A *cube* is a rectangular solid with the same length, width, and height (part b in the box below). If we call the side of a cube s and use the formula for a rectangular solid, substituting s for l, w, and h, we obtain $V = s \cdot s \cdot s = s^3$.

Volume of a Cube

$$V = s^3$$

Now consider the right circular cylinder (part c in the box below). The base is a circle with area πr^2. When we add height, h, the figure becomes a cylinder. For the same circular base, the greater the height, the greater is the volume. The volume of the right circular cylinder is found by multiplying the area of the base, πr^2, by the height h. In this book, when we use the term cylinder we mean a right circular cylinder.

Volume of a Cylinder

$$V = \pi r^2 h$$

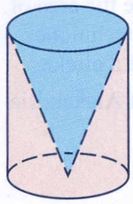


Figure 9.36

A cone is illustrated in part (d) in the box below. Imagine a cone inside a cylinder, sharing the same circle as the base. The volume of the cone is less than the volume of the cylinder that has the same base and the same height (Fig. 9.36). In fact, the volume of the cone is one-third the volume of the cylinder.

Volume of a Cone

$$V = \frac{1}{3}\pi r^2 h$$

The next shape we will discuss in this section is the sphere (part e in the box below). Basketballs, golf balls, and so on have the shape of a sphere. The formula for the volume of a sphere is as follows.

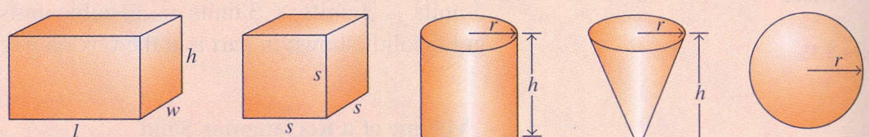
Volume of a Sphere

$$V = \frac{4}{3}\pi r^3$$

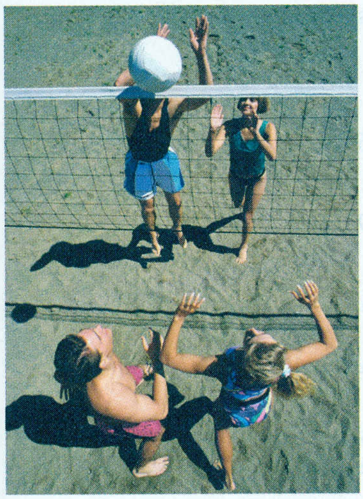
The following is a summary of the volumes of selected three-dimensional figures:

Volumes

Rectangular Solid	Cube	Cylinder	Cone	Sphere



$V = l \times w \times h$	$V = s^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{4}{3}\pi r^3$
(a)	(b)	(c)	(d)	(e)



EXAMPLE 1 *Replacing a Sand Volleyball Court*

Linda Nelson is the manager at the Colony Apartments and needs to replace the sand in the rectangular sand volleyball court. The court is 30 ft wide by 60 ft long, and the sand has a uniform depth of 18 in., see Fig. 9.37. Volleyball court sand sells for $15 per cubic yard.

a) How many cubic yards of sand does Linda need?

b) How much will the sand cost?

SOLUTION:

a) Since we are asked to find the volume in cubic yards, we will convert each measurement to yards. There are 3 ft in a yard. Thus, 30 ft equals $\frac{30}{3}$ or 10 yd, and 60 ft equals $\frac{60}{3}$ or 20 yd. There are 36 in. in a yard, so 18 in. equals $\frac{18}{36}$ or $\frac{1}{2}$ yd. The amount of sand needed is determined using the formula for the volume of a rectangular solid, $V = l \cdot w \cdot h$. In this case the height of the rectangular solid can be considered the depth of the sand.

$$V = l \cdot w \cdot h = 10 \cdot 20 \cdot \tfrac{1}{2} = 100 \text{ yd}^3$$

Note that since the measurements for length, width, and height are each in terms of yards, the answer is in terms of cubic yards.

b) One cubic yard of sand costs $15, so 100 yd^3 will cost $100 \times \$15$, or $1500.

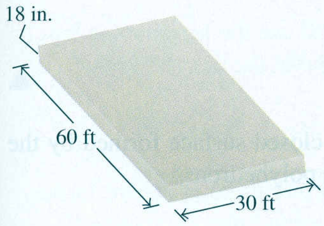


Figure 9.37

EXAMPLE 2 *Silage Storage*

Gordon Langeneger has three silos on his farm. The silos are each in the shape of a right circular cylinder (see Fig. 9.38). One silo has a 12 ft diameter and is 40 ft tall. The second silo has a 14 ft diameter and is 50 ft tall. The third silo has an 18 ft diameter and is 60 ft tall.

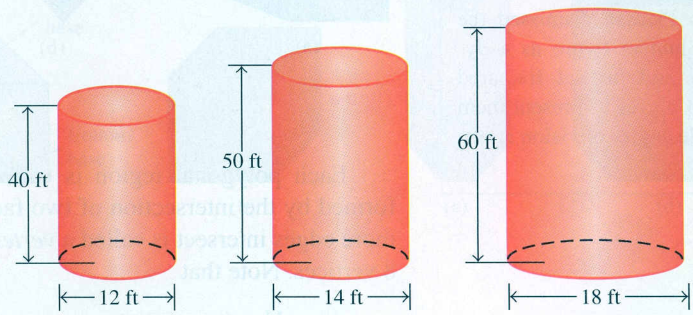


Figure 9.38

a) What is the total capacity of the three silos in cubic feet?

b) If Gordon fills all three of his silos and then feeds his cattle 150 ft^3 of silage per day, in how many days will all three silos be empty?

SOLUTION:

a) The capacity of each silo can be determined using the formula for the volume of a right circular cylinder, $V = \pi r^2 h$. Since the radius is half of the diameter, the

DID YOU KNOW

Fractured Volumes

Picasso, Pablo. *Les Demoiselles d'Avignon.* Paris (June–July 1907). Oil on canvas, 8 ft × 7 ft 8 in. (243.9 cm × 233.7 cm). The Museum of Modern Art, New York. Acquired through the Lillie P. Bliss Bequest. Photograph © 1996 The Museum of Modern Art, New York. © Estate of Pablo Picasso.

The cubist painters of the early twentieth century sought to analyze forms as geometric shapes. They were greatly influenced by Paul Cézanne who, in a famous letter, wrote of treating nature in terms of the cylinder, sphere, and cone. In one of the first paintings of the cubist period, *Les Demoiselles d'Avignon*, Pablo Picasso dismissed the idea of the human figure as a dynamic unity and instead fractured volumes and planes to present them from multiple angles of vision at one time.

radii for the three silos are 6 ft, 7 ft, and 9 ft, respectively. Now let's determine the volumes.

$$\begin{aligned}\text{Volume of the first silo} &= \pi r^2 h = \pi \cdot 6^2 \cdot 40\\ &\approx 3.14 \cdot 36 \cdot 40 \approx 4521.6 \text{ ft}^3\\ \text{Volume of the second silo} &= \pi r^2 h = \pi \cdot 7^2 \cdot 50\\ &\approx 3.14 \cdot 49 \cdot 50 \approx 7693.0 \text{ ft}^3\\ \text{Volume of the third silo} &= \pi r^2 h = \pi \cdot 9^2 \cdot 60\\ &\approx 3.14 \cdot 81 \cdot 60 \approx 15{,}260.4 \text{ ft}^3\end{aligned}$$

Therefore, the total capacity of all three silos is about $4521.6 + 7693.0 + 15{,}260.4 \approx 27{,}475.0 \text{ ft}^3$.

b) To find how long it takes to empty all three silos, we will divide the total capacity by 150 ft^3, the amount fed to Gordon's cattle every day.

$$\frac{27{,}475}{150} \approx 183.17.$$

Thus, the silos will be empty in about 183 days.

Now let's discuss polyhedrons. A ***polyhedron*** is a closed surface formed by the union of polygonal regions. Figure 9.39 illustrates some polyhedrons.

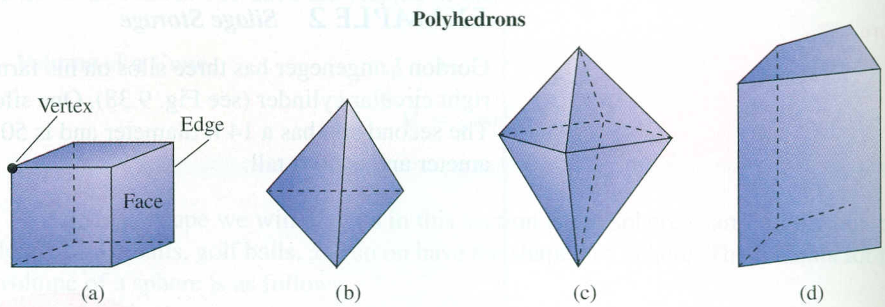


Figure 9.39

Each polygonal region is called a *face* of the polyhedron. The line segment formed by the intersection of two faces is called an *edge*. The point at which two or more edges intersect is called a *vertex*. In Fig. 9.39(a) there are 6 faces, 12 edges, and 8 vertices. Note that

$$\begin{array}{ccccccc}\text{Number of vertices} & - & \text{number of edges} & + & \text{number of faces} & = & 2\\ 8 & - & 12 & + & 6 & = & 2\end{array}$$

This formula, credited to Leonhard Euler, is true for any polyhedron.

Euler's Polyhedron Formula

Number of vertices − number of edges + number of faces = 2

We suggest that you verify this formula holds for Fig. 9.39(b), (c), and (d).

EXAMPLE 3 *Using Euler's Polyhedron Formula*

A certain polyhedron has 6 vertices and 9 edges. Determine the number of faces on this polyhedron.

SOLUTION: Since we are seeking the number of faces, we will let x represent the number of faces on the polyhedron. Next, we will use Euler's polyhedron formula to set up an equation:

$$\begin{aligned} \text{Number of vertices} - \text{number of edges} + \text{number of faces} &= 2 \\ 6 \quad - \quad 9 \quad + \quad x &= 2 \\ -3 + x &= 2 \\ x &= 5 \end{aligned}$$

Therefore, the polyhedron has 5 faces.

A ***regular polyhedron*** is one whose faces are all regular polygons of the same size and shape. Figure 9.39(a) and (b) on page 510 are regular polyhedrons.

A ***prism*** is a special type of polyhedron whose bases are congruent polygons and whose sides are parallelograms. These parallelogram regions are called the *lateral faces* of the prism. If all the lateral faces are rectangles, the prism is said to be a ***right prism***. Some right prisms are illustrated in Fig. 9.40. In this book, whenever we use the word *prism* we are referring to a right prism.

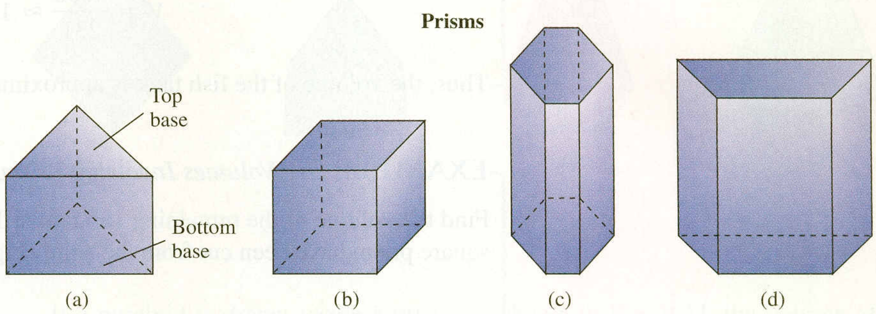


Figure 9.40

The volume of any prism can be found by multiplying the area of the base, B, by the height, h, of the prism.

Volume of a Prism

$$V = Bh$$

where B is the area of a base and h is the height.

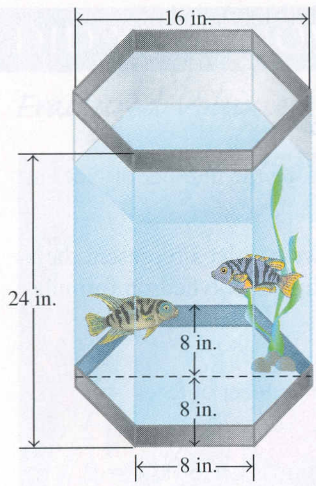


Figure 9.41

EXAMPLE 4 *Volume of a Hexagonal Prism Fish Tank*

Frank Nicolzaao's fish tank is in the shape of a hexagonal prism as shown in Fig. 9.41. Use the dimensions shown in the figure and the fact that 1 gal = 231 in.^{3} to

a) determine the volume of the fish tank in cubic inches.

b) determine the volume of the fish tank in gallons (round your answer to the nearest gallon).

SOLUTION:

a) First we will need to calculate the area of the hexagonal base of the fish tank. Notice from Fig. 9.41 that by drawing a diagonal as indicated, the base can be divided into two identical trapezoids. To find the area of the hexagonal base, we will calculate the area of one of these trapezoids and then multiply by 2.

$$\text{Area of one trapezoid} = \tfrac{1}{2}h(b_1 + b_2)$$
$$= \tfrac{1}{2}(8)(16 + 8) = 96 \text{ in.}^2$$
$$\text{Area of the hexagonal base} = 2(96) = 192 \text{ in.}^2$$

Now to determine the volume of the fish tank, we will use the formula for the volume of a prism, $V = Bh$. We already determined that the area of the base, B, is 192 in.^{2}

$$V = B \cdot h = 192 \cdot 24 = 4608 \text{ in.}^3$$

In the above calculation, the area of the base, B, was measured in square inches, and the height was measured in inches. The product of in.^{2} and in. is cubic inches, or in.^{3}

b) To determine the volume of the fish tank in gallons, we will divide the volume of the fish tank in cubic inches by 231.

$$V = \frac{4608}{231} \approx 19.95 \text{ gal}$$

Thus, the volume of the fish tank is approximately 20 gal.

EXAMPLE 5 *Volumes Involving Prisms*

Find the volume of the remaining solid after the cylinder, triangular prism, and square prism have been cut from the solid (Fig. 9.42).

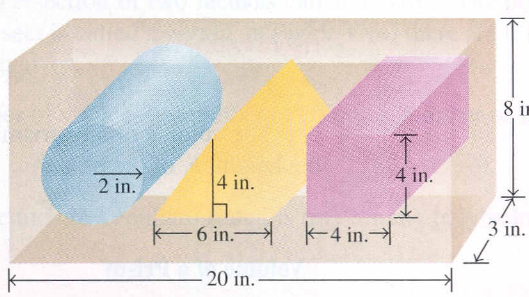


Figure 9.42

SOLUTION: To find the volume of the remaining solid, first find the volume of the rectangular solid. Then subtract the volume of the two prisms and the cylinder that were cut out.

$$\begin{aligned}
\text{Volume of rectangular solid} &= l \cdot w \cdot h \\
&= 20 \cdot 3 \cdot 8 = 480 \text{ in.}^3 \\
\text{Volume of circular cylinder} &= \pi r^2 h \\
&\approx (3.14)(2^2)(3) \\
&\approx (3.14)(4)(3) \approx 37.68 \text{ in.}^3 \\
\text{Volume of triangular prism} &= \text{area of the base} \cdot \text{height} \\
&= \tfrac{1}{2}(6)(4)(3) = 36 \text{ in.}^3 \\
\text{Volume of square prism} &= s^2 \cdot h \\
&= 4^2 \cdot 3 = 48 \text{ in.}^3 \\
\text{Volume of solid} &\approx 480 - 37.68 - 36 - 48 \\
&\approx 358.32 \text{ in.}^3
\end{aligned}$$

▲

Another special category of polyhedrons is the *pyramid*. Unlike prisms, pyramids have only one base. Some pyramids are illustrated in Fig. 9.43. Note that all but one face of a pyramid intersect at a common vertex.

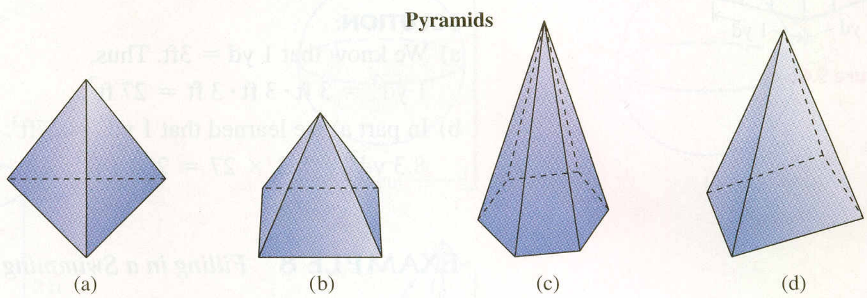


Figure 9.43

If a pyramid is drawn inside a prism, as shown in Fig. 9.44, the volume of the pyramid is less than that of the prism. In fact, the volume of the pyramid is one-third the volume of the prism.

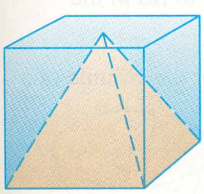


Figure 9.44

Volume of a Pyramid

$$V = \tfrac{1}{3}Bh$$

where B is the area of the base and h is the height.

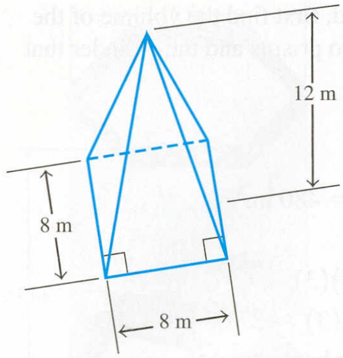


Figure 9.45

EXAMPLE 6 *Volume of a Pyramid*

Find the volume of the pyramid shown in Fig. 9.45.

SOLUTION: First find the area of the base of the pyramid. Since the base of the pyramid is a square,

$$\text{Area of base} = s^2 = 8^2 = 64 \text{ m}^2$$

Now use this information to find the volume of the pyramid.

$$\begin{aligned} V &= \tfrac{1}{3} \cdot B \cdot h \\ &= \tfrac{1}{3} \cdot 64 \cdot 12 \\ &= 256 \text{ m}^3 \end{aligned}$$

In certain situations converting volume from one cubic unit to a different cubic unit might be necessary. For example, when purchasing topsoil you might have to change the amount of topsoil from cubic feet to cubic yards prior to placing your order. Example 7 shows how this may be done.

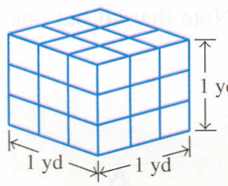


Figure 9.46

EXAMPLE 7 *Cubic Yards and Cubic Feet*

a) Convert 1 yd^3 to cubic feet. (See Fig. 9.46.)

b) Convert 8.3 yd^3 to cubic feet.

SOLUTION:

a) We know that 1 yd = 3ft. Thus,
$1 \text{ yd}^3 = 3 \text{ ft} \cdot 3 \text{ ft} \cdot 3 \text{ ft} = 27 \text{ ft}^3$.

b) In part a) we learned that $1 \text{ yd}^3 = 27\text{ft}^3$. Thus,
$8.3 \text{ yd}^3 = 8.3 \times 27 = 224.1 \text{ ft}^3$.

EXAMPLE 8 *Filling in a Swimming Pool*

Julianne Peterson recently purchased a home with a rectangular swimming pool. The pool is 30 ft long, 15 ft wide, and has a uniform depth of 4.5 ft. Julianne lives in a cold climate and so she plans to fill the pool in with dirt to make a flower garden. How many cubic yards of dirt will Julianne have to purchase to fill in the swimming pool?

SOLUTION: To find the amount of dirt, we will use the formula for the volume of a rectangular solid:

$$\begin{aligned} V &= lwh \\ &= (30)(15)(4.5) \\ &= 2025 \text{ ft}^3 \end{aligned}$$

Now, we must convert this volume from cubic feet to cubic yards. In Example 7, we learned that $1 \text{ yd}^3 = 27 \text{ ft}^3$. Therefore, $2025 \text{ ft}^3 = \frac{2025}{27} = 75 \text{ yd}^3$. Thus, Julianne needs to purchase 75 yd^3 of dirt to fill in her swimming pool.

SECTION 9.4 EXERCISES

Concept/Writing Exercises

1. In your own words, define *volume*.
2. What is solid geometry?
3. What is the difference between a polyhedron and a regular polyhedron?
4. What is the difference between a prism and a right prism?
5. In your own words, explain the difference between a prism and a pyramid.
6. In your own words, state Euler's polyhedron formula.

Practice the Skills

In Exercises 7–20, find the volume of the solid. When necessary, round your answer to hundredths.

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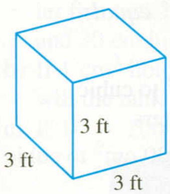

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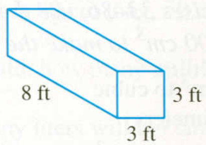

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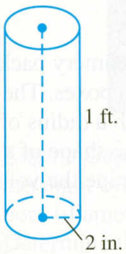

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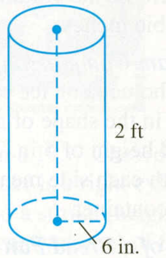

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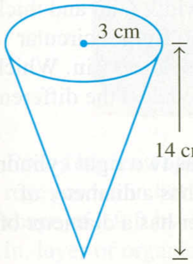

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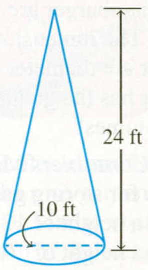

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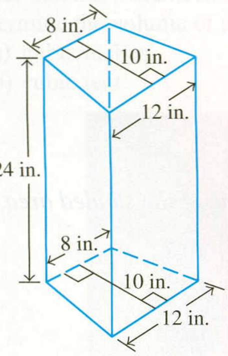

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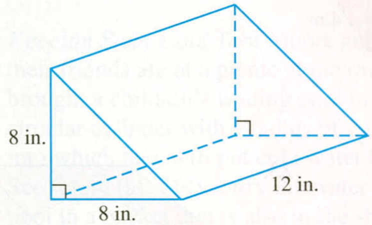

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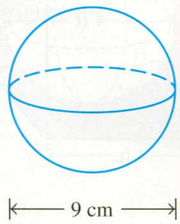

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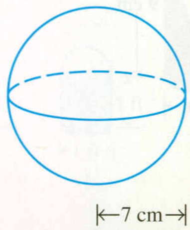

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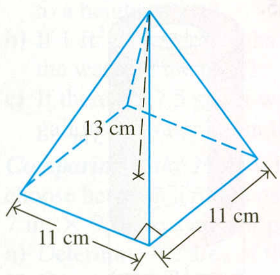

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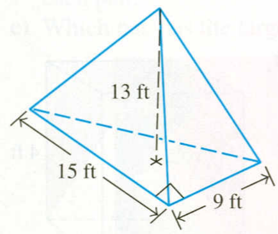

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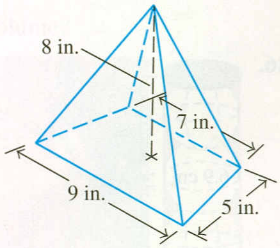

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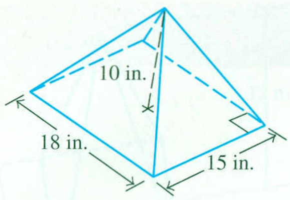

Problem Solving

In Exercises 21–28, find the volume of the shaded area. Round your answers to hundredths.

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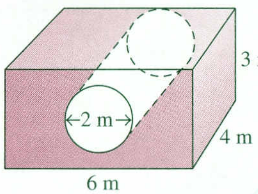

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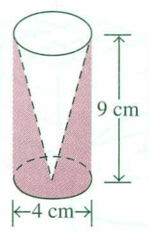

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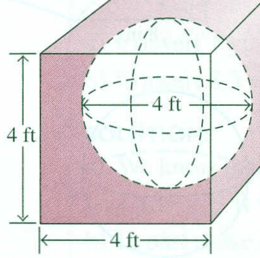

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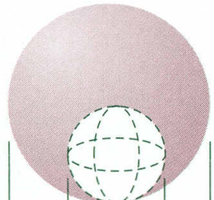

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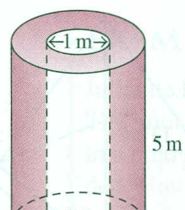

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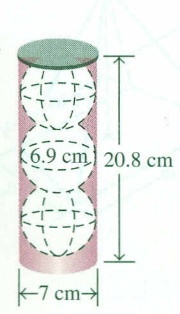

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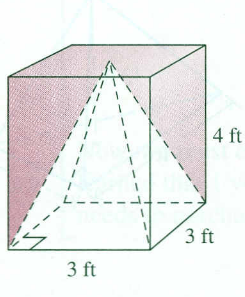

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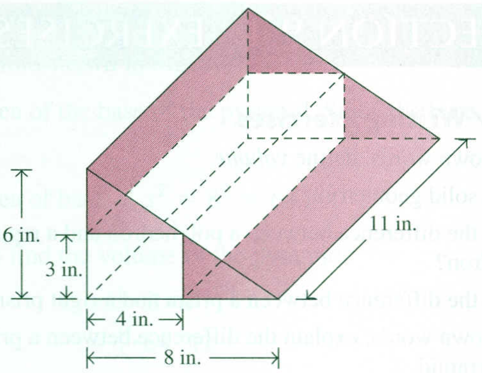


In Exercises 29–32, use the fact that 1 yd^3 equals 27 ft^3 to make the conversion.

29. 7 yd^3 to cubic feet

30. 3.8 yd^3 to cubic feet

31. 153 ft^3 to cubic yards

32. 2457 ft^3 to cubic yards

In Exercises 33–36, use the fact that 1 m^3 equals 1,000,000 cm^3 to make the conversion.

33. 5.9 m^3 to cubic centimeters

34. 17.6 m^3 to cubic centimeters

35. 3,000,000 cm^3 to cubic meters

36. 7,300,000 cm^3 to cubic meters

37. *Volume of a Freezer* The dimensions of the interior of an upright freezer are height 46 in., width 25 in., and depth 25 in. Determine the volume of the freezer
a) in cubic inches. **b)** in cubic feet.

38. *Ice Cream Comparison* The Louisburg Creamery packages its homemade ice cream in tubs and in boxes. The tubs are in the shape of a right cylinder with a radius of 3 in. and height of 5 in. The boxes are in the shape of a cube with each side measuring 5 in. Determine the volume of each container.

39. *Volume of a Bread Pan* A bread pan is 12 in. × 4 in. × 3 in. How many quarts does it hold, if 1 in.^{3} $\approx$ 0.01736 qt?

40. *Hamburger Comparison* The dimensions of a square Wendy's hamburger are length and width 4 in. and thickness $\frac{3}{16}$ in. The dimensions of a Magic Burger circular hamburger are diameter $4\frac{1}{2}$ in. and thickness $\frac{1}{4}$ in. Which hamburger has the greater volume? What is the difference in their volumes?

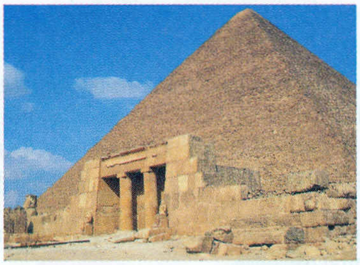
41. *Gasoline Containers* Mark Russo has two right cylindrical containers for storing gasoline. One has a diameter of 10 in. and a height of 12 in. The other has a diameter of 12 in. and a height of 10 in.
a) Which container holds the greater amount of gasoline, the taller one or the one with the greater diameter?
b) What is the difference in volume?

42. ***A Swimming Pool***

a) What is the volume of water in a rectangular swimming pool that is 15 m long and 9 m wide and has an average depth of 2 m? Give your answer in cubic meters.

b) If 1 m^3 = 1 kℓ, how many kiloliters of water will the pool hold?

43. ***The Pyramid of Cheops*** The Pyramid of Cheops in Egypt has a square base measuring 720 ft on a side. Its height is 480 ft. What is its volume?



44. ***A Fish Tank***

a) How many cubic centimeters of water will a rectangular fish tank hold if the tank is 80 cm long, 50 cm wide, and 30 cm high?

b) If 1 cm^3 holds 1 mℓ of liquid, how many milliliters will the tank hold?

c) If 1ℓ = 1000 mℓ, how many liters will the tank hold?

45. ***Engine Capacity*** The engine in a 1957 Chevrolet Corvette has eight cylinders. Each cylinder is a right cylinder with a bore (diameter) of 3.875 in. and a stroke (height) of 3 in. Determine the total displacement (volume) of this engine.

46. ***Rose Garden Topsoil*** Marisa Raffaele wishes to plant a rose garden in her backyard. The rose garden will be in the shape of a 9 ft by 18 ft rectangle. Marisa wishes to add a 4 in. layer of organic topsoil on top of the rectangular area. The topsoil sells for \$32.95 per cubic yard. Determine

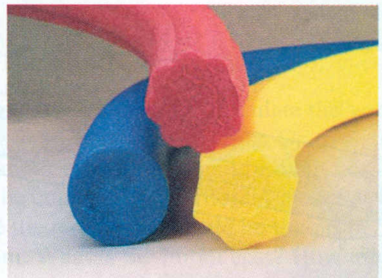
a) how many cubic yards of topsoil Marisa will need.

b) how much the topsoil will cost.

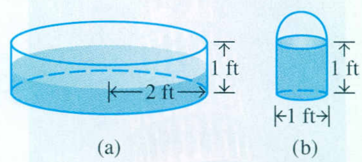
47. ***Pool Toys*** A Wacky Noodle Pool Toy, frequently referred to as a "noodle," is a cylindrical flotation device made from cell foam (see photo). One style of noodle is a cylinder that has a diameter of 2.5 in. and a length of 5.5 ft. Determine the volume of this style of noodle in

a) cubic inches.

b) cubic feet.



48. ***Keeping Soda Cold*** Tobi Moore and Tacinto Lopez and their friends are at a picnic at the town park. They have brought a children's wading pool in the shape of a right circular cylinder with a radius of 2 ft and a height of 1 ft into which they will put cold water to keep the soda cold. See figure (a). They carry the water from the faucet to the pool in a bucket that is also in the shape of a right circular cylinder, with a diameter of 1 ft and a height of 1 ft. See figure (b).



a) How many buckets of water are needed to fill the pool to a height of $\frac{1}{2}$ ft?

b) If 1 ft^3 of water weighs 62.5 lb, what is the weight of the water in the pool?

c) If there are 7.5 gal of water per cubic foot, how many gallons of water are in the pool?

49. ***Comparing Cake Pans*** When baking a cake you can choose between a round pan with a 9 in. diameter and a 7 in. × 9 in. rectangular pan.

a) Determine the area of the base of each pan.

b) If both pans are 2 in. deep, determine the volume of each pan.

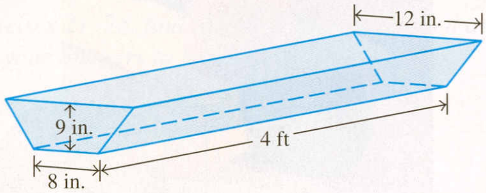
c) Which pan has the larger volume?

50. *Cake Icing* A bag used to apply icing to a cake is in the shape of a cone with a diameter of 3 in. and a height of 6 in. How much icing will this bag hold when full?

51. *Flower Box* A flower box is 4 ft long, and its ends are in the shape of a trapezoid. The upper and lower bases of the trapezoid measure 12 in. and 8 in., respectively, and the height is 9 in. Find the volume of the flower box

a) in cubic inches.

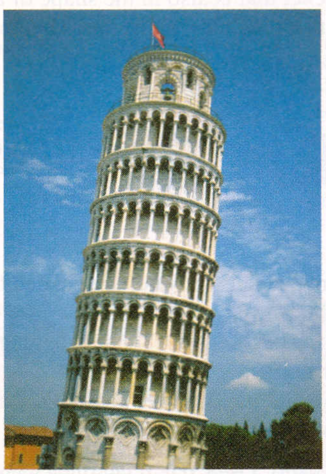
b) in cubic feet.



52. *The Leaning Tower of Pisa* The Leaning Tower of Pisa was designed to be a vertical bell tower for a cathedral. If the tower were vertical, it would be 60 meters high with a diameter of about 19.6 meters roughly in the shape of a cylinder. Use this information to find

a) the circumference of the tower.

b) the volume of tower.



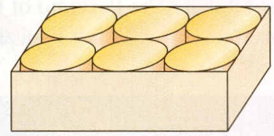
In Exercises 53–58, find the missing value indicated by the question mark. Use the following formula.

$$\begin{pmatrix}\text{Number of}\\ \text{vertices}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{edges}\end{pmatrix} + \begin{pmatrix}\text{number}\\ \text{of faces}\end{pmatrix} = 2$$

	Number of Vertices	Number of Edges	Number of Faces
53.	8	?	3
54.	12	16	?
55.	?	8	4
56.	7	12	?
57.	11	?	5
58.	?	10	4

Challenge Problems/Group Activities

59. *Packing Orange Juice* A box is packed with six cans of orange juice. The cans are touching each other and the sides of the box, as shown. What percent of the volume of the interior of the box is not occupied by the cans?



60. *Doubling the Edges of a Cube* In this exercise we will explore what happens to the volume of a cube if we double the length of each edge of the cube.

a) Choose a number between 1 and 10 and call this number s.

b) Calculate the volume of a cube with the length of each edge equal to s.

c) Now double s and call this number t.

d) Calculate the volume of a cube with the length of each edge equal to t.

e) Repeat parts (a) through (d) for a different value of s.

f) Compare the results from part (b) to the results from part (d) and explain what happens to the volume of a cube if we double the length of each edge.

61. *Doubling the Radius of a Sphere* In this exercise we will explore what happens to the volume of a sphere if we double the radius of the sphere.

a) Choose a number between 1 and 10 and call this number r.

b) Calculate the volume of a sphere with radius r (use the $\boxed{\pi}$ key on your calculator).

c) Now double r and call this number t.

d) Calculate the volume of a sphere with radius t.

e) Repeat parts (a) through (d) for a different value of r.

f) Compare the results from part (b) to the results from part (d) and explain what happens to the volume of a sphere if we double the radius.

62. *Cost of a Dripping Faucet* Leah Quintero has a faucet in her home that drips at a rate of 42 drops per minute. There are approximately 20 drops in 1 mℓ, 1000 mℓ in 1 ℓ, and approximately 3.79 ℓ in 1 gal. Assume water costs about $0.11 per gallon.

a) Determine the number of drops of water wasted over a 1-year period.

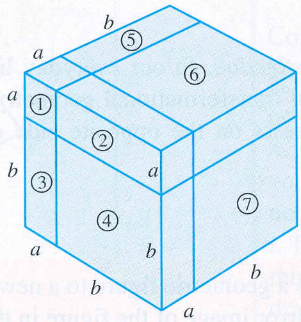
b) Determine the volume of water wasted over a 1-year period in milliliters, liters, and gallons.

c) Estimate the cost of water wasted over a 1-year period.

63. **a)** Explain how to demonstrate, using the cube shown on page 519, that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

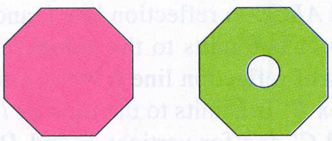
b) What is the volume in terms of a and b of each numbered piece in the figure?



c) An eighth piece is not illustrated. What is its volume?

Recreational Mathematics

64. ***More Pool Toys*** Wacky Noodle Pool Toys (see Exercise 47) come in many different shapes and sizes.



Base for part (a) Base for part (b)

a) Determine the volume, in cubic inches, of a noodle that is in the shape of a 5.5-ft-long solid octagonal prism whose base has an area of 5 in.^{2}

b) Determine the volume, in cubic inches, of a hollow noodle that has the same shape as the noodle described in part (a) except that a right circular cylinder of diameter 0.75 in. has been removed from the center.

Internet/Research Activities

65. ***Air Conditioner Selection*** Calculate the volume of the room in which you sleep or study. Go to a store that sells room air conditioners and find out how many cubic feet can be cooled by the different models available. Describe the model that would be the proper size for your room. What is the initial cost? How much does that model cost to operate? If you moved to a room that had twice the amount of floor space and the same height, would the air conditioner you selected still be adequate? Explain.

66. Pappus of Alexandria (ca. A.D. 350) was the last of the well-known ancient Greek mathematicians. Write a paper on his life and his contributions to mathematics.

9.5 TRANSFORMATIONAL GEOMETRY, SYMMETRY, AND TESSELLATIONS

In our study of geometry, we have thus far focused on definitions, axioms, and theorems that are used in the study of *Euclidean geometry*. We will now introduce a second type of geometry called *transformational geometry*. In ***transformational geometry***, we study various ways to move a geometric figure without altering the shape or size of the figure. When discussing transformational geometry we often use the term *rigid motion*.

> The act of moving a geometric figure from some starting position to some ending position without altering its shape or size is called a **rigid motion** (or **transformation**).

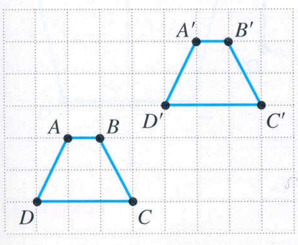


Figure 9.47

Consider trapezoid *ABCD* in Figure 9.47. If we move each point on this trapezoid 4 units to the right and 3 units up, the trapezoid is in the location specified by trapezoid $A'B'C'D'$. This figure illustrates one type of rigid motion. When studying rigid motions we are only concerned about the starting and ending positions of the figure and not what happens in between. When discussing rigid motions of two-dimensional figures, there are four types of rigid motions: reflections, rotations, translations, and glide reflections. We call these four types of rigid motions the ***basic rigid motions in a plane***. After we discuss the four rigid motions we will discuss symmetry of geometric figures and tessellations.

Reflections

The first rigid motion we will study is *reflection*. In our everyday life we are quite familiar with the concept of reflection. In transformational geometry a reflection is an image of a geometric figure that appears on the opposite side of a designated line.

A **reflection** is a rigid motion that moves a geometric figure to a new position such that the figure in the new position is a mirror image of the figure in the starting position. In two dimensions the figure and its mirror image are equidistant from a line called the **reflection line** or the **axis of reflection.**

Figure 9.48 below shows trapezoid $ABCD$, a reflection line l, and the reflected trapezoid $A'B'C'D'$. Notice that vertex A is 6 units to the *left* of reflection line l and that vertex A' is 6 units to the *right* of reflection line l. Next notice that vertex B is 2 units to the *left* of l and that vertex B' is 2 units to the right of l. A similar relationship holds true for vertices C and C' and for vertices D and D'. It is important to see that the trapezoid is not simply *moved* to the other side of the reflection line, but instead it is *reflected*. Notice in the trapezoid $ABCD$ that the longer base BC is on the right side of the trapezoid, but in the reflected trapezoid $A'B'C'D'$ the longer base $B'C'$ is on the left side of the trapezoid. Finally, notice the colors of the sides of the two trapezoids. Side AB in trapezoid $ABCD$ and side $A'B'$ in the reflected trapezoid are both blue. Sides BC and sides $B'C'$ are both red, sides CD and $C'D'$ are both gold, and sides DA and $D'A'$ are both green. In this section we will occasionally use such color coding to help you visualize the effect of a rigid transformation on a figure.

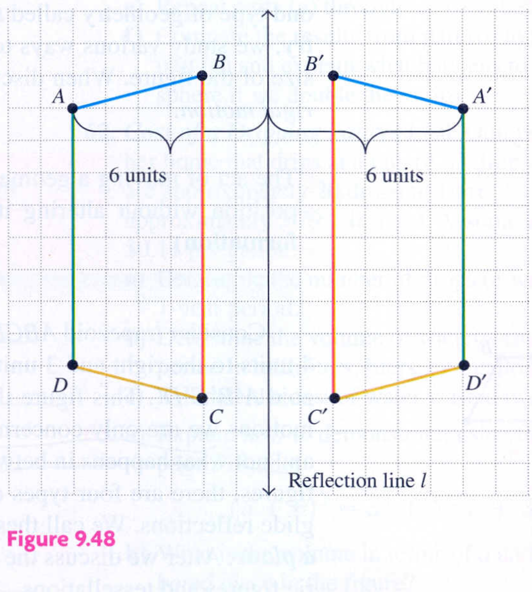


Figure 9.48

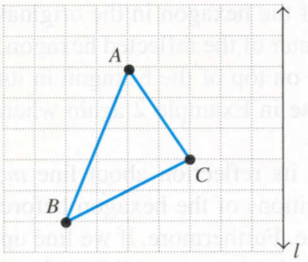


Figure 9.49

EXAMPLE 1 *Reflection of a Triangle*

Construct the reflection of triangle ABC, shown in Fig. 9.49, about reflection line l.

SOLUTION: The reflection of triangle ABC will be called $A'B'C'$. To determine the position of the reflection, we first examine vertex A in Fig. 9.49. Notice that vertex A is 5 units to the *left* of reflection line l. Thus, in the reflected triangle $A'B'C'$, vertex A' must also be 5 units away from, but to the *right* of, reflection line l. Next, notice that vertex B is 7 units to the left of l and that vertex C is 3 units to the left of l. Thus, in the reflection, vertex B' must be 7 units to the right of l and vertex C' must be 3 units to the right of l. Fig. 9.50 shows vertices A', B', and C'. Finally, we draw line segments between vertices A', B', and C' to form the sides of the reflection, triangle $A'B'C'$, as illustrated in Fig. 9.50.

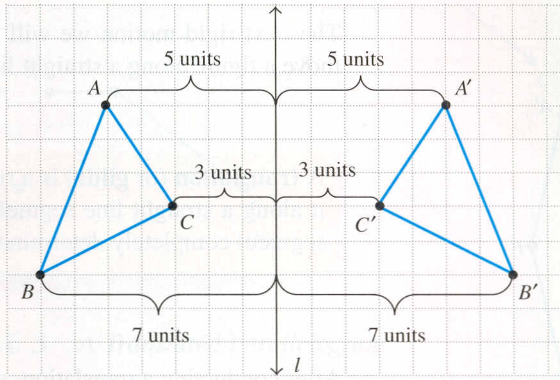


Figure 9.50

In Example 1, the reflection line did not intersect the figure being reflected. We now will study an example where the reflection line goes directly through the figure to be reflected.

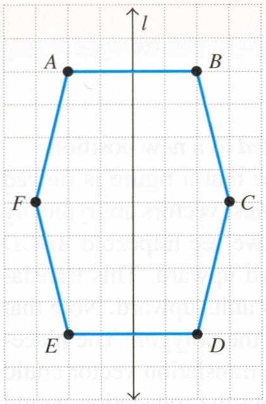


Figure 9.51

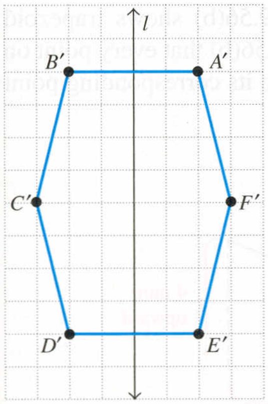


Figure 9.52

EXAMPLE 2 *Reflection of a Hexagon*

Construct the reflection of hexagon $ABCDEF$, shown in Fig. 9.51, about reflection line l.

SOLUTION: From Figure 9.51 we see that vertex A in hexagon $ABCDEF$ is 2 units to the left of reflection line l. Thus, vertex A' in the reflected hexagon will be 2 units to the right of l (see Figure 9.52). Notice that vertex A' of the reflected hexagon is in the same location as vertex B of hexagon $ABCDEF$ in Figure 9.51.

We next see that vertex B in hexagon $ABCDEF$ is 2 units to the right of l. Thus, vertex B' in the reflected hexagon will be 2 units to the left of l. Notice that vertex B' of the reflected hexagon is in the same location as vertex A of hexagon $ABCDEF$. We continue this process to determine the locations of vertices C', D', E', and F' of the reflected hexagon. Notice once again that each vertex of the reflected hexagon is in the same location as a vertex of hexagon $ABCDEF$. Finally, we draw the line segments to complete the reflected hexagon $A'B'C'D'E'F'$ (see Figure 9.52). For this example, we see that other than the vertex labels, the positions of the hexagon before and after the reflection are identical.

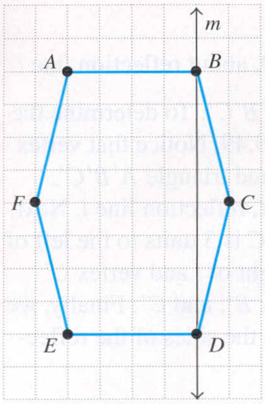


Figure 9.53

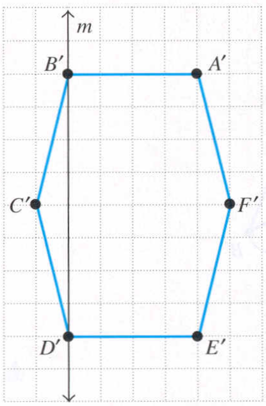


Figure 9.54

In Example 2 the reflection line was in the center of the hexagon in the original position. As a result, the reflection line was also in the center of the reflected hexagon. In this particular case the reflected hexagon lies directly on top of the hexagon in its original position. We will revisit reflections such as those in Example 2 again when we discuss *reflective symmetry* later in this section.

Now consider hexagon $ABCDEF$ in Fig. 9.53 and its reflection about line m, hexagon $A'B'C'D'E'F'$ in Fig. 9.54. Notice that the positions of the hexagon before and after the reflection, relative to line m, are not the same. Furthermore, if we line up reflection line m in Fig. 9.53 and Fig. 9.54, we would see that hexagon $ABCDEF$ and hexagon $A'B'C'D'E'F'$ are in different positions.

Translations

The next rigid motion we will discuss is the *translation*. In a translation we simply move a figure along a straight line to a new position.

A **translation** (or **glide**) is a rigid motion that moves a geometric figure by sliding it along a straight line segment in the plane. The direction and length of the line segment completely determine the translation.

After conducting a translation, we say the figure was *translated* to a new position.

A concise way to indicate the direction and the distance that a figure is moved during a translation is with a *translation vector*. In mathematics, vectors are typically represented with boldface letters. For example, in Fig. 9.55 we see trapezoid $ABCD$ and a translation vector, **v**, which is pointing to the right and upward. This translation vector indicates a translation of 9 units to the right and 4 units upward. Note that in Fig. 9.55 the translated vector appears on the right side of the polygon. The placement of the translation vector does not matter. Therefore, the translation vector could have been placed to the left, above, or below the polygon, and the translation would not change. When trapezoid $ABCD$ is translated using **v**, every point on trapezoid $ABCD$ is moved 9 units to the right and 4 units upward. This movement is demonstrated for vertex A in Fig. 9.56(a) on page 523. Figure 9.56(b) shows trapezoid $ABCD$ and the translated trapezoid $A'B'C'D'$. Note in Fig. 9.56(b) that every point on trapezoid $A'B'C'D'$ is 9 units to the right and 4 units up from its corresponding point on trapezoid $ABCD$.

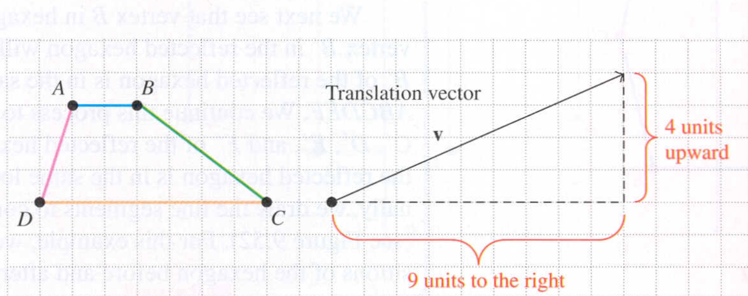


Figure 9.55

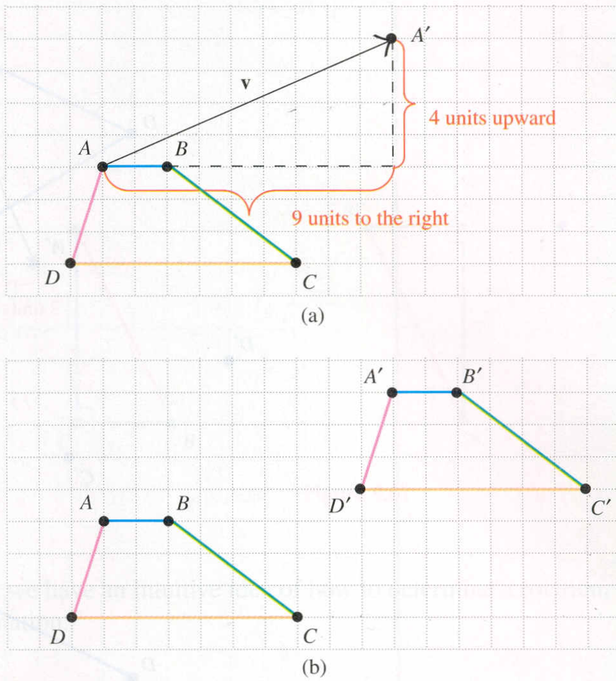


Figure 9.56

EXAMPLE 3 *A Translated Parallelogram*

Given parallelogram $ABCD$ and translation vector, **v**, shown in Fig. 9.57, construct the translated parallelogram $A'B'C'D'$.

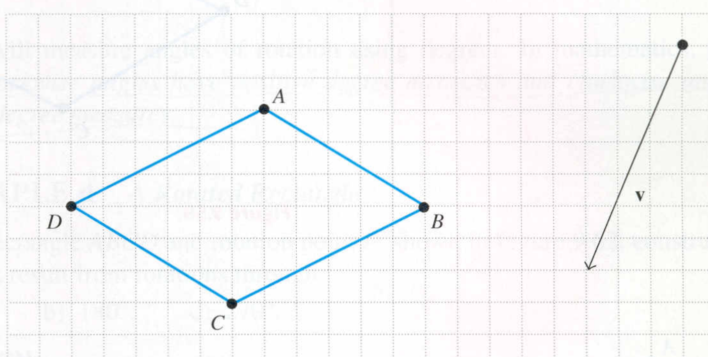


Figure 9.57

SOLUTION: The translated figure will be a parallelogram of the same size and shape as parallelogram $ABCD$. We notice that the translation vector, **v**, points 7 units downward and 3 units to the left. We next examine vertex A. To determine the location of vertex A' of the translated parallelogram start at vertex A of parallelogram $ABCD$ and move down 7 units and to the left 3 units. We label this vertex A' (see Fig. 9.58a on page 524). We determine vertices B', C', and D' in a similar manner by moving down 7 units and to the left 3 units from vertices B, C, and D, respectively. Figure 9.58(b) shows parallelogram $ABCD$ and the translated parallelogram $A'B'C'D'$. Note in Fig. 9.58(b) that every point on parallelogram $A'B'C'D'$ is 7 units down and 3 units to the left of its corresponding point on parallelogram $ABCD$.

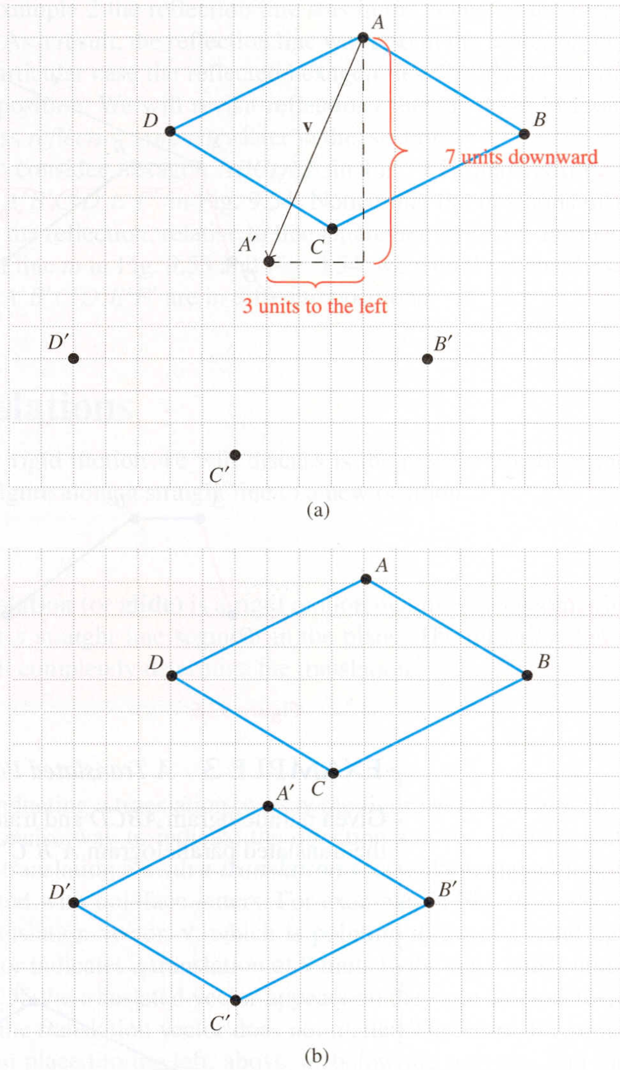


Figure 9.58

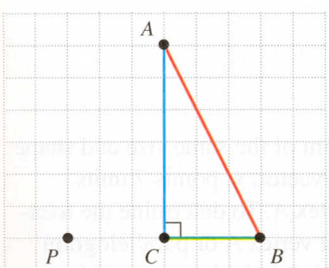


Figure 9.59

Rotations

The next rigid motion we will discuss is *rotation*. To help visualize a rotation examine Fig. 9.59, which shows right triangle ABC and point P about which right triangle ABC is to be rotated.

Imagine that this page was removed from this book and attached to a bulletin board with a single pin through point P. Next imagine rotating the page 90° in the *counterclockwise* direction. The triangle would now appear as triangle $A'B'C'$ shown in Fig. 9.60. Next, imagine rotating the original triangle 180° in a counterclockwise direction. The triangle would now appear as triangle $A''B''C''$ shown in Fig. 9.61.

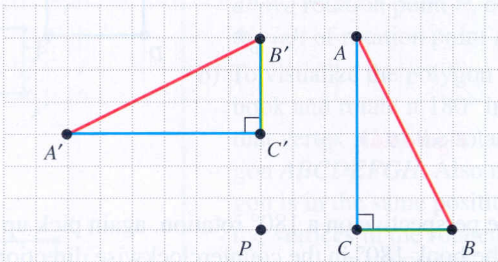


Figure 9.60

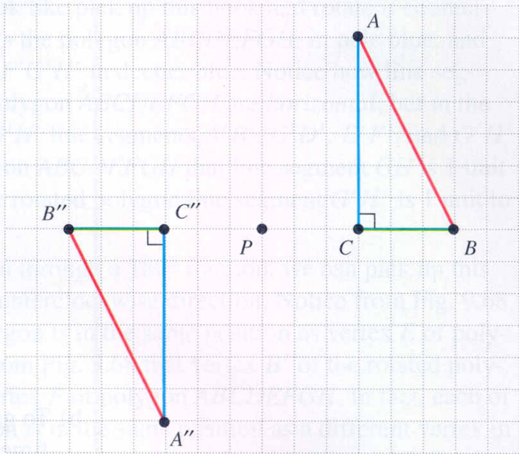


Figure 9.61

Now that we have an intuitive idea of how to determine a rotation, we give the definition of rotation.

> A **rotation** is a rigid motion performed by rotating a geometric figure in the plane about a specific point, called the **rotation point** or the **center of rotation.** The angle through which the object is rotated is called the **angle of rotation.**

We will measure angles of rotation using degrees. In mathematics, generally, *counterclockwise angles have positive degree measures and clockwise angles have negative degree measures.*

EXAMPLE 4 *A Rotated Rectangle*

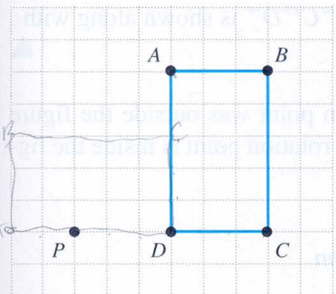


Figure 9.62

Given rectangle $ABCD$ and rotation point P, shown in Figure 9.62, construct rectangles that result from rotations through

a) 90°. b) 180°. c) 270°.

SOLUTION:

a) First, since 90 is a *positive* number, we will rotate the figure in a counterclockwise direction. We also note that the rotated rectangle will be the same size and shape as rectangle $ABCD$. To get an idea of what the rotated rectangle will look like, pick up this book and rotate it counterclockwise 90°. Fig. 9.63 on page 526 shows rectangle $ABCD$ and the rectangle rotated 90°, $A'B'C'D'$, about point P. Notice how line segment AB in rectangle $ABCD$ is horizontal, but in the rotated rectangle in Fig. 9.63 line segment $A'B'$ is vertical. Also notice that in rectangle $ABCD$ vertex D is 3 units to the *right* of rotation point P, but in the rotated rectangle vertex D' is 3 units *above* rotation point P.

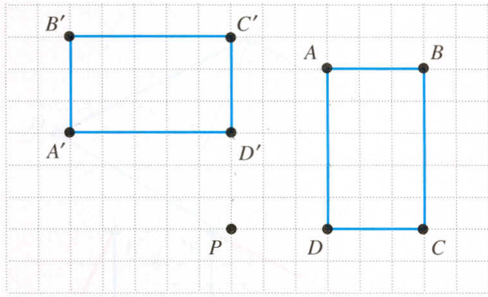


Figure 9.63

b) To gain some perspective on a 180° rotation, again pick up this book, but this time rotate the book 180° in the counterclockwise direction. The rotated rectangle $A''B''C''D''$ is shown along with the rectangle $ABCD$ in Fig. 9.64.

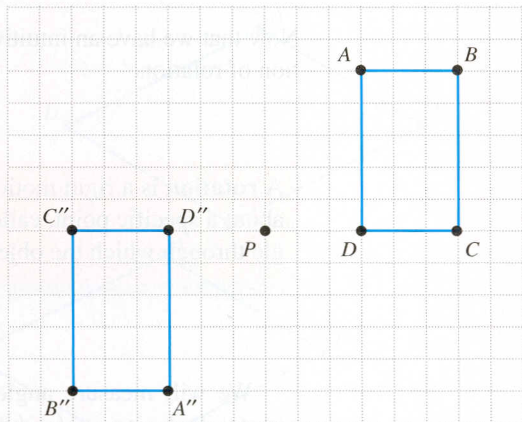


Figure 9.64

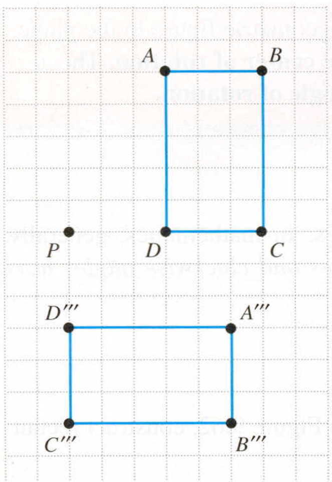


Figure 9.65

c) To gain some perspective on a 270° rotation, rotate this book 270° in the counterclockwise direction. The rotated rectangle $A'''B'''C'''D'''$ is shown along with rectangle $ABCD$ in Fig. 9.65.

Thus far, in our examples of rotations, the rotation point was outside the figure being rotated. We now will study an example where the rotation point is inside the figure to be rotated.

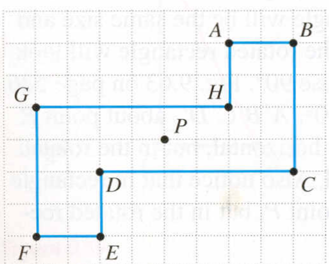


Figure 9.66

EXAMPLE 5 *A Rotation Point Inside a Polygon*

Given polygon $ABCDEFGH$ and rotation point P, shown in Fig. 9.66, construct polygons that result from rotations through

a) 90°. b) 180°.

SOLUTION:

a) We will rotate the polygon 90° in a counterclockwise direction. The resulting polygon will be the same size and shape as polygon $ABCDEFGH$. To visualize what the rotated polygon will look like pick up this book and rotate it counterclockwise 90°. Figure 9.67 shows the polygon *ABCDEFGH,* in pale blue, and the rotated polygon $A'B'C'D'E'F'G'H'$ in deeper blue. Notice how line segments *AB, CD, EF,* and *GH* in polygon *ABCDEFGH* are *horizontal*, but in the rotated polygon $A'B'C'D'E'F'G'H'$ line segments $A'B'$, $C'D'$, $E'F'$, and $G'H'$ are *vertical*. Also notice in polygon *ABCDEFGH* that line segment *GH* is 1 unit *above* rotation point *P*, but in the rotated polygon line segment $G'H'$ is 1 unit to the *left* of rotation point *P*.

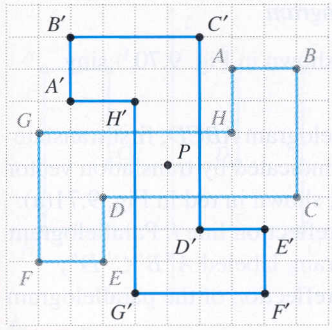


Figure 9.67

b) To visualize the polygon obtained through a 180° rotation, we can pick up this book and rotate it 180° in the counterclockwise direction. Notice from Fig. 9.68 that vertex A' of the rotated polygon is in the same position as vertex *E* of polygon *ABCDEFGH*. Also notice from Fig. 9.68 that vertex B' of the rotated polygon is in the same position as vertex *F* of polygon *ABCDEFGH*. In fact, each of the vertices in the rotated polygon is in the same position as a different vertex in polygon *ABCDEFGH*. From Fig. 9.68 we see that, other than vertex labels, the position of rotated polygon $A''B''C''D''E''F''G''H''$ is the same as the position of polygon *ABCDEFGH*.

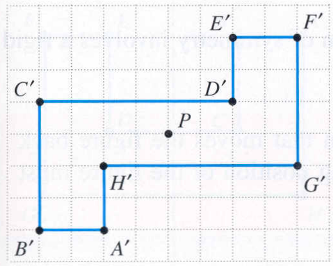


Figure 9.68

The polygon used in Example 5 will be discussed again later when we discuss *rotational symmetry*. The three rigid motions we have discussed thus far are reflection, translation, and rotation. Now we will discuss the fourth rigid motion, *glide reflection.*

A **glide reflection** is a rigid motion formed by performing a *translation* (or *glide*) followed by a *reflection.*

A glide reflection, as its name suggests, is a translation (or glide) followed by a reflection. Both translations and reflections were discussed earlier in this section. Consider triangle *ABC* (shown in blue), translation vector **v**, and reflection line *l* in Fig. 9.69. The translation of triangle *ABC*, obtained using translation vector **v**, is triangle $A'B'C'$ (shown in red). The reflection of triangle $A'B'C'$ about reflection line *l* is triangle $A''B''C''$ (shown in green). Thus, triangle $A''B''C''$ is the glide reflection of triangle *ABC* using translation vector **v** and reflection line *l*.

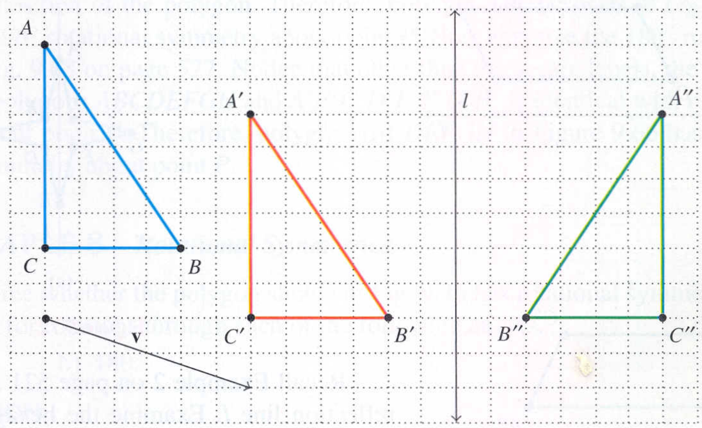


Figure 9.69

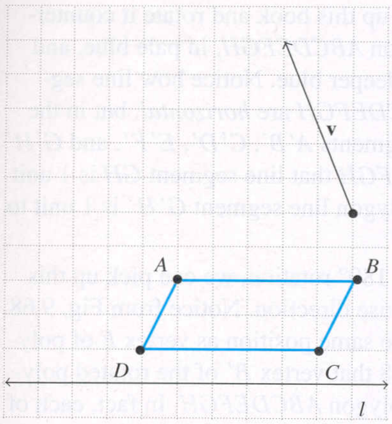


Figure 9.70

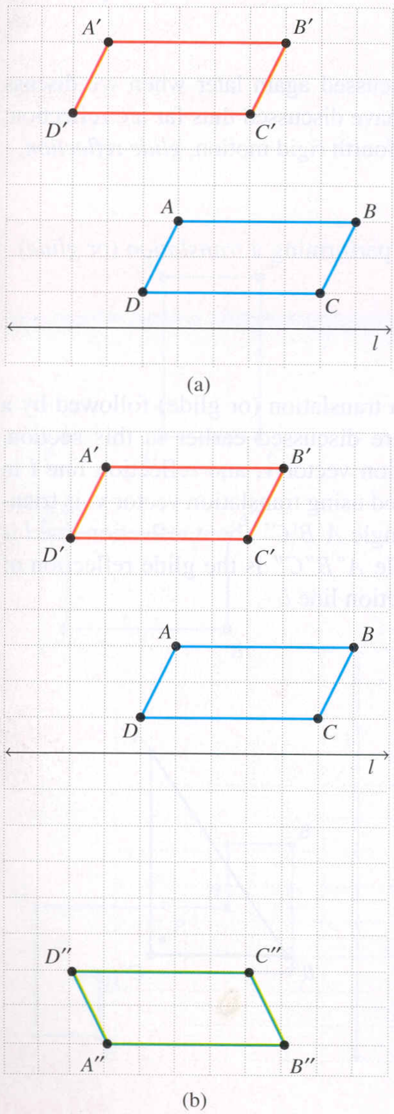


Figure 9.71

EXAMPLE 6 *A Glide Reflection of a Parallelogram*

Construct a glide reflection of parallelogram $ABCD$, shown in Fig. 9.70, using translation vector **v** and reflection line l.

SOLUTION: To construct the glide reflection of parallelogram $ABCD$, first translate the parallelogram 2 units to the left and 5 units up as indicated by translation vector **v**. This translated parallelogram is labeled $A'B'C'D'$, shown in red in Fig. 9.71(a). Next, we will reflect parallelogram $A'B'C'D'$ about reflection line l. Parallelogram $A'B'C'D'$, shown in red, and the reflected parallelogram, labeled $A''B''C''D''$, shown in green, are shown in Fig. 9.71(b). The glide reflection of the parallelogram $ABCD$ is parallelogram $A''B''C''D''$.

Symmetry

We are now ready to discuss symmetry. Our discussion of symmetry involves a rigid motion of an object.

> A **symmetry** of a geometric figure is a rigid motion that moves the figure back onto itself. That is, the beginning position and ending position of the figure must be identical.

Suppose we start with a figure in a specific position and perform a rigid motion on this figure. If the position of the figure after the rigid motion is identical to the position of the figure before the rigid motion (if the beginning and ending positions of the figure coincide), then the rigid motion is a symmetry and we say that the figure has symmetry. For a two-dimensional figure there are four types of symmetries: reflective symmetry, rotational symmetry, translational symmetry, and glide reflective symmetry. In this textbook, however, we will only discuss reflective symmetry and rotational symmetry.

Consider the polygon and reflection line l shown in Fig. 9.72(a). If we use the rigid motion of reflection and reflect the polygon $ABCDEFGH$ about line l, we get polygon $A'B'C'D'E'F'G'H'$. Note that the ending position of the polygon is identical to the starting position as shown in Fig. 9.72(b). Compare Fig. 9.72(a) with Fig. 9.72(b). Although the vertex labels are different, the reflected polygon is in the same position as the polygon in the original position. Thus, we say that the polygon has **reflective symmetry** about line l. We refer to line l as a **line of symmetry.**

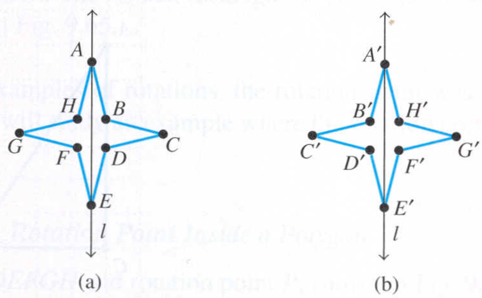


Figure 9.72

Recall Example 2 on page 521 in which hexagon $ABCDEF$ was reflected about reflection line l. Examine the hexagon in the original position (Fig. 9.51) and the hexagon in the final position after being reflected about line l (Fig. 9.52). Other than the labels of the vertices, the beginning and ending positions of the hexagon are identical. Therefore, hexagon $ABCDEF$ has reflective symmetry about line l.

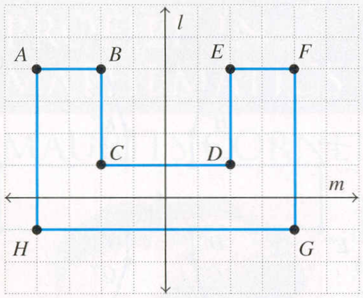


Figure 9.73

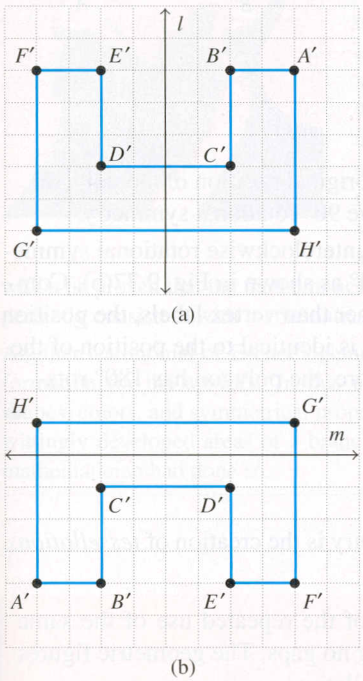


Figure 9.74

EXAMPLE 7 *Reflective Symmetries of Polygons*

Determine whether the polygon shown in Fig. 9.73 has reflective symmetry about each of the following lines.

a) Line l b) Line m

SOLUTION:

a) Examine the reflection of the polygon about line l as seen in Fig. 9.74(a). Notice that other than the vertex labels, the beginning and ending positions of the polygon are identical. Thus, the polygon has reflective symmetry about line l.

b) Examine the reflection of the polygon about line m as seen in Fig. 9.74(b). Notice that the position of the reflected polygon is different from the original position of the polygon. Thus, the polygon does not have reflective symmetry about line m. ▲

We will now discuss a second type of symmetry, rotational symmetry. Consider the polygon and rotation point P shown in Fig. 9.75(a). The rigid motion of rotation of polygon $ABCDEFGH$ through a 90° angle about point P gives polygon $A'B'C'D'E'F'G'H'$ shown in Fig. 9.75(b). Compare Fig. 9.75(a) to Fig. 9.75(b). Although the vertex labels are different, the position of the polygon before and after the rotation is identical. Thus, we say that the polygon has 90° **rotational symmetry** about point P. We refer to point P as the **point of symmetry.**

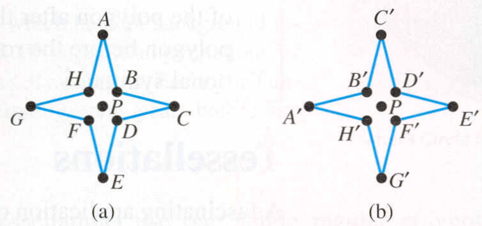


Figure 9.75

Recall Example 5 on page 526 in which polygon $ABCDEFGH$ was rotated 90° about point P in part (a) and 180° in part (b). First examine the polygon in the original position in Fig. 9.66 on page 526 and the 90° rotated polygon in Fig. 9.67 on page 527. Notice the position of the polygon after the 90° rotation is different from the original position of the polygon. Therefore, polygon $ABCDEFGH$ in Fig. 9.67 does not have 90° rotational symmetry about point P. Now examine the 180° rotated polygon in Fig. 9.68 on page 527. Notice that other than the vertex labels, the position of the two polygons $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ is identical with regard to rotation about point P. Therefore, polygon $ABCDEFGH$ in Figure 9.66 has 180° rotational symmetry about point P.

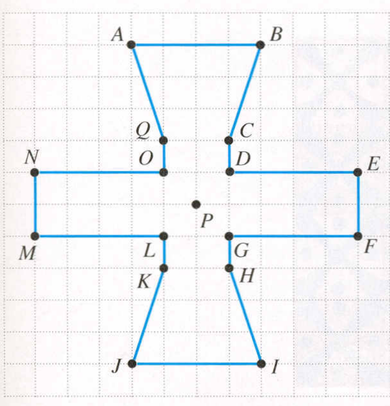


Figure 9.76

EXAMPLE 8 *Rotational Symmetries*

Determine whether the polygon shown in Fig. 9.76 has rotational symmetry about point P for rotations through each of the following angles.

a) 90° b) 180°

SOLUTION:

a) To determine whether the polygon has 90° counterclockwise rotational symmetry about point P we rotate the polygon 90° as shown in Fig. 9.77(a) on page 530. Compare Fig. 9.77(a) with Fig. 9.76. Notice that the position of the polygon after

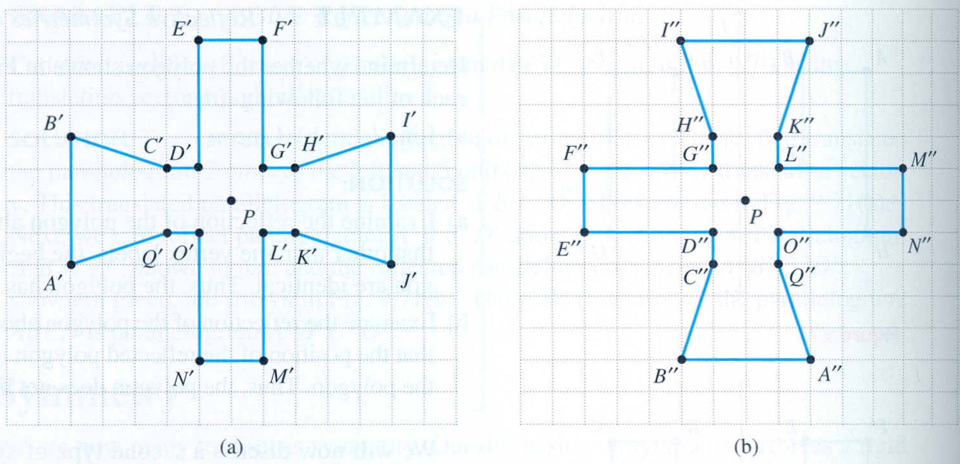


Figure 9.77

the rotation in Fig. 9.77(a) is different than the original position of the polygon (Fig. 9.76). Therefore, the polygon does not have 90° rotational symmetry.

b) To determine whether the polygon has 180° counterclockwise rotational symmetry about the point P we rotate the polygon 180° as shown in Fig. 9.77(b). Compare Fig. 9.77(b) with Fig. 9.76. Notice that other than vertex labels, the position of the polygon after the rotation in Fig. 9.77(b) is identical to the position of the polygon before the rotation (Fig. 9.76). Therefore, the polygon has 180° rotational symmetry. ▲

Tessellations

A fascinating application of transformational geometry is the creation of *tessellations*.

A **tessellation** (or **tiling**) is a pattern consisting of the repeated use of the same geometric figures to entirely cover a plane, leaving no gaps. The geometric figures used are called the **tessellating shapes** of the tessellation.

Figure 9.78 shows an example of a tessellation from ancient Egypt. Perhaps the most famous person to incorporate tessellations into his work is M. C. Escher (see Profiles in Mathematics on the next page).

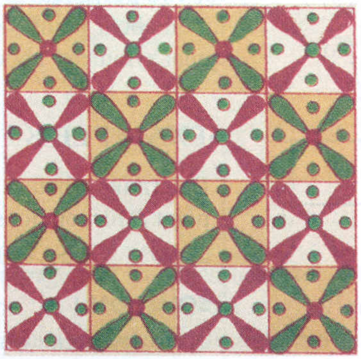


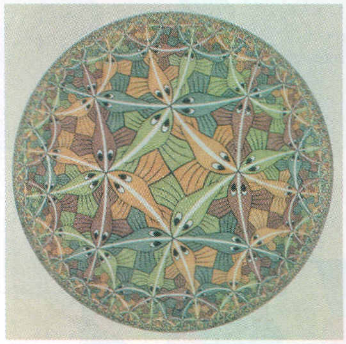
Figure 9.78

PROFILE IN MATHEMATICS

MAURITS CORNELIUS ESCHER

In addition to being wonderfully engaging art, the work of M. C. Escher (1898–1972) also displays some of the more beautiful and intricate aspects of mathematics. Escher's work involves Euclidean, non-Euclidean (to be studied shortly), and transformational geometries. Amazingly, Escher had no formal training in higher mathematics.

In 1936 Escher visited the Alhambra Palace in Granada, Spain, and became fascinated with Moorish tilings. Thereafter, Escher became obsessed with creating art that used objects to cover a plane so as to leave no gaps. Escher's brother recognized the mathematics depicted in this art and gave Escher a list of mathematics articles he felt would be of assistance to the artist. That was Escher's first exposure to formal mathematics. Among the mathematicians whose work influenced Escher were George Polya and Donald Coxeter. Polya's work with symmetry became a cornerstone of Escher's famous tessellations. Coxeter's work with non-Euclidean geometry was key to Escher's later work involving infinity, multiple dimensions, and hyperbolic and spherical tessellations. In 1995 Coxeter published a paper in which he proved that the mathematics Escher displayed in the etching *Circle Limit III* was indeed perfectly consistent with mathematical theory.



Escher's *Circle Limit III*

Another testimonial of Escher's genius is found in a notebook in which he kept background information for his artwork. In this notebook Escher categorized all possible combinations of shapes, colors, and symmetrical properties of polygons in the plane. By doing so, Escher had unwittingly developed areas of a branch of mathematics called *crystallography*, years before any mathematician had done so.

The simplest tessellations use one single regular polygon as the tessellating shape. Recall that a *regular polygon* is one whose sides are all the same length and whose interior angles all have the same measure. A tessellation that uses one single regular polygon as the tessellating shape is called a *regular tessellation*. It can be shown that only three regular tessellations exist: those that use an equilateral triangle, a square, or a regular hexagon as the tessellating shape. Figure 9.79 shows each of these regular tessellations. Notice that each of these tessellations can be obtained from a single tessellating shape through the use of reflections, translations, or rotations.

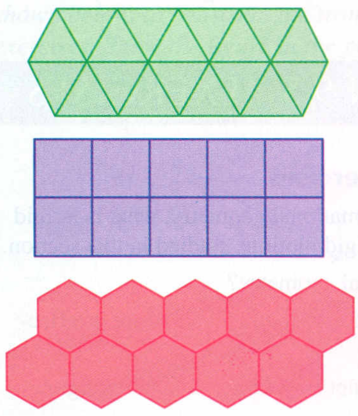


Figure 9.79

We will now learn how to create unique tessellations. We will do so by constructing a unique tessellating shape from a square. We could also construct other tessellating shapes using an equilateral triangle or a regular hexagon. If you wish to follow along with our construction, you will need some lightweight cardboard, a ruler, cellophane tape, and a pair of scissors. We will start by measuring and cutting out a square 2 in. by 2 in. from the cardboard. We next cut the square into two parts by cutting it from top to bottom using any kind of cut. One example is shown in Fig. 9.80. We then rearrange the pieces and tape the two vertical edges together as shown in Fig. 9.81. Next we cut this new shape into two parts by cutting it from left to right using any kind of cut as shown in Fig. 9.82. We then rearrange the pieces and tape the two horizontal edges together as shown in Fig. 9.83. This completes our tessellating shape.

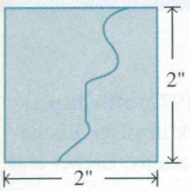


Figure 9.80

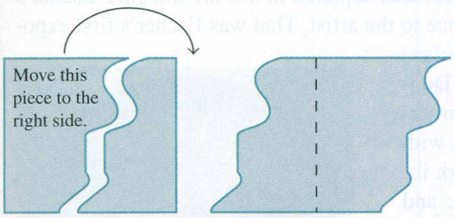


Figure 9.81

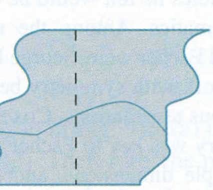


Figure 9.82

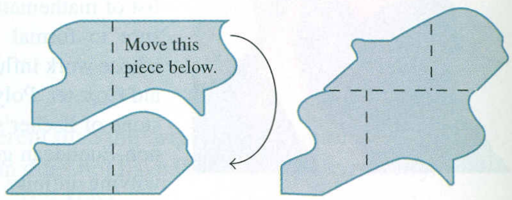


Figure 9.83

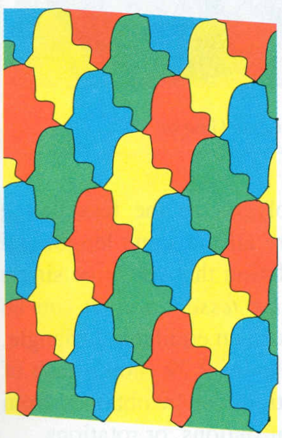


Figure 9.84

We now set the cardboard tessellating shape in the middle of a blank piece of paper (the tessellating shape can be rotated to any position as a starting point) and trace the outline of the shape onto the paper. Next move the tessellating shape so that it lines up with the figure already drawn and trace the outline again. Continue to do that until the page is completely covered. Once the page is covered with the tessellation, we can add some interesting colors or even some unique sketches to the tessellation. Figure 9.84 shows one tessellation created using the tessellation shape in Fig. 9.83. In Fig. 9.84 the tessellation shape was rotated about 45° counterclockwise.

An infinite number of different tessellations can be created using the method described by altering the cuts made. We could also create different tessellations using an equilateral triangle, a regular hexagon, or other types of polygons. There are also other, more complicated ways to create the tessellating shape. The Internet has many sites devoted to the creation of tessellations by hand. Many computer programs that generate tessellations are also available.

SECTION 9.5 EXERCISES

Concept/Writing Exercises

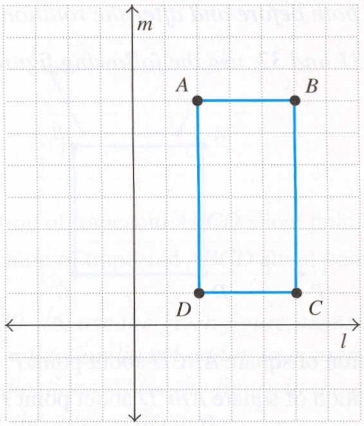
1. In the study of transformational geometry, what is a rigid motion? List the four rigid motions studied in this section.
2. What is transformational geometry?
3. In terms of transformational geometry, describe a reflection.
4. Describe how to construct a reflection of a given figure about a given line.
5. In terms of transformational geometry, describe a translation.
6. Describe how to construct a translation of a given figure using a translation vector.
7. In terms of transformational geometry, describe a rotation.
8. Describe how to construct a rotation of a given figure, about a given point, through a given angle.
9. In terms of transformational geometry, describe a glide reflection.

10. Describe how to construct a glide reflection of a given figure using a given translation vector and a given reflection line.
11. Describe what it means for a figure to have reflective symmetry about a given line.
12. Describe what it means for a figure to have rotational symmetry about a given point.
13. What is a tessellation?
14. Describe one way to make a unique tessellation from a 2-in. by 2-in. cardboard square.

Practice the Skills/Problem Solving

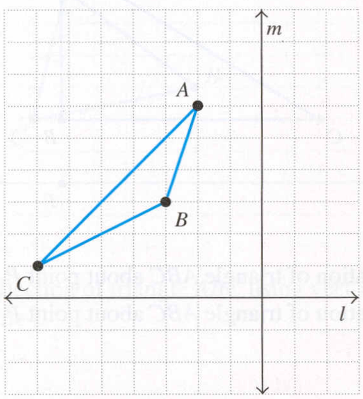
In Exercises 15–22, use the given figure and lines of reflection to construct the indicated reflections. Show the figure in the positions both before and after the reflection.

In Exercises 15 and 16, use the following figure. Construct



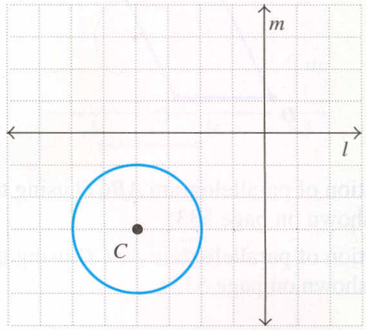
15. the reflection of rectangle *ABCD* about line *m*.
16. the reflection of rectangle *ABCD* about line *l*.

In Exercises 17 and 18, use the following figure. Construct



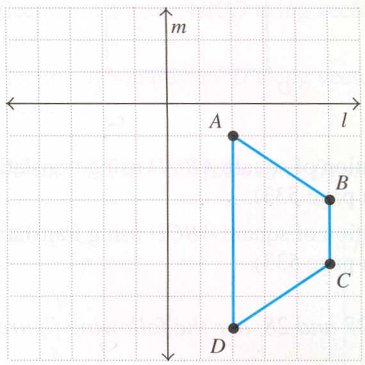
17. the reflection of triangle *ABC* about line *l*.
18. the reflection of triangle *ABC* about line *m*.

In Exercises 19 and 20, use the following figure. Construct



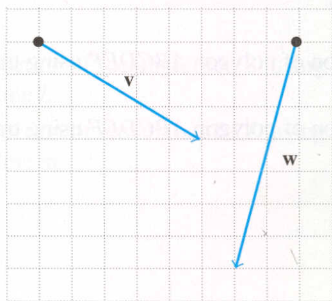
19. the reflection of circle *C* about line *l*.
20. the reflection of circle *C* about line *m*.

In Exercises 21 and 22, use the following figure. Construct

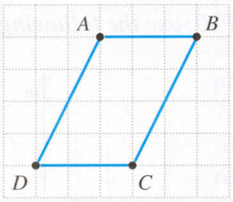


21. the reflection of trapezoid *ABCD* about line *m*.
22. the reflection of the trapezoid *ABCD* about line *l*.

In Exercises 23–30, use the translation vectors, **v** *and* **w** *shown below, to construct the translations indicated in the exercises. Show the figure in the positions both before and after the translation.*



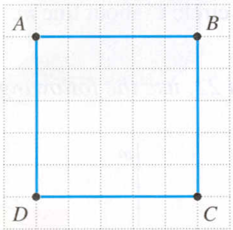
In Exercises 23 and 24, use the following figure. Construct



23. the translation of parallelogram *ABCD* using translation vector **v** (shown on page 533).

24. the translation of parallelogram *ABCD* using translation vector **w** (shown on page 533).

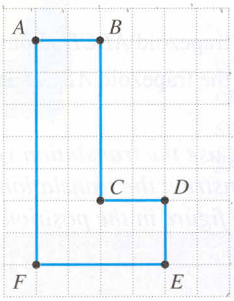
In Exercises 25 and 26, use the following figure. Construct



25. the translation of square *ABCD* using translation vector **w** (shown on page 533).

26. the translation of square *ABCD* using translation vector **v** (shown on page 533).

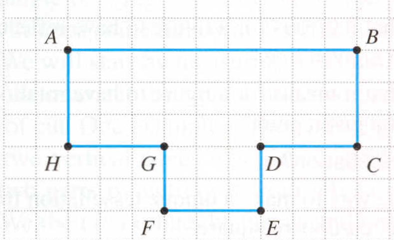
In Exercises 27 and 28, use the following figure. Construct



27. the translation of polygon *ABCDEF* using translation vector **v**.

28. the translation of polygon *ABCDEF* using translation vector **w**.

In Exercises 29 and 30, use the following figure. Construct

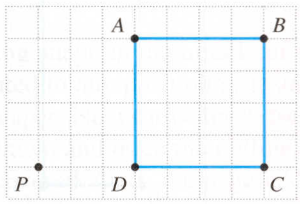


29. the translation of polygon *ABCDEFGH* using translation vector **w**.

30. the translation of polygon *ABCDEFGH* using translation vector **v**.

In Exercises 31–38, use the given figure and rotation point P to construct the indicated rotations. Show the figure in the positions both before and after the rotation.

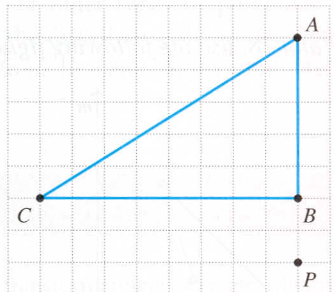
In Exercises 31 and 32, use the following figure. Construct



31. a 90° rotation of square *ABCD* about point *P*.

32. a 180° rotation of square *ABCD* about point *P*.

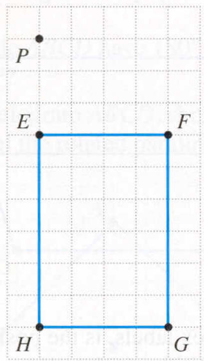
In Exercises 33 and 34, use the following figure. Construct



33. a 180° rotation of triangle *ABC* about point *P*.

34. a 270° rotation of triangle *ABC* about point *P*.

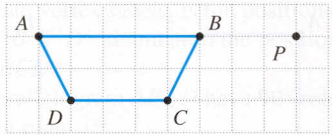
In Exercises 35 and 36, use the following figure. Construct



35. a 270° rotation of rectangle $EFGH$ about point P.

36. a 180° rotation of rectangle $EFGH$ about point P.

In Exercises 37 and 38, use the following figure. Construct

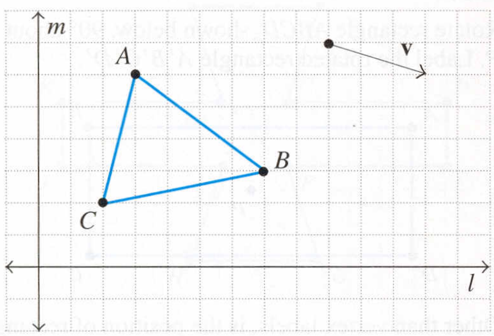


37. a 90° rotation of trapezoid $ABCD$ about point P.

38. a 270° rotation of trapezoid $ABCD$ about point P.

In Exercises 39–46, use the given figure, translation vectors **v** *and* **w**, *and reflection lines l and m to construct the indicated glide reflections. Show the figure in the positions before and after the glide reflection.*

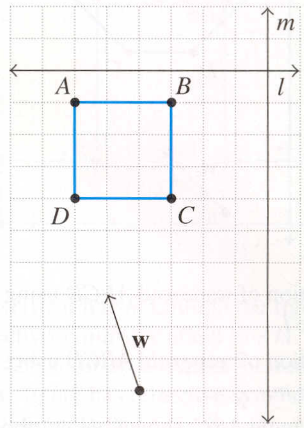
In Exercises 39 and 40, use the following figure. Construct



39. a glide reflection of triangle ABC using vector **v** and reflection line l.

40. a glide reflection of triangle ABC using vector **v** and reflection line m.

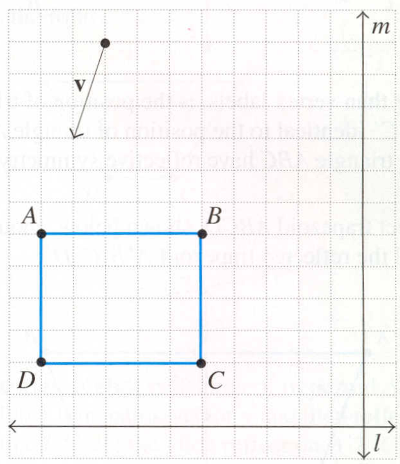
In Exercises 41 and 42, use the following figure. Construct



41. a glide reflection of square $ABCD$ using vector **w** and reflection line l.

42. a glide reflection of square $ABCD$ using vector **w** and reflection line m.

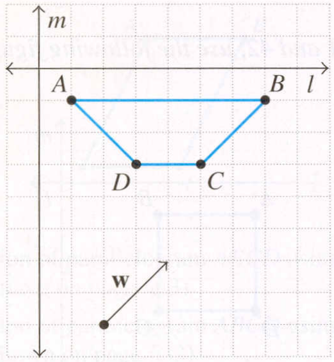
In Exercises 43 and 44, use the following figure. Construct



43. a glide reflection of rectangle $ABCD$ using vector **v** and reflection line l.

44. a glide reflection of rectangle $ABCD$ using vector **v** and reflection line m.

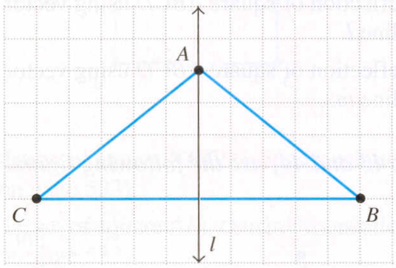
In Exercises 45 and 46, use the following figure. Construct



45. a glide reflection of trapezoid *ABCD* using vector **w** and reflection line *l*.

46. a glide reflection of trapezoid *ABCD* using vector **w** and reflection line *m*.

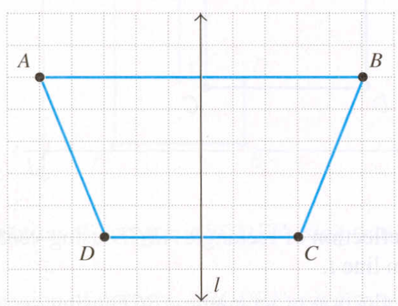
47. a) Reflect triangle *ABC*, shown below, about line *l*. Label the reflected triangle $A'B'C'$.



b) Other than vertex labels, is the position of triangle $A'B'C'$ identical to the position of triangle *ABC*?

c) Does triangle *ABC* have reflective symmetry about line *l*?

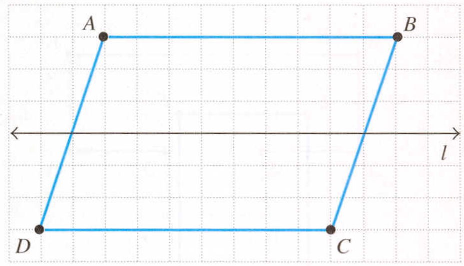
48. a) Reflect trapezoid *ABCD*, shown below, about line *l*. Label the reflected trapezoid $A'B'C'D'$.



b) Other than vertex labels, is the position of trapezoid $A'B'C'D'$ identical to the position of trapezoid *ABCD*?

c) Does trapezoid *ABCD* have reflective symmetry about line *l*?

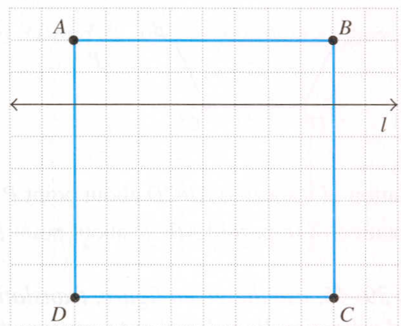
49. a) Reflect parallelogram *ABCD*, shown below, about line *l*. Label the reflected parallelogram $A'B'C'D'$.



b) Other than vertex labels, is the position of parallelogram $A'B'C'D'$ identical to the position of parallelogram *ABCD*?

c) Does parallelogram *ABCD* have reflective symmetry about line *l*?

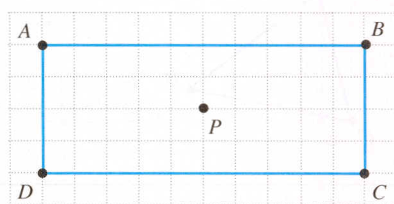
50. a) Reflect square *ABCD*, shown below, about line *l*. Label the reflected square $A'B'C'D'$.



b) Other than vertex labels, is the position of square $A'B'C'D'$ identical to the position of square *ABCD*?

c) Does square *ABCD* have reflective symmetry about line *l*?

51. a) Rotate rectangle *ABCD*, shown below, 90° about point *P*. Label the rotated rectangle $A'B'C'D'$.



b) Other than vertex labels, is the position of rectangle $A'B'C'D'$ identical to the position of rectangle *ABCD*?

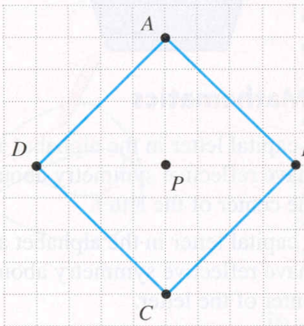
c) Does rectangle *ABCD* have 90° rotational symmetry about point *P*?

d) Now, rotate the rectangle in the original position, rectangle *ABCD*, 180° about point *P*. Label the rotated rectangle $A''B''C''D''$.

e) Other than vertex labels, is the position of rectangle $A''B''C''D''$ identical to the position of rectangle $ABCD$?

f) Does rectangle $ABCD$ have 180° rotational symmetry about point P?

52. a) Rotate parallelogram $ABCD$, shown below, 90° about point P. Label the rotated parallelogram $A'B'C'D'$.



b) Other than vertex labels, is the position of parallelogram $A'B'C'D'$ identical to the position of parallelogram $ABCD$?

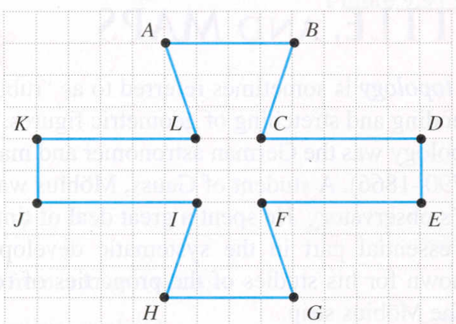
c) Does parallelogram $ABCD$ have 90° rotational symmetry about point P?

d) Now rotate the parallelogram in the original position, parallelogram $ABCD$, shown above, 180° about point P. Label the rotated parallelogram $A''B''C''D''$.

e) Other than vertex labels, is the position of parallelogram $A''B''C''D''$ identical to the position of parallelogram $ABCD$?

f) Does parallelogram $ABCD$ have 180° rotational symmetry about point P?

53. Consider the following figure.



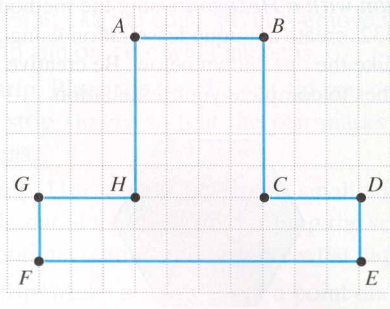
a) Insert a vertical line m through the figure so the figure has reflective symmetry about line m.

b) Insert a horizontal line l through the figure so the figure has reflective symmetry about line l.

c) Insert a point P within the figure so the figure has 180° rotational symmetry about point P.

d) Is it possible to insert a point P within the figure so the figure has 90° rotational symmetry about point P? Explain your answer.

54. Consider the following figure.



a) Insert a vertical line m through the figure so the figure has reflective symmetry about line m.

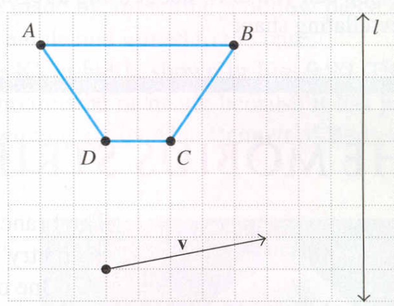
b) Is it possible to insert a horizontal line l through the figure so the figure has reflective symmetry about line l? Explain your answer.

c) Is it possible to insert a point P within the figure so the figure has 90° rotational symmetry about point P? Explain your answer.

d) Is it possible to insert a point P within the figure so the figure has 180° rotational symmetry about point P? Explain your answer.

Challenge Problems/Group Activities

55. ***Glide Reflection, Order*** Examine the figure below and then do the following:



a) Determine a glide reflection of trapezoid $ABCD$ by first applying translation vector **v** and then reflecting about the line l. Label the glide reflection $A'B'C'D'$.

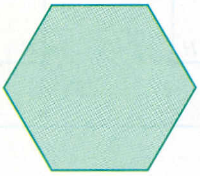
b) In this step we will reverse the order of the translation and the reflection. First reflect trapezoid $ABCD$ about the line l and then translate the reflection using vector **v**. Label the resulting figure $A''B''C''D''$.

c) Is figure $A'B'C'D'$ in the same position as figure $A''B''C''D''$?

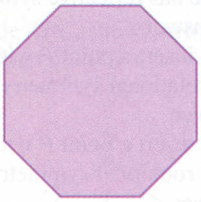
d) What can be said about the order of the translation and the reflection used in a glide reflection? Is the figure obtained in part (a) or part (b) the glide reflection?

56. *Tessellation with a Square* Create a unique tessellation from a square piece of cardboard by using the method described on page 532 of the text. Be creative using color and sketches to complete your tessellation.

57. *Tessellation with a Hexagon* Using the method described on page 532, create a unique tessellation using a regular hexagon like the one shown below. Be creative using color and sketches to complete your tessellation.

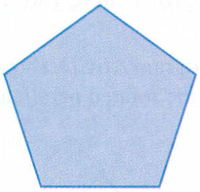


58. *Tessellation with an Octagon?* **a)** Trace the regular octagon, shown below, onto a separate piece of paper.



b) Try to create a regular tessellation by tracing this octagon repeatedly. Attempt to cover the entire piece of paper where no two octagons overlap each other. What conclusion can you draw about using a regular octagon as a tessellating shape?

59. *Tessellation with a Pentagon?* Repeat Exercise 58 using the regular pentagon below instead of a regular octagon.



Recreational Mathematics

60. Examine each capital letter in the alphabet and determine which letters have reflective symmetry about a horizontal line through the center of the letter.

61. Examine each capital letter in the alphabet and determine which letters have reflective symmetry about a vertical line through the center of the letter.

62. Examine each capital letter in the alphabet and determine which letters have 180° rotational symmetry about a point in the center of the letter.

Internet/Research Activities

63. In the study of biology reflective symmetry is called *bilateral symmetry* and rotational symmetry is called *radial symmetry*. Do research and write a report on the role symmetry plays in the study of biology.

64. Write a paper on the mathematics displayed in the artwork of M. C. Escher. Include such topics as tessellations, optical illusions, perspective, and non-Euclidean geometry.

9.6 THE MÖBIUS STRIP, KLEIN BOTTLE, AND MAPS

The branch of mathematics called ***topology*** is sometimes referred to as "rubber sheet geometry" because it deals with bending and stretching of geometric figures.

One of the first pioneers of topology was the German astronomer and mathematician August Ferdinand Möbius (1790–1866). A student of Gauss, Möbius was the director of the University of Leipzig's observatory. He spent a great deal of time studying geometry and he played an essential part in the systematic development of projective geometry. He is best known for his studies of the properties of one-sided surfaces, including the one called the Möbius strip.

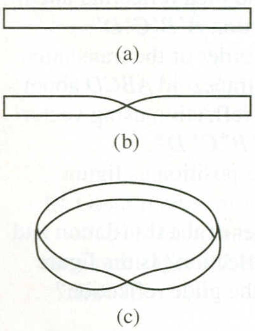


Figure 9.85

Möbius Strip

If you place a pencil on one surface of a sheet of paper and do not remove it from the sheet, you must cross the edge to get to the other surface. Thus, a sheet of paper has one edge and two surfaces. The sheet retains these properties even when crumpled into a ball. The ***Möbius strip***, also called a ***Möbius band***, is a one-sided, one-edged surface. You can construct one, as shown in Figure 9.85, by (a) taking a strip of paper, (b) giving one end a half twist, and (c) taping the ends together.

The Möbius strip has some very interesting properties. To better understand these properties, perform the following experiments.

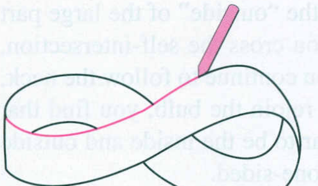


Figure 9.86

Experiment 1 Make a Möbius strip using a strip of paper and tape as illustrated in Fig. 9.85. Place the point of a felt-tip pen on the edge of the strip (Fig. 9.86). Pull the strip slowly so that the pen marks the edge; do not remove the pen from the edge. Continue pulling the strip and observe what happens.

Experiment 2 Make a Möbius strip. Place the tip of a felt-tip pen on the surface of the strip (Fig. 9.87). Pull the strip slowly so that the pen marks the surface. Continue and observe what happens.

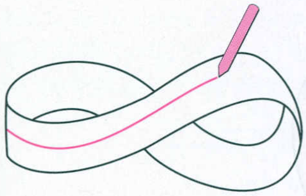


Figure 9.87

Experiment 3 Make a Möbius strip. Use scissors to make a small slit in the middle of the strip. Starting at the slit, cut along the strip, keeping the scissors in the middle of the strip (Fig. 9.88). Continue cutting and observe what happens.

Experiment 4 Make a Möbius strip. Make a small slit at a point about one-third of the width of the strip. Cut along the strip, keeping the scissors the same distance from the edge (Fig. 9.89). Continue cutting and observe what happens.

If you give a strip of paper several twists, you get variations on the Möbius strip. To a topologist the important distinction is between an odd number of twists, which leads to a one-sided surface, and an even number of twists, which leads to a two-sided surface. All strips with an odd number of twists are topologically the same as a Möbius strip, and all strips with an even number of twists are topologically the same as an ordinary cylinder, which has no twists.

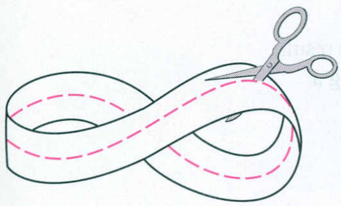


Figure 9.88

Klein Bottle

Another topological object is the punctured ***Klein bottle***; see Fig. 9.90. This object, named after Felix Klein (1849–1925), resembles a bottle but only has one side.

A punctured Klein bottle can be made by stretching a hollow piece of glass tubing. The neck is then passed through a hole and joined to the base.

Look closely at the model of the Klein bottle shown in Fig. 9.90. The punctured Klein bottle has only one edge and no outside or inside because it has just one side. Figure 9.91 shows a Klein bottle blown in glass by Alan Bennett of Bedford, England.

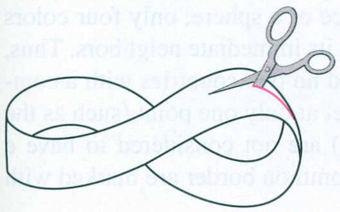


Figure 9.89

Limericks from unknown writers:

"A mathematician confided
That a Möbius band is one-sided,
And you'll get quite a laugh
If you cut one in half
For it stays in one piece when divided."

"A mathematician named Klein
Thought the Möbius band was divine.
He said, 'If you glue
the edges of two
You'll get a weird bottle like mine.' "

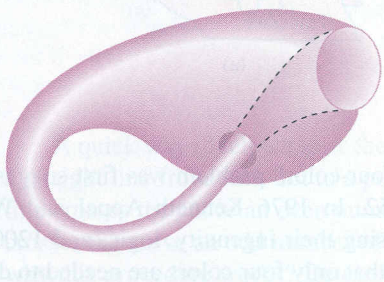


Figure 9.90

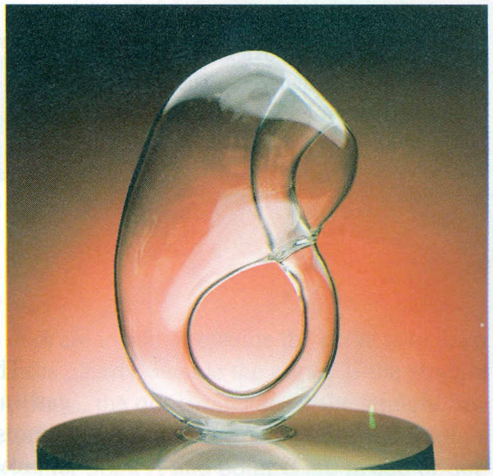
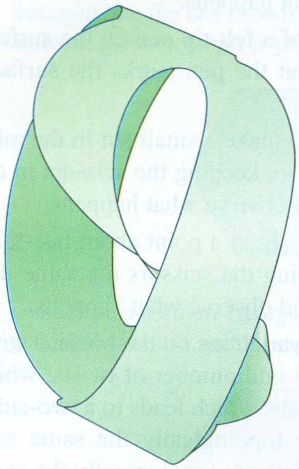


Figure 9.91 ***Klein bottle***, a one-sided surface, blown in glass by Alan Bennett.

DID YOU KNOW

His Vision Came in a Dream



Paper-strip Klein bottle

The Life, the Times, and the Art of Branson Graves Stevenson, by Herbert C. Anderson Jr. (Janher Publishing, 1979), reports that "in response to a challenge from his son, Branson made his first Klein bottle. . . . He failed in his first try, until the famous English potter, Wedgwood, came to Branson in a dream and showed him how to make the Klein bottle." That was around 50 years ago. Branson's study of claywork and pottery eventually led to the formation of the Archie Bray Foundation in Helena, Montana.

People have made Klein bottles from all kinds of materials. There is a knitting pattern for a woolly Klein bottle and even a paper Klein bottle with a hole.

Imagine trying to paint a Klein bottle. You start on the "outside" of the large part and work your way down the narrowing neck. When you cross the self-intersection, you have to pretend temporarily that it is not there, so you continue to follow the neck, which is now inside the bulb. As the neck opens up, to rejoin the bulb, you find that you are now painting the inside of the bulb! What appear to be the inside and outside of a Klein bottle connect together seamlessly since it is one-sided.

It is interesting to note that if a Klein bottle is cut along a curve, the results are two (one-twist) Möbius strips, see Fig. 9.92. Thus, a Klein bottle could also be made by gluing together two Möbius strips along the edges.

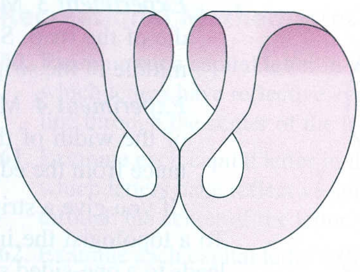


Figure 9.92 **Two Möbius strips** result from cutting a Klein bottle along a curve.

Maps

Maps have fascinated topologists for years because of the many challenging problems they present. Mapmakers have known for a long time that regardless of the complexity of the map and whether it is drawn on a flat surface or a sphere, only four colors are needed to differentiate each country (or state) from its immediate neighbors. Thus, every map can be drawn by using only four colors, and no two countries with a common border will have the same color. Regions that meet at only one point (such as the states of Arizona, Colorado, Utah, and New Mexico) are not considered to have a common border. In Fig. 9.93(a) no two states with a common border are marked with the same color.

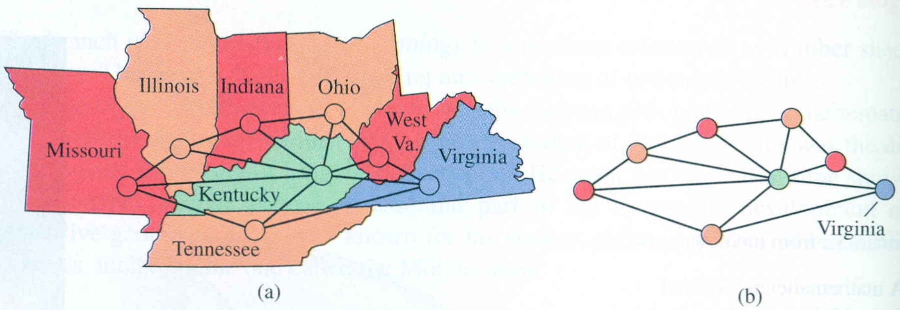


Figure 9.93

The "four-color" problem was first suggested by a student of Augustus DeMorgan in 1852. In 1976 Kenneth Appel and Wolfgang Haken of the University of Illinois—using their ingenuity, logic, and 1200 hours of computer time—succeeded in proving that only four colors are needed to draw a map. They solved the four-color map problem by reducing any map to a series of points and connecting line segments.

They replaced each country with a point. They connected two countries having a common border with a straight line; see Fig. 9.93(b). They then showed that the points of any graph in the plane could be colored by using only four colors in such a way that no two points connected by the same line were the same color.

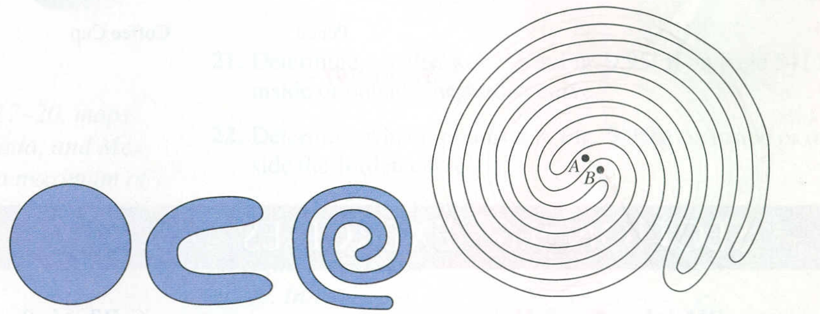
Mathematicians have shown that, on different surfaces, more than four colors may be needed to draw a map. For example, a map drawn on a Möbius strip requires a maximum of six colors as in Fig. 9.94(a). A map drawn on a torus (the shape of a doughnut) requires a maximum of seven colors as in Fig. 9.94(b).

(a) (b)

Figure 9.94

Jordan Curves

A *Jordan curve* is a topological object that can be thought of as a circle twisted out of shape; see Fig. 9.95 (a)–(d). Like a circle, it has an inside and an outside. To get from one side to the other at least one line must be crossed. Consider the Jordan curve in Fig. 9.95(d). Are points A and B inside or outside the curve?



(a) (b) (c) (d)

Figure 9.95

A quick way to tell whether the two dots are inside or outside the curve is to draw a straight line from each dot to a point that is clearly outside the curve. If the straight line crosses the curve an even number of times, the dot is outside. If the straight line crosses the curve an odd number of times, the dot is inside the curve. Can you explain why this procedure works? Determine whether point A and point B are inside or outside the curve (see Exercises 21 and 22 at the end of this section).

Topological Equivalence

Someone once said that a topologist is a person who does not know the difference between a doughnut and a coffee cup. Two geometric figures are said to be ***topologically equivalent*** if one figure can be elastically twisted, stretched, bent, or shrunk into the other figure without puncturing or ripping the original figure. If a doughnut is made of elastic material, it can be stretched, twisted, bent, shrunk, and distorted until it resembles a coffee cup with a handle, as shown in Fig. 9.96. Thus, the doughnut and coffee cup are topologically equivalent.

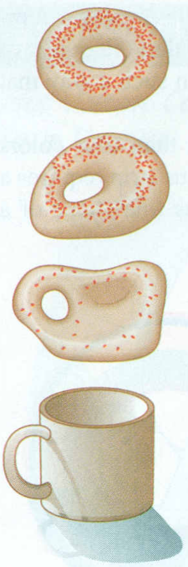


Figure 9.96

In topology figures are classified according to their *genus*. The ***genus*** of an object is determined by the number of holes in the object. A cup and a doughnut each have one hole and are of genus 1. A kettle and scissors each have two holes and are of genus 2. Figure 9.97 illustrates this type of classification.

Figure 9.97

SECTION 9.6 EXERCISES

Concept/Writing Exercises

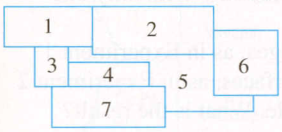
1. Explain why topology is sometimes referred to as "rubber sheet geometry."
2. What is a Möbius strip?
3. Explain how to make a Möbius strip.
4. What is a Klein bottle?
5. What is the maximum number of colors needed to create a map on a flat surface if no two regions colored the same are to share a common border?
6. What is the maximum number of colors needed to create a map if no two regions colored the same are to share a common border if the surface is a
   a) Möbius strip?
   b) torus?
7. What is a Jordan curve?
8. When testing to determine whether a point is inside or outside a Jordan curve, explain why if you count an odd number of lines, the point is inside the curve, and if you count an even number of lines, the point is outside the curve.

9. How is the genus of a figure determined?
10. When are two figures topologically equivalent?

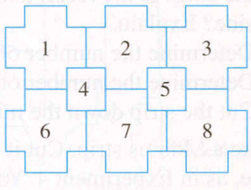
Practice the Skills

In Exercises 11–16, color the map by using a maximum of four colors so that no two regions with a common border have the same color.

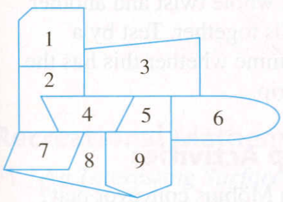
11.



12.



13.



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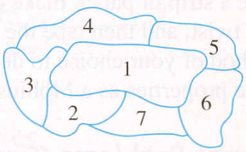

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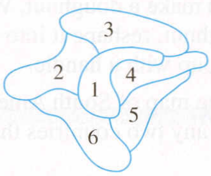

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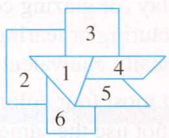
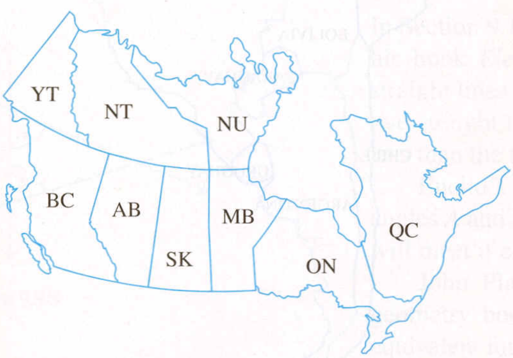


Using the Four-Color Theorem *In Exercises 17–20, maps show certain areas of the United States, Canada, and Mexico. Shade in the states (or provinces) using a maximum of four colors so that no two states (or provinces) with a common border have the same color.*

17.



18.

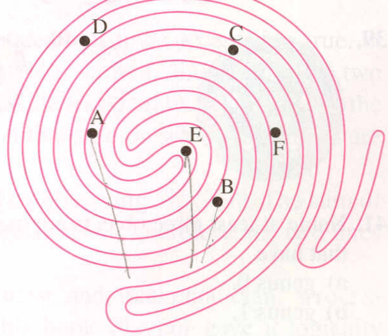
19.

20.

21. Determine whether point A in Fig. 9.95(d) on page 541 is inside or outside the Jordan curve.
22. Determine whether point B in Fig. 9.95(d) is inside or outside the Jordan curve.

At right is a Jordan curve. In Exercises 23–28, determine if the point is inside or outside of the curve.

23. Point A
24. Point B
25. Point C
26. Point D
27. Point E
28. Point F

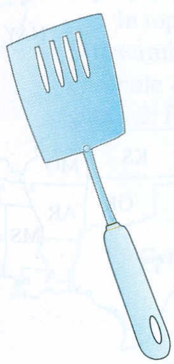


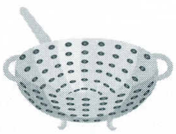
In Exercises 29–40, give the genus of the object. If the object has a genus larger than 5, write "larger than 5."

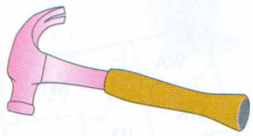
29. 

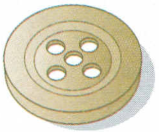
30. 

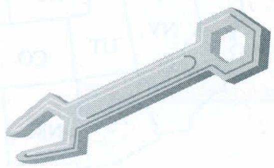
31. 

32. 

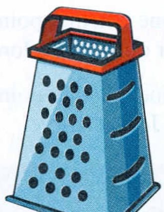
33. 

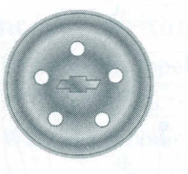
34. 

35. 

36. 

37. 

38. 

39. 

40. 

41. Name at least three objects not mentioned in this section that have
  a) genus 0.
  b) genus 1.
  c) genus 2.
  d) genus 3 or more.

42. Use the result of Experiment 1 on page 539 to find the number of edges on a Möbius strip.

43. Use the result of Experiment 2 on page 539 to find the number of surfaces on a Möbius strip.

44. How many separate strips are obtained in Experiment 3 on page 539?

45. How many separate strips are obtained in Experiment 4 on page 539?

46. **a)** Take a strip of paper, give it one full twist, and connect the ends. Is the result a Möbius strip with only one side? Explain.
  b) Determine the number of edges, as in Experiment 1.
  c) Determine the number of surfaces, as in Experiment 2.
  d) Cut the strip down the middle. What is the result?

47. Make a Möbius strip. Cut it one-third of the way from the edge, as in Experiment 4. You should get two loops, one going through the other. Determine whether either (or both) of these loops is itself a Möbius strip.

48. Take a strip of paper, make one whole twist and another half twist, and then tape the ends together. Test by a method of your choice to determine whether this has the same properties as a Möbius strip.

Challenge Problems/Group Activities

49. Can you see any advantage in a Möbius conveyor belt? Explain.

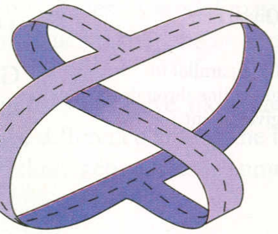
50. Using clay (or glazing compound) make a doughnut. Without puncturing or tearing the doughnut, reshape it into a topologically equivalent figure, a cup with a handle.

51. Using at most four colors, color the map of South America. Do not use the same color for any two countries that share a common border.

52. Using at most four colors, color the following map of the counties of Arizona. Do not use the same color for any two counties that share a common border.

Recreational Mathematics

53. ***An Interesting Surface*** Construct the following surface using two strips of paper, scissors, and tape, then answer the following questions.



a) How many sides does this surface have?
b) How many edges does this surface have?
c) Attempt to cut the surface "in half" by making a small slit in the middle of the paper surface. Then cut along the surface (see dashed line in the figure), keeping the scissors the same distance from the edge. In your own words, describe what happens.

It is interesting to note that although the surface shown in the figure above shares some of the same traits as a Möbius strip, this surface is not topologically equivalent to the Möbius strip.

Internet/Research Activity

54. Use the Internet to find a map of your state that shows the outline of all the counties within your state. Print this map and, using at most four colors, color it. Do not use the same color for any two counties that share a common border.

9.7 NON-EUCLIDEAN GEOMETRY AND FRACTAL GEOMETRY

Non-Euclidean Geometry

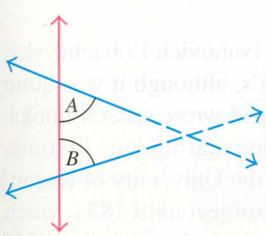


Figure 9.98

In Section 9.1 we stated postulates or axioms are statements to be accepted as true. In his book *Elements,* Euclid's fifth postulate was, "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles."

Euclid's fifth axiom may be better understood by observing Fig. 9.98. The sum of angles A and B is less than the sum of two right angles (180°). Therefore, the two lines will meet if extended.

John Playfair (1748–1819), a Scottish physicist and mathematician, wrote a geometry book that was published in 1795. In his book Playfair gave a logically equivalent interpretation of Euclid's fifth postulate. This version is often referred to as Playfair's postulate or the Euclidean parallel postulate.

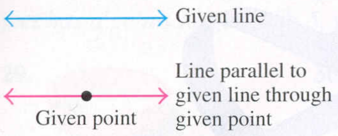


Figure 9.99

The Euclidean Parallel Postulate

Given a line and a point not on the line, one and only one line can be drawn through the given point parallel to the given line (Fig. 9.99)

The Euclidean parallel postulate may be better understood by looking at Fig. 9.99. Many mathematicians after Euclid believed that this postulate was not as self-evident as the other nine. Others believed this postulate could be proved from the other nine postulates and therefore was not needed at all. Of the many attempts to prove that the fifth postulate was not needed, the most noteworthy one was presented by Girolamo Saccheri (1667–1733), a Jesuit priest in Italy. In the course of his elaborate chain of deductions, Saccheri proved many of the theorems of what is now called hyperbolic geometry. However, Saccheri did not realize what he had done. He believed that Euclid's geometry was the only "true" geometry and concluded that his own work was in error. Thus, Saccheri narrowly missed receiving credit for a great achievement: the founding of ***non-Euclidean geometry***.

Over time, geometers became more and more frustrated at their inability to prove Euclid's fifth postulate. One of them, a Hungarian named Farkos Bolyai, in a letter to his son, Janos Bolyai, wrote, "I entreat you leave the science of parallels alone. . . . I have traveled past all reefs of this infernal dead sea and have always come back with a broken mast and torn sail." The son, refusing to heed his father's advice, continued to think about parallels until, in 1823, he saw the whole truth and enthusiastically declared, "I have created a new universe from nothing." He recognized that geometry branches in two directions, depending on whether Euclid's fifth postulate is applied. He recognized two different geometries and published his discovery as a 24-page appendix to a textbook written by his father. The famous mathematician George Bruce Halsted called it "the most extraordinary two dozen pages in the whole history of thought." Farkos Bolyai proudly presented a copy of his son's work to his friend Carl Friedrich Gauss, then Germany's greatest mathematician, whose reply to the father had a devastating effect on the son. Gauss wrote, "I am unable to praise this work. . . . To praise it would be to praise myself. Indeed, the whole content of the work, the path taken by your son, the results to which he is led, coincides almost entirely with my meditations which occupied my mind partly for the last thirty or thirty-five years." We now know from his earlier correspondence that Gauss had indeed been familiar with ***hyperbolic geometry*** even before Janos was born. In his letter Gauss also indicated that it was his intention not to let his theory be published during his lifetime, but to record it so that the theory would not perish with him. It is believed that the reason Gauss did not publish his work was that he feared being ridiculed by other prominent mathematicians of his time.

Nikolay Ivanovich Lobachevsky

At about the same time as Bolyai's publication, Nikolay Ivanovich Lobachevsky, a Russian, published a paper that was remarkably like Bolyai's, although it was quite independent of it. Lobachevsky made a deeper investigation and wrote several books. In marked contrast to Bolyai, who received no recognition during his lifetime, Lobachevsky received great praise and became a professor at the University of Kazan.

G. F. Bernhard Riemann

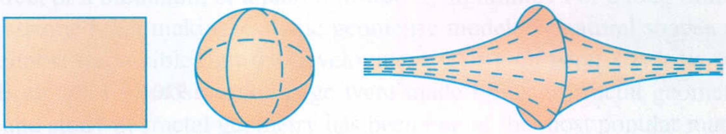
After the initial discovery, little attention was paid to the subject until 1854, when G. F. Bernhard Riemann (1826–1866), a student of Gauss, suggested a second type of non-Euclidean geometry, which is now called ***spherical***, ***elliptical***, or ***Riemannian geometry***. The hyperbolic geometry of his predecessors was synthetic; that is, it was not based on or related to any concrete model when it was developed. Riemann's geometry was closely related to the theory of surfaces. A ***model*** may be considered a physical interpretation of the undefined terms that satisfies the axioms. A model may be a picture or an actual physical object.

The two types of non-Euclidean geometries we have mentioned are elliptical geometry and hyperbolic geometry. The major difference among the three geometries lies in the fifth axiom. The fifth axiom of the three geometries is summarized here.

The Fifth Axiom of Geometry

Euclidean	Elliptical	Hyperbolic
Given a line and a point not on the line, one and only one line can be drawn parallel to the given line through the given point.	Given a line and a point not on the line, no line can be drawn through the given point parallel to the given line.	Given a line and a point not on the line, two or more lines can be drawn through the given point parallel to the given line.

To understand the fifth axiom of the two non-Euclidean geometries, remember that the term *line* is undefined. Thus, a line can be interpreted differently in different geometries. A model for Euclidean geometry is a plane, such as a blackboard (Fig. 9.100a). A model for elliptical geometry is a sphere (Fig. 9.100b). A model for hyperbolic geometry is a pseudosphere (Fig. 9.100c). A pseudosphere is similar to two trumpets placed bell to bell. Obviously, a line on a plane cannot be the same as a line on either of the other two figures. The curved red lines in Fig. 9.101(a) and both colored lines in Fig. 9.102, on page 548, are examples of lines in elliptical and hyperbolic geometry, respectively.



(a) Plane (b) Sphere (c) Pseudosphere

Figure 9.100

Elliptical Geometry

A circle on the surface of a sphere is called a great circle if it divides the sphere into two equal parts. If we were to cut through a sphere along a great circle, we would have two identical pieces. If we interpret a line to be a great circle, we can see the fifth axiom of elliptical geometry is true. Two great circles on a sphere must intersect; hence, there can be no parallel lines (Fig. 9.101a).

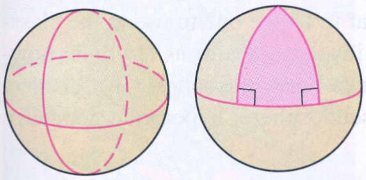
If we were to construct a triangle on a sphere, the sum of its angles would be greater than 180° (Fig. 9.101b). The theorem, "The sum of the measures of the angles of a triangle is greater than 180°," has been proven by means of the axioms of elliptical geometry. The sum of the measures of the angles varies with the area of the triangle and gets closer to 180° as the area decreases.

MATHEMATICS *Everywhere*

Mapping The Brain

Medical researchers and mathematicians are currently attempting to capture an image of the three-dimensional human brain on a two-dimensional map. In some respects the task is like capturing the image of Earth on a two-dimensional map, but because of the many folds and fissures on the surface of the brain, the task is much more complex. Points of the brain that are at different depths can appear too close in a flat image. Therefore, to develop an accurate mapping, researchers use topology, hyperbolic geometry, and elliptical geometry to create an image known as a *conformal mapping*. Researchers use conformal mappings to precisely identify the parts of the brain that correspond to specific functions.

Photograph courtesy of Dr. Monica K. Hurdal (mhurdal@math.fsu.edu) Dept. of Mathematics, Florida State University



(a) (b)

Figure 9.101

Hyperbolic Geometry

The lines in hyperbolic geometry are represented by geodesics on the surface of the pseudosphere. A *geodesic* is the shortest and least-curved arc between two points on a surface. Figure 9.102 illustrates two lines on the surface of a pseudosphere.

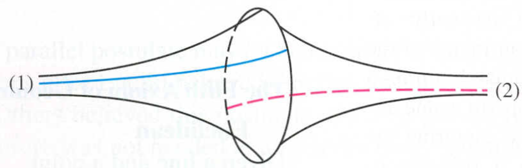


Figure 9.102

Figure 9.103(a) illustrates the fifth axiom of hyperbolic geometry.* Note that, through the given point, two lines are drawn parallel to the given line. If we were to construct a triangle on a pseudosphere, the sum of the measures of the angles would be less than 180° (Fig. 9.103b). The theorem, "The sum of the measures of the angles of a triangle is less than 180°," has been proven by means of the axioms of hyperbolic geometry.

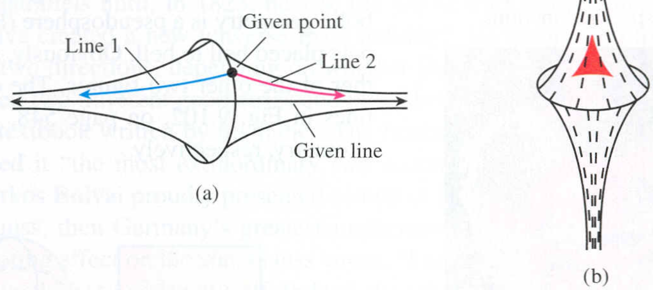


Figure 9.103

We have stated that the sum of the measures of the angles of a triangle is 180°, is greater than 180°, and is less than 180°. Which statement is correct? Each statement is correct *in its own system.* Many theorems hold true for all three geometries; vertical angles still have the same measure, we can uniquely bisect a line segment with a straightedge and compass alone, and so on.

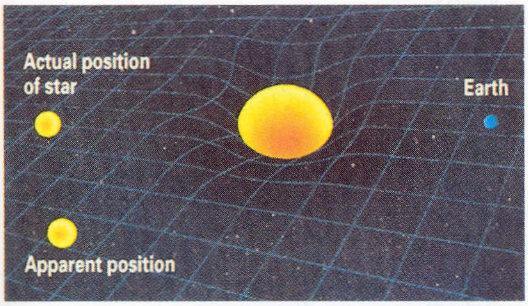
The many theorems based on the fifth postulate may differ in each geometry. It is important for you to realize each theorem proved is true *in its own system* because each is logically deduced from the given set of axioms of the system. No one system is the "best" system. Euclidean geometry may appear to be the one to use in the classroom, where the blackboard is flat. In discussions involving Earth as a whole, however, elliptical geometry may be the most useful, since Earth is a sphere. If the object under consideration has the shape of a saddle or pseudosphere, hyperbolic geometry may be the most useful.

*A formal discussion of hyperbolic geometry is beyond the scope of this text.

DID YOU KNOW

It's All Relative

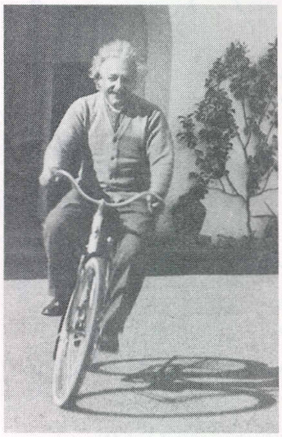
Albert Einstein's general theory of relativity, published in 1916, approached space and time differently from our everyday understanding of them. Einstein's theory unites the three dimensions of space with one of time in a four-dimensional space, time continuum. His theory dealt with the path that light and objects take while moving through space under the force of gravity. Einstein conjectured that mass (such as stars and planets) caused space to be curved. The greater the mass, the greater the curvature. Also, in the region nearer to the mass, the curvature of the space is greater.



You can visualize Einstein's theory by thinking of space as a rubber sheet pulled taut on which a mass is placed, causing the rubber sheet to bend.

To prove his conjecture, Einstein exposed himself to Riemann's non-Euclidean geometry. Einstein believed that the trajectory of a particle in space represents not a straight line but the straightest curve possible, a geodesic.

Space–time is now thought to be a combination of three different types of curvature: spherical (described by Riemannian geometry), flat (described by Euclidean geometry), and saddle-shaped (described by hyperbolic geometry).

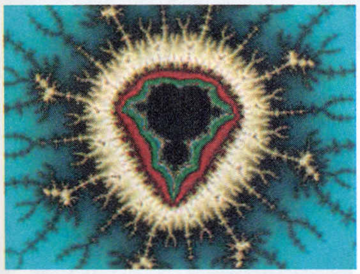


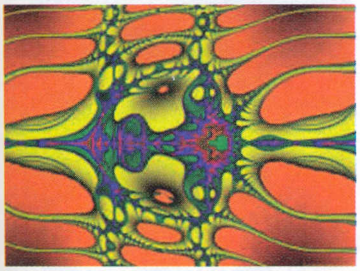
Einstein's theory was confirmed by the solar eclipses of 1919 and 1922.

"The Great Architect of the universe now appears to be a great mathematician."
British physicist Sir James Jeans

Fractal Geometry

We are familiar with one-, two-, and three-dimensional figures. Many objects, however, are difficult to categorize as one-, two-, or three-dimensional. For example, how would you classify the irregular shapes we see in nature such as a coastline, or the bark on a tree, or a mountain, or a path followed by lightning? For a long time mathematicians assumed that making realistic geometric models of natural shapes and figures was almost impossible, but the development of ***fractal geometry*** now makes it possible. Both color photos on this page were made by using fractal geometry. The discovery and study of fractal geometry has been one of the most popular mathematical topics in recent times.





Fractal images

The word *fractal* (from the Latin word *fractus,* "broken up, fragmented") was first used in the mid-1970s by mathematician Benoit Mandelbrot to describe shapes that had several common characteristics, including some form of "self-similarity," as will be seen shortly in the Koch snowflake.

Typical fractals are extremely irregular curves or surfaces that "wiggle" enough so that they are not considered one-dimensional. Fractals do not have integer dimensions; their dimensions are between 1 and 2. For example, a fractal may have a dimension of 1.26. Fractals are developed by applying the same rule over and over again, with the end point of each simple step becoming the starting point for the next step, in a process called ***recursion***.

Using the recursive process, we will develop a famous fractal called the ***Koch snowflake*** named after Helga von Koch, a Swedish mathematician who first discovered its remarkable characteristics. The Koch snowflake illustrates a property of all fractals called *self-similarity*; that is, each smaller piece of the curve resembles the whole curve.

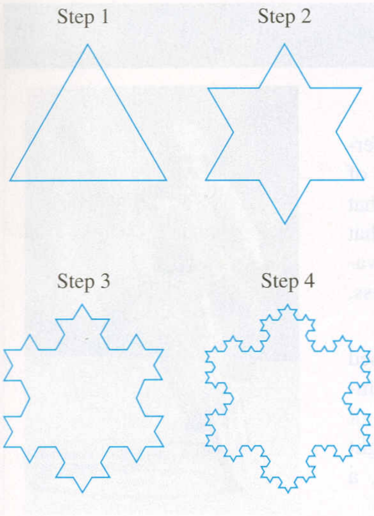


Figure 9.104

To develop the Koch snowflake:

1. Start with an equilateral triangle (step 1, Fig. 9.104).
2. Whenever you see an edge —— replace it with _/_ (steps 2–4).

What is the perimeter of the snowflake in Fig. 9.104, and what is its area? A portion of the boundary of the Koch snowflake known as the Koch curve or the snowflake curve is represented in Fig. 9.105.

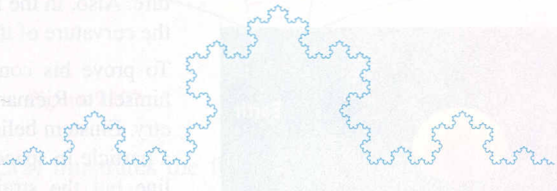


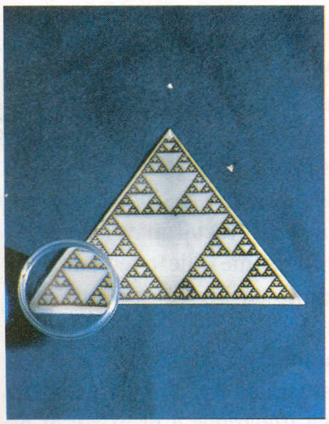
Figure 9.105

Because the Koch curve consists of infinitely many pieces of the form _/_, the perimeter is also infinite. The area of the Koch snowflake is 1.6 times the area of the starting equilateral triangle. Thus, the area of the snowflake is finite. The Koch snowflake has a finite area enclosed by an infinite boundary! This fact may seem difficult to accept, but it is true. However, the Koch snowflake, like other fractals, is not an everyday run-of-the-mill geometric shape.

Let us look at a few more fractals made using the recursive process. We will now construct what is known as a *fractal tree.* Start with a tree trunk (Fig. 9.106a on page 551). Draw two branches, each one a bit smaller than the trunk (Fig. 9.106b). Draw two branches from each of those branches, and continue; see Figs. 9.106c and Fig. 9.106d. Ideally we continue the process forever.

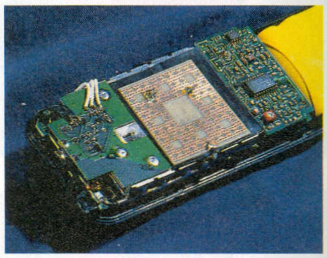
DID YOU KNOW

Fractal Antennas



Fractal triangle can act as a miniaturized antenna.

Recently researchers have begun to use antennas made using fractal designs. Many antennas that look like a simple unit, including most radar antennas, are actually arrays of up to thousands of small antennas. Scientists have discovered that a fractal arangement can result in antennas being as powerful as traditional antennas, using only a quarter of the number of elements. Dwight Jasserd of the University of Pennsylvania says, "Fractals bridge the gap; they have short-range disorder and long-range order." Fractal antennas are 25% more efficient than the rubbery "stubby" antennas found on most phones. Why do fractal antennas work so well? It has been proven mathematically that for an antenna to work equally well at all frequencies, it must be self-similar, having the same basic appearance at every scale. That is, it has to be fractal!



Hidden inside a cordless phone, a square fractal antenna (center board) replaces the usual rubbery stalk.

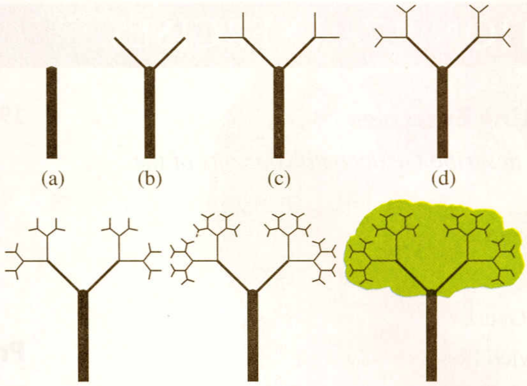


Figure 9.106 The fractal tree

If you take a little piece of any branch and zoom in on it, it will look exactly like the original tree. Fractals are *scale independent,* which means that you cannot really tell whether you are looking at something very big or something very small because the fractal looks the same whether you are close to it or far from it.

In Figs. 9.107 and 9.108 we develop two other fractals through the process of recursion. Figure 9.107 shows a fractal called the Sierpinski triangle, and Fig. 9.108 shows a fractal called the Sierpinski carpet. Both fractals are named after Waclaw Sierpinski, a Polish mathematician who is best known for his work with fractals and space-filling curves.

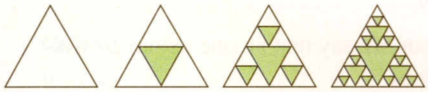


Figure 9.107 Sierpinski triangle

Fractals provide a way to study natural forms such as coastlines, trees, mountains, galaxies, polymers, rivers, weather patterns, brains, lungs, and blood supply. Fractals also help explain that which appears chaotic. The blood supply in the body is one example. The branching of arteries and veins appear chaotic, but closer inspection reveals the same type of branching occurs for smaller and smaller blood vessels, down to the capillaries. Thus, fractal geometry provides a geometric structure for chaotic processes in nature. The study of chaotic processes is called *chaos theory*.

Fractals nowadays have a potentially important role to play in characterizing weather systems and in providing insight into various physical processes such as the occurrence of earthquakes or the formation of deposits that shorten battery life. Some scientists view fractal statistics as a doorway to unifying theories of medicine, offering a powerful glimpse of what it means to be healthy.

Fractals lie at the heart of current efforts to understand complex natural phenomena. Unraveling their intricacies could reveal the basic design principles at work in our world. Until recently, there was no way to describe fractals. Today, we are beginning to see such features everywhere. Tomorrow, we may look at the entire universe through a fractal lens.

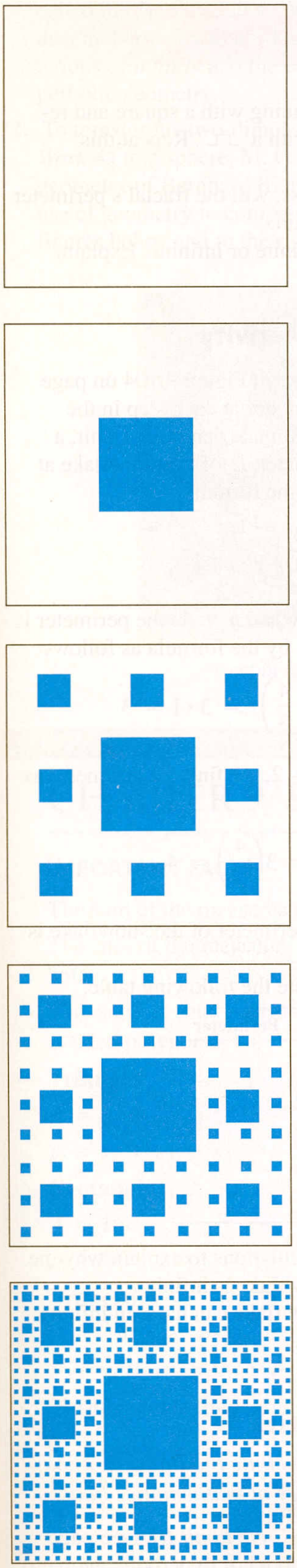


Figure 9.108 Sierpinski carpet

SECTION 9.7 EXERCISES

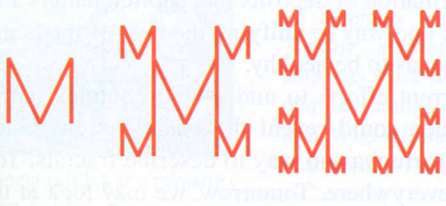
Concept/Writing Exercises

In Exercises 1–6, describe the accomplishments of the mathematician.

1. Girolamo Saccheri
2. Janos Bolyai
3. Carl Friedrich Gauss
4. Nikolay Ivanovich Lobachevsky
5. G. F. Bernhard Riemann
6. Benoit Mandelbrot
7. State the fifth axiom of
   a) Euclidean geometry. b) hyperbolic geometry.
   c) elliptical geometry.
8. State the theorem concerning the sum of the measures of the angles of a triangle in
   a) Euclidean geometry. b) hyperbolic geometry.
   c) elliptical geometry.
9. What model is often used in describing and explaining Euclidean geometry?
10. What model is often used in describing and explaining elliptical geometry?
11. What model is often used in describing and explaining hyperbolic geometry?
12. What do we mean when we say that no axiomatic system of geometry is "best"?
13. List the three types of curvature of space and the types of geometry that correspond to them.
14. List at least five natural forms that appear chaotic that we can study using fractals.

Practice the Skills

In the following we show a fractal-like figure made using a recursive process with the letter "M." Use this fractal-like figure as a guide in constructing fractal-like figures with the letter given in Exercises 15–18. Show three steps, as is done here.



15. I **16.** W **17.** V **18.** H

19. a) Develop a fractal by beginning with a square and replacing each side —— with a ⎍. Repeat this process twice.
    b) If you continue this process, will the fractal's perimeter be finite or infinite? Explain.
    c) Will the fractal's area be finite or infinite? Explain.

Problem Solving/Group Activity

20. In forming the Koch snowflake in Figure 9.104 on page 550 the perimeter becomes greater at each step in the process. If each side of the original triangle is 1 unit, a general formula for the perimeter, L, of the snowflake at any step, n, may be found by the formula

$$L = 3\left(\frac{4}{3}\right)^{n-1}$$

For example, at the first step when $n = 1$, the perimeter is 3 units, which can be verified by the formula as follows:

$$L = 3\left(\frac{4}{3}\right)^{1-1} = 3\left(\frac{4}{3}\right)^{0} = 3 \cdot 1 = 3$$

At the second step, when $n = 2$, we find the perimeter as follows:

$$L = 3\left(\frac{4}{3}\right)^{2-1} = 3\left(\frac{4}{3}\right) = 4$$

Thus, at the second step the perimeter of the snowflake is 4 units.

a) Use the formula to complete the following table.

Step	Perimeter
1	
2	
3	
4	
5	
6	

b) Use the results of your calculations to explain why the perimeter of the Koch snowflake is infinite.
c) Explain how the Koch snowflake can have an infinite perimeter, but a finite area.

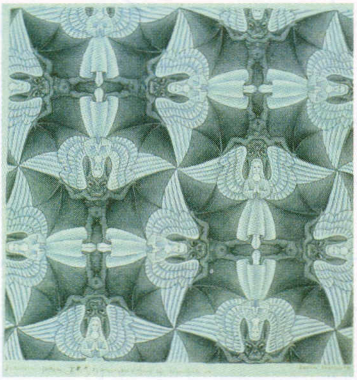
Internet/Research Activities

In Exercises 21–23, references include the Internet, books on art, encyclopedias, and history of mathematics books.

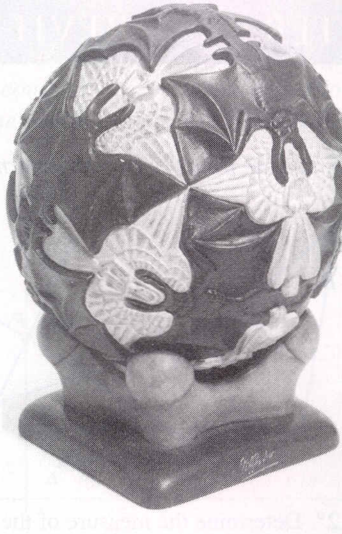
21. To complete his masterpiece *Circle Limit III*, (see page 531) M. C. Escher studied a model of hyperbolic geometry

called the *Poincaré disk*. Write a paper on the Poincaré disk and how it was used in Escher's art. Include representations of *infinity* and the concepts of *point* and *line* in hyperbolic geometry.

22. To transfer his two-dimensional tiling known as *Symmetry Work 45* to a sphere, M. C. Escher used the spherical geometry of Bernhard Riemann. Write a paper on Escher's use of geometry to complete this masterpiece (see the figures below and to the right).



Symmetry Work 45



23. Go to the website *Fantastic Fractals* and study the information about fractals given there. Print copies, in color if a color printer is available, of the Mandlebrot set and the Julia set.

CHAPTER 9 SUMMARY

IMPORTANT FACTS

The sum of the measures of the angles of a triangle is 180°.

The sum of the measures of the angles of a quadrilateral is 360°.

The sum of the measures of the interior angles of an n-sided polygon is $(n - 2)180°$.

Triangle
$A = \frac{1}{2}bh$
$p = s_1 + s_2 + s_3$

Square
$A = s^2$
$p = 4s$

Rectangle
$A = lw$
$p = 2l + 2w$

Parallelogram
$A = bh$
$p = 2b + 2w$

Trapezoid
$A = \frac{1}{2}h(b_1 + b_2)$
$p = s_1 + s_2 + b_1 + b_2$

Pythagorean theorem
$a^2 + b^2 = c^2$

Circle
$A = \pi r^2$; $C = 2\pi r$ or $C = \pi d$

Cube
$V = s^3$

Rectangular solid
$V = lwh$

Cylinder
$V = \pi r^2 h$

Cone
$V = \frac{1}{3}\pi r^2 h$

Sphere
$V = \frac{4}{3}\pi r^3$

Prism
$V = Bh$, where B is the area of the base

Pyramid
$V = \frac{1}{3}Bh$, where B is the area of the base

Fifth postulate in Euclidean geometry

Given a line and a point not on the line, only one line can be drawn through the given point parallel to the given line.

Fifth postulate in elliptical geometry

Given a line and a point not on the line, no line can be drawn through the given point parallel to the given line.

Fifth postulate in hyperbolic geometry

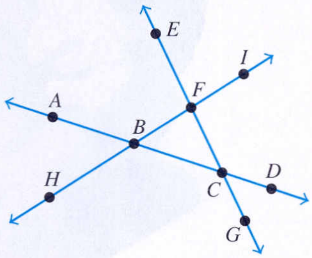
Given a line and a point not on the line, two or more lines can be drawn through the given point parallel to the given line.

CHAPTER 9 REVIEW EXERCISES

9.1

In Exercises 1–6, use the figure shown to determine the following.

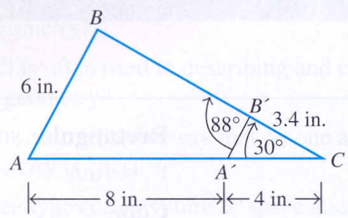
1. $\measuredangle EFI \cap \measuredangle BFC$
2. $\overline{BF} \cup \overline{FC} \cup \overline{BC}$
3. $\overleftrightarrow{AB} \cup \overrightarrow{BC}$
4. $\overset{\circ\!\!\rightarrow}{BH} \cup \overset{\circ\!\!\rightarrow}{HB}$
5. $\overleftrightarrow{HI} \cap \overleftrightarrow{EG}$
6. $\overset{\circ\!\!\rightarrow}{CF} \cap \overset{\circ\!\!\rightarrow}{CG}$



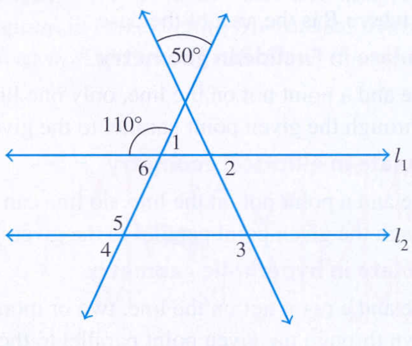
7. $m\measuredangle A = 51.2°$. Determine the measure of the complement of $\measuredangle A$.
8. $m\measuredangle B = 124.7°$. Determine the measure of the supplement $\measuredangle B$.

9.2

In Exercises 9–12, use the similar triangles ABC and A′B′C shown to determine the following.



9. The length of $\overline{BC}$
10. The length of $\overline{A'B'}$
11. $m\measuredangle BAC$
12. $m\measuredangle ABC$
13. In the following figure, l_1 and l_2 are parallel lines. Determine $m\measuredangle 1$ through $m\measuredangle 6$.

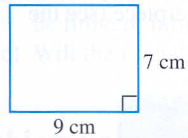


14. Determine the sum of the measures of the interior angles of a hexagon.

9.3

In Exercises 15–19, determine the area of the figure.

15.



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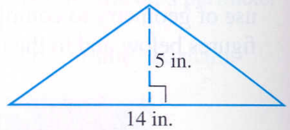

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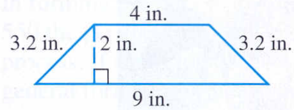

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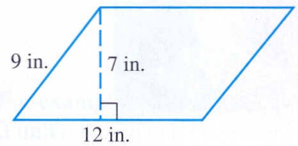

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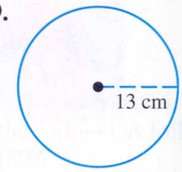
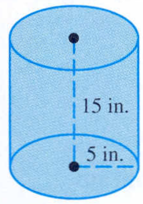


20. **Cost of Kitchen Tile** Determine the total cost of covering a 14 ft by 16 ft kitchen floor with ceramic tile. The cost of the tile selected is \$2.75 per square foot.

9.4

In Exercises 21–26, determine the volume of the figure. Round your answers to hundredths.

21.



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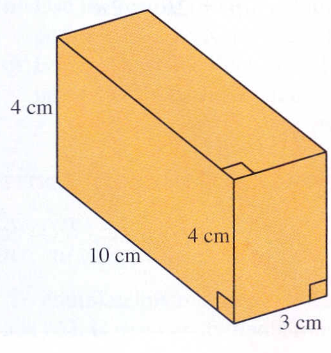

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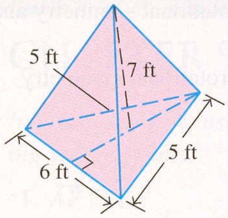

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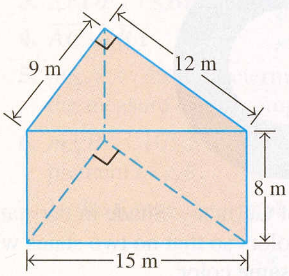

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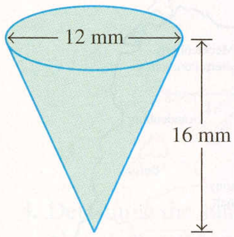

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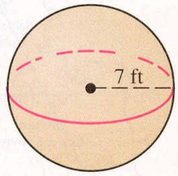
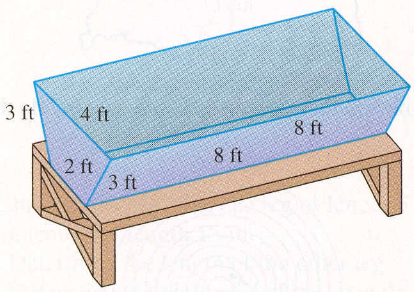


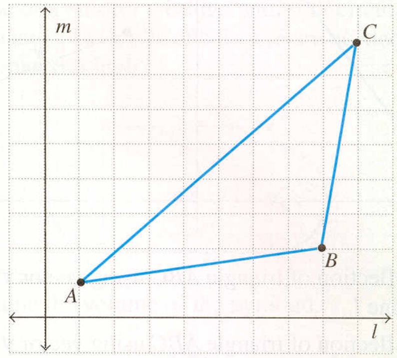
27. *Water Trough* Steven Dale has a water trough whose ends are trapezoids and whose sides are rectangles, as illustrated. He is afraid that the base it is sitting on will not support the weight of the trough when it is filled with water. He knows that the base will support 4800 lb.



a) Find the number of cubic feet of water contained in the trough.
b) Find the total weight, assuming that the trough weighs 375 lb and the water weighs 62.5 lb per cubic foot. Is the base strong enough to support the trough filled with water?
c) If 1 gal of water weighs 8.3 lb, how many gallons of water will the trough hold?

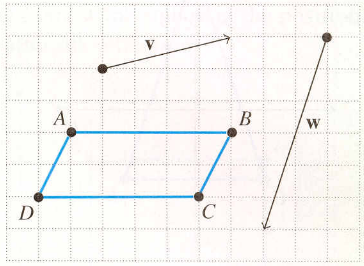
9.5

In Exercises 28 and 29, use the given triangle and lines of reflection to construct the indicated reflections. Show the triangle in the positions both before and after the reflection. Construct



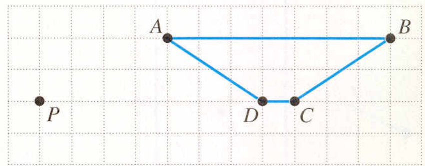
28. the reflection of triangle ABC about line l.
29. the reflection of triangle ABC about line m.

*In Exercises 30 and 31, use translation vectors **v** and **w** to construct the indicated translations. Show the parallelogram in the positions both before and after the translation. Construct*



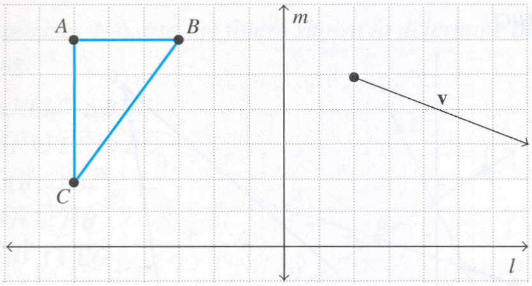
30. the translation of parallelogram $ABCD$ using translation vector **v**.
31. the translation of parallelogram $ABCD$ using translation vector **w**.

In Exercises 32–34, use the given figure and rotation point P to construct the indicated rotations. Show the trapezoid in the positions both before and after the rotation. Construct



32. a 90° rotation of trapezoid $ABCD$ about point P.
33. a 180° rotation of trapezoid $ABCD$ about point P.
34. a 270° rotation of trapezoid $ABCD$ about point P.

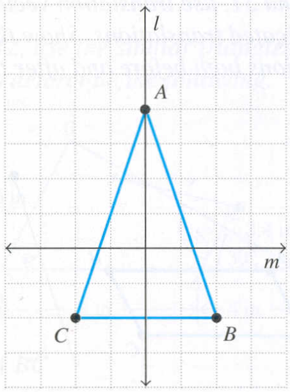
In Exercises 35 and 36, use the given figure, translation vector **v**, *and reflection lines l and m to construct the indicated glide reflections. Show the triangle in the positions both before and after the glide reflection. Construct*



35. a glide reflection of triangle ABC using vector **v** and reflection line l.

36. a glide reflection of triangle ABC using vector **v** and reflection line m.

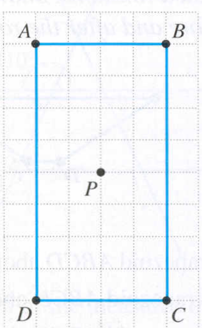
In Exercises 37 and 38, use the following figure to answer the following questions.



37. Does triangle ABC have reflective symmetry about line l? Explain.

38. Does triangle ABC have reflective symmetry about line m? Explain.

In Exercises 39 and 40, use the following figure to answer the following questions.



39. Does rectangle $ABCD$ have 90° rotational symmetry about point P? Explain.

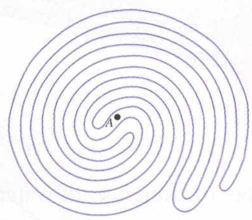
40. Does rectangle $ABCD$ have 180° rotational symmetry about point P? Explain.

9.6

41. Give the genus of the object:

42. The map shows the states of Germany. Shade in the states using a maximum of four colors so that no two states with a common border have the same color.

43. Determine whether point A is inside or outside the Jordan curve.



9.7

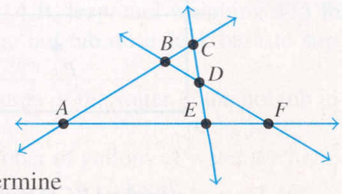
44. State the fifth axiom of Euclidean, elliptical, and hyperbolic geometry.

45. Develop a fractal by beginning with a square and replacing each side —— with a ⎍. Repeat this process twice.

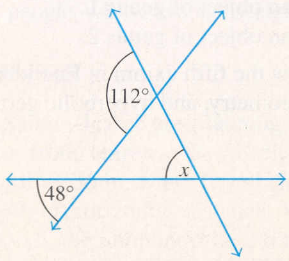
CHAPTER 9 TEST

In Exercises 1–4, use the figure to describe the following sets of points:

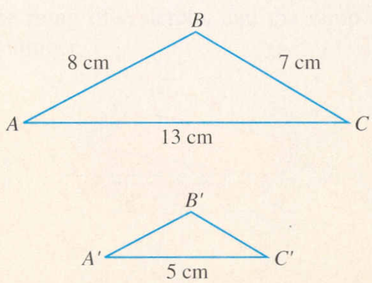
1. $\overleftrightarrow{AF} \cap \overrightarrow{EF}$
2. $\overline{BC} \cup \overline{CD} \cup \overline{BD}$
3. $\measuredangle EDF \cap \measuredangle BDC$
4. $\overrightarrow{AC} \cup \overrightarrow{BA}$



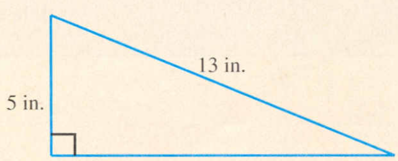
5. $m\measuredangle A = 36.9°$. Determine the measure of the complement of $\measuredangle A$.
6. $m\measuredangle B = 101.5°$. Determine the measure of the supplement of $\measuredangle B$.
7. In the figure, determine the measure of $\measuredangle x$.



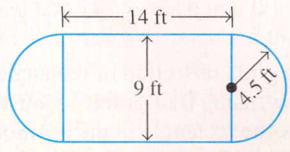
8. Determine the sum of the measures of the interior angles of an octagon.
9. Triangles ABC and $A'B'C'$ are similar figures. Determine the length of side $\overline{B'C'}$.



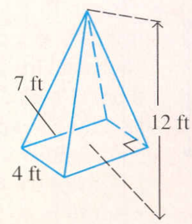
10. Right triangle ABC has one leg of length 5 in. and a hypotenuse of length 13 in.
    a) Determine the length of the other leg.
    b) Determine the perimeter of the triangle.
    c) Determine the area of the triangle.



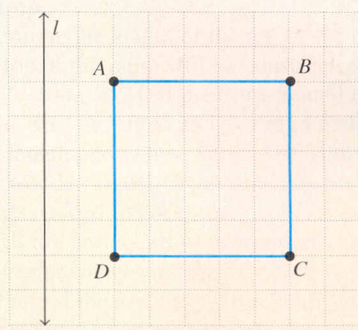
11. Determine the volume of a sphere of diameter 16 cm.
12. **Building a Pier** The sketch shows the dimensions of the base of a pier for a bridge with semicircular ends. How many cubic yards of concrete are needed to build a pier 6 ft high?



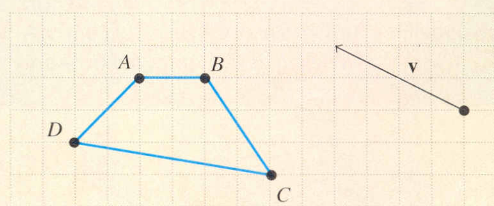
13. Determine the volume of the pyramid.



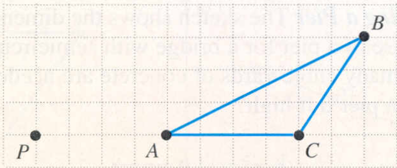
14. Construct a reflection of square $ABCD$, shown below, about line l. Show the square in the positions both before and after the reflection.



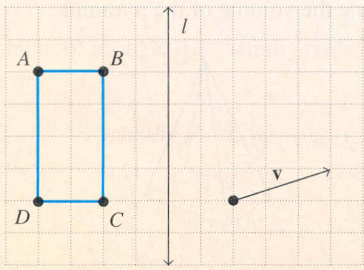
15. Construct a translation of quadrilateral $ABCD$, shown below, using translation vector **v**. Show the quadrilateral in the positions both before and after the translation.



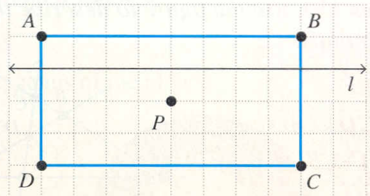
16. Construct a 180° rotation of triangle ABC, shown below, about rotation point P. Show the triangle in the positions both before and after the rotation.



17. Construct a glide reflection of rectangle $ABCD$, shown below, using translation vector **v** and reflection line l. Show the rectangle in the positions both before and after the glide reflection.



18. Use the figure below to answer the following questions.



a) Does rectangle $ABCD$ have reflective symmetry about line l? Explain.
b) Does rectangle $ABCD$ have 180° rotational symmetry about point P? Explain.

19. What is a Möbius strip?

20. a) Sketch an object of genus 1.
b) Sketch an object of genus 2.

21. Explain how the fifth axiom in Euclidean geometry, elliptical geometry, and hyperbolic geometry differ.

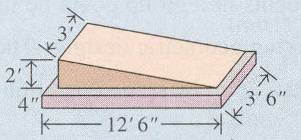
GROUP PROJECTS

Supporting a Jacuzzi

1. Samantha Saraniti is thinking of buying a circular hot tub 12 ft in diameter, 4 ft deep, and weighing 475 lb. She wants to place the hot tub on a deck built to support 30,000 lb.
   a) Determine the volume of the water in the hot tub in cubic feet.
   b) Determine the number of gallons of water the hot tub will hold. Note 1 $\text{ft}^3 \approx 7.5$ gal.
   c) Determine the weight of the water in the hot tub. (*Hint:* Fresh water weighs about 52.4 lb/ft^3.)
   d) Will the deck support the weight of the hot tub and water?
   e) Will the deck support the weight of the hot tub, water, and four people, whose average weight is 115 lb?

Designing a Ramp

2. David and Sandra Jessee are planning to build a ramp so that their front entrance is wheelchair accessible. The ramp will be 36 in. wide. It will rise 2 in. for each foot of length of horizontal distance. Where the ramp meets the porch, the ramp must be 2 ft high. To provide stability for the ramp, the Jessees will install a slab of concrete 4 in. thick and 6 in. longer and wider than the ramp (see accompanying figure). The top of the slab will be level with the ground. The ramp may be constructed of concrete or pressure-treated lumber. You are to estimate the cost of materials for constructing the slab, the ramp of concrete, and the ramp of pressure-treated lumber.



Slab

a) Determine the length of the base of the ramp.
b) Determine the dimensions of the concrete slab on which the ramp will set.
c) Determine the volume of the concrete in cubic yards needed to construct the slab.
d) If ready-mix concrete costs \$45 per cubic yard, determine the cost of the concrete needed to construct the slab.

Concrete Ramp

e) To build the ramp of concrete a form in the shape of the ramp must be framed. The two sides of the form are triangular, and the shape of the end, which is against the porch is rectangular. The form will be framed from $\frac{3}{4}$ in. plywood, which comes in 4 ft × 8 ft sheets. Determine the number of sheets of plywood needed. Assume that the entire sheet(s) will be used to make the sides and the end of the form and that there is no waste.
f) If the plywood costs \$18.95 for a 4 ft × 8 ft sheet, determine the cost of the plywood.
g) To brace the form, the Jessees will need two boards 2 in. × 4 in. × 8 ft (referred to as 8 ft 2 × 4's) and six pieces of lumber 2 in. × 4 in. × 3 ft. These six pieces of lumber will be cut from 8 ft 2 × 4 boards. Determine the number of 8 ft 2 × 4 boards needed.
h) Determine the cost of the 8 ft 2 × 4 boards needed in part (g) if one board costs \$2.14.
i) Determine the volume, in cubic yards, of concrete needed to fill the form.
j) Determine the cost of the concrete needed to fill the form.
k) Determine the total cost of materials for building the ramp of concrete by adding the results in parts (d), (f), (h), and (j).

Wooden Ramp

l) Determine the length of the top of the ramp.
m) The top of the ramp will be constructed of $\frac{5}{4}$ in. × 6 in. × 10 ft pressure-treated lumber. The boards will be butted end to end to make the necessary length and will be supported from underneath by a wooden frame. Determine the number of boards needed to cover the top of the ramp. The boards are laid lengthwise on the ramp.
n) Determine the cost of the boards to cover the top of the ramp if the price of a 10 ft length is \$6.47.
o) To support the top of the ramp, the Jessees will need 10 pieces of 8 ft 2 × 4's. The price of a pressure-treated 8 ft 2 × 4 is \$2.44. Determine the cost of the supports.
p) Determine the cost of the materials for building a wooden ramp by adding the amounts from parts (d), (n), and (o).
q) Are the materials for constructing a concrete ramp or a wooden ramp less expensive?